Contribution en analyse et mécanique unilatérale

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Équipe d'Analyse et Mécanique Unilatérale

HABILITATION A DIRIGER DES RECHERCHES

Spécialité : **Mathématiques Appliquées**

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Présentée par

Mohamed ROCHDI

Le 21 novembre 2000

Composition du jury :

Directeur de Recherches :

Daniel Goeleven	Professeur	Université de La Réunion, France

Rapporteurs :

Meir Shillor	Professeur	Oakland University, Michigan, USA
Ken L. Kuttler	Professeur	Brigham Young University, Utah, USA
Michel Raous	Directeur de Recherches	CNRS, Université de la méditerrannée, Marseille

Examinateurs :

Michel Frémond	Directeur de Recherches	Laboratoire Central des Ponts et Chaussées, Paris
Bernard Brogliato	Chargé de Recherches	CNRS, Université Joseph Fourier, Grenoble

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Introduction

Comme l'indique le titre, les travaux présentés dans ce mémoire constituent une contribution dans le domaine de l'analyse et la mécanique unilatérale. Il s'agit du fruit de six années de recherches commencées en septembre 1994. La plupart des travaux présentés portent sur le développement de méthodes d'analyse pour l'étude théorique des problèmes de la mécanique du contact. Ces problèmes sont modélisés sous forme de systèmes d'équations et d'inéquations aux dérivées partielles mettant en jeu plusieurs sortes de phénomènes non linéaires tels que les phénomènes de frottement, d'impact, de délamination, d'adhérence, d'usure, d'endommagement, etc. L'objectif de ces travaux est de trouver des formulations faibles pour les modèles étudiés et de procéder à l'étude théorique de ces formulations sous certaines hypotèses. Les formulations faibles auxquelles on aboutit dans ces travaux sont en général des couplages entre des équations et des inéquations variationnelles. Il se peut que l'on aboutisse à une inéquation hémivariationnelle comme c'est le cas dans l'un des travaux. Dans les études théoriques menées, différents outils mathématiques ont été explorés : analyse fonctionnelle, analyse convexe, théorie des semi-groupes, etc.

Les travaux contenus dans ce mémoire sont soit parus, soit acceptés, soit soumis pour publication. Ils sont présentés dans l'ordre chronologique de leur réalisation. Chaque article est présenté dans sa version parue ou acceptée et il est précédé d'un court descriptif en français résumant son contenu technique. On peut regrouper ces articles en trois catégories de problèmes qui montrent le cheminement progressif que j'ai suivi dans le domaine.

La première catégorie concerne les problèmes sans frottement pour les matériaux viscoplastiques. Il s'agit d'un premier travail qui consiste à généraliser tout un groupe de problèmes quasistatiques ou dynamiques du type déplacement-traction sous une même forme suivi de quelques travaux portant sur des problèmes de contact sans frottement suivant les conditions de Signorini.

La seconde catégorie traite de quelques problèmes quasistatiques de contact avec frottement pour les matériaux élastiques, viscoélastiques ou viscoplastiques. Plusieurs types de contact avec frottement sont considérés : loi de Tresca, loi de Coulomb, compliance normale, usure, etc. Ce travail a été réalisé pour une majeure partie après ma thèse de Doctorat.

La troisième catégorie coïncide avec mon arrivée à l'Université de La Réunion. Les problèmes traités pendant cette période sont divers. Une partie du travail concerne des problèmes quasistatiques ou dynamiques de contact avec frottement pour des matériaux viscoélastiques tenant compte de lois de phénomènes plus complexes faisant intervenir entre autres la température, le dégagement de chaleur au niveau de la surface de contact, l'endommagement du matériau, le phénomène d'adhérence, etc. Une autre partie du travail porte sur des problèmes unilatéraux relevant de la mécanique des systèmes. Il s'agit là de l'investigation de problèmes ponctuels sur lesquels une étude théorique suivie d'une étude numérique conduisant à des simulations numériques ont été menés. Dans ces problèmes, les phénomènes de frottement ainsi que d'impact ont été pris en compte en se basant sur des travaux très récents dans ce domaine. Les simulations numériques dans ce cadre ont été concluantes.

On Rate-Type Viscoplastic Problems with Linear Boundary Conditions

M. ROCHDI et M. SOFONEA

On Rate-Type Viscoplastic Problems with Linear Boundary Conditions

M. Rochdi et M. Sofonea

Descriptif

Dans cet article, on introduit le concept abstrait de "*conditions aux limites linéaires*" qui est destiné à généraliser un bon nombre de conditions aux limites classiques telles que celles de déplacement-traction.

On considère un corps occupant le domaine Ω de $\mathbb{R}^N$ $(N = 1, 2, 3)$ et on introduit l'espace $\mathcal{H}_1$ défini par

$$\mathcal{H}_1 = \Big\{ \sigma = (\sigma_{ij}) \in L^2(\Omega)_s^{N\times N} \mid \sigma_{ij,j} \in L^2(\Omega)^N,\ i = 1, \ldots, N \Big\}.$$

On note par ν la normale unitaire sortante à Ω et par $\gamma : H^1(\Omega)^N \longrightarrow H^{\frac{1}{2}}(\Gamma)^N$ l'application trace. On dit que ce corps est soumis à des conditions aux limites linéaires si le champ de ses déplacements u et le champ de ses contraintes σ vérifient les relations

(1) $$u \in \hat{u} + V,$$

(2) $$\sigma \in \hat{\sigma} + \mathcal{V},$$

où $\hat{u} \in H^1(\Omega)^N$ et $\hat{\sigma} \in \mathcal{V}$ ainsi que V qui est un sous-espace fermé non vide de $H^1(\Omega)^N$ sont des données relatives au type de conditions aux limites mécaniques considérées, alors que $\mathcal{V}$ est donné par

$$\mathcal{V} = \Big\{ \sigma \in \mathcal{H}_1 \mid \langle \sigma\nu, \gamma v \rangle_{H^{-\frac{1}{2}}(\Gamma)^N \times H^{\frac{1}{2}}(\Gamma)^N} = 0 \quad \forall v \in V \Big\}.$$

Ce travail concerne l'étude du problème portant sur un milieu continu viscoplastique ayant une loi de comportement de la forme

$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u)),$$

soumis à des conditions aux limites linéaires. Ici, $\mathcal{E}$ et G sont des fonctions constitutives, et $\varepsilon(u)$ est le tenseur des petites déformations linéarisé. Afin de justifier ce nouveau concept, on commence par citer plusieurs exemples de conditions aux limites mécaniques concrètes pouvant se mettre sous la forme (1)–(2). On poursuit ensuite avec l'analyse du

problème aux limites en question dans le cas quasistatique. L'existence et l'unicité de la solution est alors prouvée moyennant une méthode de point fixe. Le cas dynamique est ensuite considéré et, là aussi, l'existence et l'unicité de la solution est établie, en utilisant des arguments de la théorie des semi-groupes d'opérateurs linéaires. Commme application des résultats généraux obtenus, aussi bien dans le cas quasistatique que dans le cas dynamique, des exemples concrets de problèmes aux limites sont présentés et des interprétations mécaniques données.

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On Rate–Type Viscoplastic Problems with Linear Boundary Conditions

By M. Rochdi and M. Sofonea of Perpignan

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Abstract. In this paper we introduce the abstract concept of "linear boundary conditions" in the study of deformable bodies. We establish two existence and uniqueness results concerning respectively quasistatic and dynamic problems involving such type of boundary conditions. We also apply these existence results in the study of viscoplastic problems involving classical boundary conditions.

1. Introduction

In this paper we consider initial and boundary value problems for rate–type viscoplastic models of the form

$$\dot{\sigma} = \mathcal{E}\dot{\varepsilon} + G(\sigma, \varepsilon) \tag{1.1}$$

in which σ denotes the stress tensor and ε represents the small strain tensor. Here and everywhere in this paper the dot above represents the derivative with respect to the time variable.

Constitutive laws of the form (1.1) were proposed in order to describe real materials like rubbers, metals, rocks and so on. Various results and mechanical interpretations concerning such type of models may be found for instance in [CRISU].

Existence and uniqueness results for initial and boundary value problems involving models of the form (1.1) were given by I. R. Ionescu and M. Sofonea [IONSO1], S. Djabi and M. Sofonea [DJASO] in the quasistatic case and by I. R. Ionescu [IONES] in the dynamic case. A more detailed presentation concerning various functional and numerical methods in the study of the models (1.1) as well as some complete

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applications in engineering sciences may be found in [IONSO2]. In all the above quotated works only problems involving classical displacement – traction boundary conditions were considered, in which the displacement field is imposed on a given part of the edge of the viscoplastic body and the stress field is imposed on the complementary part of the boundary (see Example 3.3 in Section 3).

The purpose of this paper is to study some problems associated to viscoplastic models of the form (1.1) and involving more general boundary conditions. Both the quasistatic and dynamic cases are discussed in the context of small perturbations theory. The paper is structured as follows: in Section 2, we summerize some known facts concerning Sobolev – type functional spaces including several results on the trace maps; in Section 3, the concept of "linear boundary conditions" is defined and the interpretation of this concept in the case of some classical examples are considered. The following two sections are dedicated to the study of initial and boundary value problems involving linear boundary conditions; so, in Section 4 the quasistatic case is considered and an existence and uniqueness result is obtained by using fixed point method (Theorem 4.1); in Section 5, the dynamic case is studied and here, the existence and uniqueness of the solution is obtained by using arguments of semigroups of linear operators theory (Theorem 5.1). In both these two last sections concrete examples and interpretations of the linear boundary conditions are presented (Corollaries 4.4 and 5.5).

2. Notations and preliminaries

In all this paper, we denote by S_N the space of second order symmetric tensors on $\mathbb{R}^N$ $(N = 1, 2, 3)$ while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on S_N and $\mathbb{R}^N$. Let us now consider a bounded domain $\Omega \subset \mathbb{R}^N$ with boundary Γ. We suppose that Ω belongs to the class $C^{1,1}$ (see for example [NECAS]) and we denote by ν the unit outer normal on Γ. The following notations are also used:

$$\begin{aligned} H &= \{ u = (u_i) \mid u_i \in L^2(\Omega),\ i = \overline{1,N} \}, \\ \mathcal{H} &= \{ \sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega),\ i,j = \overline{1,N} \}, \\ H_1 &= \{ u = (u_i) \mid u_i \in H^1(\Omega),\ i = \overline{1,N} \}, \\ \mathcal{H}_1 &= \{ \sigma \in \mathcal{H} \mid \sigma_{ij,j} \in H,\ i = \overline{1,N} \}. \end{aligned}$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products given by

$$\begin{aligned} \langle u, v \rangle_H &= \int_\Omega u_i v_i \, dx, \\ \langle \sigma, \tau \rangle_{\mathcal{H}} &= \int_\Omega \sigma_{ij} \tau_{ij} \, dx, \\ \langle u, v \rangle_{H_1} &= \langle u, v \rangle_H + \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}, \\ \langle \sigma, \tau \rangle_{\mathcal{H}_1} &= \langle \sigma, \tau \rangle_{\mathcal{H}} + \langle \operatorname{Div} \sigma, \operatorname{Div} \tau \rangle_H, \end{aligned}$$

where $\varepsilon : H_1 \to \mathcal{H}$ and Div : $\mathcal{H}_1 \to H$ are respectively the deformation and the divergence operators defined by

$$\varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}),$$
$$\mathrm{Div}\,\sigma = (\sigma_{ij,j})_i, \quad i,j = \overline{1,N}.$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are respectively denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$.

Let us now denote by $\mathcal{R}$ the set of rigid displacements defined by

$$\mathcal{R} = \{ u \in H_1 \mid \varepsilon(u) = 0 \}$$

and let V be a closed subspace of H_1. If

(2.1) $$V \cap \mathcal{R} = \{0\},$$

then Korn's inequality holds:

(2.2) $$|\varepsilon(u)|_{\mathcal{H}} \geq C\,|u|_{H_1} \quad \text{for all} \quad u \in V$$

where C is a strictely positive constant (see for instance [NEHLA] p. 79).

Let $H_\Gamma = [H^{\frac{1}{2}}(\Gamma)]^N$ and $\xi \in H_\Gamma$. We define the normal and the tangential components of ξ by

(2.3) $$\xi_\nu = \xi \cdot \nu \quad \text{and} \quad \xi_\tau = \xi - \xi_\nu \nu.$$

So, denoting by H_τ the closed subspace of H_Γ defined by

$$H_\tau = \{ \xi \in H_\Gamma \mid \xi_\nu = 0 \text{ a.e. on } \Gamma \},$$

it can be proved that the mapping $\xi \mapsto (\xi_\nu, \xi_\tau)$ is an isomorphism from H_Γ onto $H^{\frac{1}{2}}(\Gamma) \times H_\tau$ (see for example [HUNLI] or [PANAG] p. 32).

In the sequel, we denote by H'_Γ and H'_τ respectively the strong dual spaces of H_Γ and H_τ. We also denote by $\langle \cdot, \cdot \rangle_\Gamma$, $\langle \cdot, \cdot \rangle_{1/2}$, $\langle \cdot, \cdot \rangle_\tau$ the duality pairing mappings between H'_Γ and H_Γ, $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$, H'_τ and H_τ.

For all $\xi' \in H'_\Gamma$ the normal and tangential components of ξ' are respectively defined by

(2.4) $$\langle \xi'_\nu, \xi \rangle_{1/2} = \langle \xi', \xi\nu \rangle_\Gamma \quad \text{for all} \quad \xi \in H^{\frac{1}{2}}(\Gamma),$$

(2.5) $$\langle \xi'_\tau, \xi \rangle_\tau = \langle \xi', \xi \rangle_\Gamma \quad \text{for all} \quad \xi \in H_\tau.$$

The mapping $\xi' \mapsto (\xi'_\nu, \xi'_\tau)$ is an isomorphism from H'_Γ onto $H^{-\frac{1}{2}}(\Gamma) \times H'_\tau$ (see again [HUNLI] or [PANAG] p. 32) and using (2.3) – (2.5) it result

(2.6) $$\langle \xi', \xi \rangle_\Gamma = \langle \xi'_\nu, \xi_\nu \rangle_{1/2} + \langle \xi'_\tau, \xi_\tau \rangle_\tau \quad \text{for all} \quad \xi' \in H'_\Gamma, \ \xi \in H_\Gamma.$$

Let $\gamma : H_1 \to H_\Gamma$ be the trace map. It is well known that γ is a linear, continuous and surjective map and there exists $z : H_\Gamma \to H_1$ a linear and continuous map such that

(2.7) $$\gamma(z(\xi)) = \xi \quad \text{for all} \quad \xi \in H_\Gamma.$$

For all $u \in H_1$ we define $\gamma_\nu u \in H^{\frac{1}{2}}(\Gamma)$ and $\gamma_\tau u \in H_\tau$ given by

$$\gamma_\nu u = (\gamma u)_\nu, \quad \gamma_\tau u = (\gamma u)_\tau .$$

Let us also recall that

$$\gamma u = u_{|\Gamma}, \quad \gamma_\nu u = u_{|\Gamma} \cdot \nu, \quad \gamma_\tau u = u_{|\Gamma} - (u_{|\Gamma} \cdot \nu)\nu \tag{2.8}$$

for all $u \in C^1(\overline{\Omega})^N$. For this reason and for simplicity, we shall use sometimes the notations u, u_ν, u_τ instead of γu, $\gamma_\nu u$, $\gamma_\tau u$ for all $u \in H_1$.

Let now $\sigma \in \mathcal{H}_1$; there exists an element $\overline{\gamma}\sigma \in H'_\Gamma$ such that

$$\langle \overline{\gamma}\sigma, \gamma u\rangle_\Gamma = \langle \sigma, \varepsilon(u)\rangle_{\mathcal{H}} + \langle \operatorname{Div} \sigma, u\rangle_H \quad \text{for all} \quad u \in H_1 . \tag{2.9}$$

The map $\overline{\gamma} : \mathcal{H}_1 \to H'_\Gamma$ is linear, continuous, surjective and there exists a linear continuous map $\overline{z} : H'_\Gamma \to \mathcal{H}_1$ such that

$$\overline{\gamma}(\overline{z}(\Sigma)) = \Sigma \quad \text{for all} \quad \Sigma \in H'_\Gamma . \tag{2.10}$$

We also define $\overline{\gamma}_\nu \sigma \in H^{-\frac{1}{2}}(\Gamma)$ and $\overline{\gamma}_\tau \sigma \in H'_\tau$ by

$$\overline{\gamma}_\nu \sigma = (\overline{\gamma}\sigma)_\nu, \quad \overline{\gamma}_\tau \sigma = (\overline{\gamma}\sigma)_\tau .$$

Let us also recall that if $\sigma \in C^1(\overline{\Omega})_s^{N\times N}$ then, from (2.9), (2.4) and (2.5), it results

$$\overline{\gamma}\sigma = \sigma_{|\Gamma}\nu, \quad \overline{\gamma}_\nu \sigma = (\sigma_{|\Gamma}\nu)_\nu, \quad \overline{\gamma}_\tau \sigma = \sigma_{|\Gamma}\nu - (\sigma_{|\Gamma}\nu)\nu . \tag{2.11}$$

For this reason and for simplicity, we shall use sometimes the notations $\sigma\nu$, σ_ν, σ_τ instead of $\overline{\gamma}\sigma$, $\overline{\gamma}_\nu\sigma$, $\overline{\gamma}_\tau\sigma$ for all $\sigma \in \mathcal{H}_1$. So, using (2.6) and (2.9) we obtain

$$\begin{aligned} \langle \sigma\nu, \gamma u\rangle_\Gamma &= \langle \sigma_\nu, u_\nu\rangle_{1/2} + \langle \sigma_\tau, u_\tau\rangle_\tau \\ &= \langle \sigma, \varepsilon(u)\rangle_{\mathcal{H}} + \langle \operatorname{Div} \sigma, u\rangle_H \end{aligned} \tag{2.12}$$

for all $u \in H_1$ and $\sigma \in \mathcal{H}_1$.

Let us now consider a partition of Γ into two disjoint measurable sets R and S ($\Gamma = R \cup S$, $R \cap S = \emptyset$) and let $\sigma \in \mathcal{H}_1$. Everywhere in this paper we use the following definitions:

$$\sigma\nu = 0 \text{ on } R \iff \langle \sigma\nu, \gamma u\rangle_\Gamma = 0 \text{ for all } u \in H_1 \text{ such that } u = 0 \text{ on } S . \tag{2.13}$$

$$\sigma_\nu = 0 \text{ on } R \iff \langle \sigma\nu, \gamma u\rangle_\Gamma = 0 \text{ for all } u \in H_1 \text{ such that } \begin{cases} u_\nu = 0 \text{ on } S, \\ u_\tau = 0 \text{ on } \Gamma. \end{cases} \tag{2.14}$$

$$\sigma_\tau = 0 \text{ on } R \iff \langle \sigma\nu, \gamma u\rangle_\Gamma = 0 \text{ for all } u \in H_1 \text{ such that } \begin{cases} u_\tau = 0 \text{ on } S, \\ u_\nu = 0 \text{ on } \Gamma. \end{cases} \tag{2.15}$$

We say that $\sigma\nu = h$ on R if $\sigma\nu - h = 0$ on R and the equalities $\sigma_\nu = h$ on R and $\sigma_\tau = h$ on R are defined in a similar way.

Finally, let us notice that if X is one of the above real Hilbert spaces, $1 \le p \le \infty$, $k \in \mathbb{N}$ and $T > 0$, we use the classical notations $L^\infty(0,T,X)$, $W^{k,p}(0,T,X)$ and we denote by $|\cdot|_{\infty,X}$, $|\cdot|_{1,\infty,X}$ the norms on the spaces $L^\infty(0,T,X)$ and respectively $W^{1,\infty}(0,T,X)$. We also recall that if X and Y are two real Hilbert spaces, we denote by $\mathcal{L}(X,Y)$ the space of linear continuous operators from X into Y.

3. Linear boundary conditions

In what follows, we suppose that a deformable body occupies a bounded domain $\Omega \subset \mathbb{R}^N$ $(N = 1, 2, 3)$ which is assumed to belong to the class $C^{1,1}$. The evolution of this body is modeled by an initial and boundary value problem involving the constitutive law, the strain-displacement relation, the equilibrium or the motion equation as well as the initial and boundary conditions. The unknowns of this problem are the displacement field $u \in H_1$ and the stress field $\sigma \in \mathcal{H}_1$.

In this section, we present an abstract formulation for boundary conditions which includes a number of known classical conditions as well as the displacement-traction conditions already studied in [DJASO] and [IONES] - [IONSO2]. So, we say that the displacement vector u and the stress tensor σ are submitted to "linear boundary conditions" if there exists $\hat{u} \in H_1$, $\hat{\sigma} \in \mathcal{H}_1$ and a closed subspace V of H_1 such that

(3.1) $$u \in \hat{u} + V,$$

(3.2) $$\sigma \in \hat{\sigma} + \mathcal{V},$$

where $\mathcal{V}$ is the closed subspace of $\mathcal{H}_1$ defined by

(3.3) $$\mathcal{V} = \{ \sigma \in \mathcal{H}_1 \mid \langle \sigma\nu, \gamma v \rangle_\Gamma = 0 \text{ for all } v \in V \}.$$

We now study the link between the previous definition and a number of classical boundary conditions.

Example 3.1. Let Γ_i, $i = \overline{1,4}$, be measurable sets such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j$ and let us consider the following boundary conditions:

(3.4) $$\begin{cases} u = g_1 & \text{on } \Gamma_1, \\ \sigma\nu = h_2 & \text{on } \Gamma_2, \\ u_\nu = g_3, \quad \sigma_\tau = h_3 & \text{on } \Gamma_3, \\ u_\tau = g_4, \quad \sigma_\nu = h_4 & \text{on } \Gamma_4. \end{cases}$$

We suppose that there exists $g \in H_\Gamma$ and $h \in L^2(\Gamma)^N$ such that

(3.5) $$g = g_1 \text{ on } \Gamma_1, \quad g_\nu = g_3 \text{ on } \Gamma_3, \quad g_\tau = g_4 \text{ on } \Gamma_4,$$

(3.6) $$h = h_2 \text{ on } \Gamma_2, \quad h_\tau = h_3 \text{ on } \Gamma_3, \quad h_\nu = h_4 \text{ on } \Gamma_4.$$

Then, taking $\hat{u}$, $\hat{\sigma}$ and V given by

(3.7) $$\hat{u} = z g,$$

(3.8) $$\hat{\sigma} = \overline{z} h,$$

(3.9) $$V = \{ v \in H_1 \mid v = 0 \text{ on } \Gamma_1, \ v_\nu = 0 \text{ on } \Gamma_3, \ v_\tau = 0 \text{ on } \Gamma_4 \},$$

we have the following result:

Lemma 3.2. *If u and σ are regular (say C^1) functions satisfying* (3.4), *then* (3.1) *and* (3.2) *hold. Conversely, if (u,σ) satisfies the linear boundary conditions* (3.1) *and* (3.2), *then* (3.4) *holds in the sense of traces given in Section* 2.

Proof. Let (u,σ) be a regular couple of functions satisfying (3.4). Using (3.7), (2.7), (3.5) and (3.4) we obtain $u-\hat{u}\in V$, i. e., (3.1) holds. In the same way, from (3.8), (2.10), (3.6) and (3.4) it follows

$$\langle\sigma\nu-\hat{\sigma}\nu,\gamma v\rangle_\Gamma \;=\; \int_\Gamma(\sigma\nu-h)v\,da \;=\; 0 \quad\text{for all}\quad v\in V$$

hence by (3.3) we get $\sigma-\hat{\sigma}\in\mathcal{V}$, i.e., (3.2) holds.

Conversely, let $u\in H_1$, $\sigma\in\mathcal{H}_1$ such that (3.1) and (3.2) hold. Using (3.7) and (3.5) we obtain

$$u = g_1 \quad\text{on}\quad \Gamma_1\,,\qquad u_\nu = g_3 \quad\text{on}\quad \Gamma_3\,,\qquad u_\tau = g_4 \quad\text{on}\quad \Gamma_4 \tag{3.10}$$

in the sense traces on Γ. Moreover, having in mind (3.3), (3.8) and (2.10) we obtain

$$\langle\sigma\nu-h,\gamma v\rangle_\Gamma \;=\; 0 \quad\text{for all}\quad v\in V\,. \tag{3.11}$$

Taking now $v\in H_1$ such that $v=0$ on $\Gamma_1\cup\Gamma_2\cup\Gamma_3$, from (3.11) and (2.13) it results $\sigma\nu-h=0$ on Γ_2. In a similar way, using (2.14) and (2.15) we obtain $\sigma_\tau-h_\tau=0$ on Γ_3, $\sigma_\nu-h_\nu=0$ on Γ_4. Using now (3.6) it results

$$\sigma\nu = h_2 \quad\text{on}\quad \Gamma_2\,,\qquad \sigma_\tau = h_3 \quad\text{on}\quad \Gamma_3\,,\qquad \sigma_\nu = h_4 \quad\text{on}\quad \Gamma_4\,. \tag{3.12}$$

Lemma 3.2 follows now from (3.10) and (3.12). □

Example 3.3. Let Γ_1,Γ_2 be two measurable subsets of Γ such that $\Gamma=\Gamma_1\cup\Gamma_2$, $\Gamma_1\cap\Gamma_2=\emptyset$ and let us consider the following boundary conditions:

$$\begin{cases} u = g & \text{on}\quad \Gamma_1\,,\\ \sigma\nu = h & \text{on}\quad \Gamma_2\,.\end{cases} \tag{3.13}$$

We suppose that $g\in H_\Gamma$ and $h\in H'_\Gamma$. Then, using the same technique as in the proof of Lemma 3.2 we obtain that the conditions (3.13) are equivalent to the linear boundary conditions (3.1) – (3.2) where $\hat{u}=zg$, $\hat{\sigma}=\tilde{z}h$ and $V=\{v\in H_1 \mid v=0 \text{ on } \Gamma_1\}$.

Example 3.4. Let Γ_1,Γ_2 be two measurable subsets of Γ such that $\Gamma=\Gamma_1\cup\Gamma_2$, $\Gamma_1\cap\Gamma_2=\emptyset$ and let us consider the following boundary conditions:

$$\begin{cases} u_\nu = g\,,\quad \sigma_\tau = h_1 & \text{on}\quad \Gamma_1\,,\\ \sigma\nu = h_2 & \text{on}\quad \Gamma_2\,.\end{cases} \tag{3.10}$$

We suppose that $g\in H^{\frac{1}{2}}(\Gamma)$, $h_1\in L^2(\Gamma_1)^N$, $h_2\in L^2(\Gamma_2)^N$ and let $h\in L^2(\Gamma)^N$ be a function such that $h_\tau=h_1$ on Γ_1, $h=h_2$ on Γ_2. Taking $\hat{u}$, $\hat{\sigma}$ and V given by $\hat{u}=z(g\nu)$, $\hat{\sigma}=\tilde{z}\,h$, $V=\{v\in H_1 \mid v_\nu=0 \text{ on } \Gamma_1\}$, as in Example 3.1 we obtain that if u and σ are regular functions satisfying the boundary conditions (3.14) then (u,σ) satisfies the linear boundary conditions (3.1) – (3.2). Conversely, if (u,σ) satisfies the linear boundary conditions (3.1) – (3.2), then (3.14) holds in the sense of traces.

Example 3.5. Let Γ_1, Γ_2 be two measurable subsets of Γ such that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$ and let us consider the following boundary conditions:

$$\text{(3.11)} \qquad \begin{cases} u_\tau = g, & \sigma_\nu = h_1 \quad \text{on} \quad \Gamma_1, \\ \sigma\nu = h_2 & \text{on} \quad \Gamma_2. \end{cases}$$

We suppose that $g \in H_\tau$, $h_1 \in L^2(\Gamma_1)$, $h_2 \in L^2(\Gamma_2)^N$ and let $h \in L^2(\Gamma)^N$ be a function such that $h = h_1\nu$ on Γ_1, $h = h_2$ on Γ_2. Taking $\hat{u}$, $\hat{\sigma}$ and V given by $\hat{u} = zg$, $\hat{\sigma} = \bar{z}h$, $V = \{v \in H_1 \mid v_\tau = 0 \text{ on } \Gamma_1\}$, as in Example 3.1 we obtain that if u and σ are regular functions satisfying the boundary conditions (3.15) then (u,σ) satisfies the linear boundary conditions (3.1) – (3.2). Conversely, if (u,σ) satisfies the linear boundary conditions (3.1) – (3.2), then (3.15) holds in the sense of traces.

Let us finally remark that in the case of evolution problems, the dependence of the boundary conditions in time is taken into account by considering time dependence given by the data $\hat{u}$ and $\hat{\sigma}$ while V is a fixed subspace of H_1 and $\mathcal{V}$ is defined by (3.3). So, in the sequel, the linear boundary conditions (3.1) – (3.2) will be considered for all $t \in [0,T]$, T representing the duration of the evolution processes.

4. Quasistatic processes for rate-type viscoplastic materials

In this section, we consider the quasitatic evolution of a viscoplastic body submitted to linear boundary conditions. This evolution is given by the following mixed problem: find the displacement field $u : \Omega \times [0,T] \to \mathbb{R}^N$ and the stress field $\sigma : \Omega \times [0,T] \to S_N$ such that:

$$\text{(4.1)} \qquad \dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u)) \quad \text{in} \quad \Omega \times (0,T),$$

$$\text{(4.2)} \qquad \operatorname{Div}\sigma + f = 0 \quad \text{in} \quad \Omega \times (0,T),$$

$$\text{(4.3)} \qquad u \in \hat{u} + V \quad \text{on} \quad (0,T),$$

$$\text{(4.4)} \qquad \sigma \in \hat{\sigma} + \mathcal{V} \quad \text{on} \quad (0,T),$$

$$\text{(4.5)} \qquad u(0) = u_0, \quad \sigma(0) = \sigma_0 \quad \text{in} \quad \Omega.$$

The evolution equation (4.1) represents the constitutive law already presented in Section 1 in which $\varepsilon = \varepsilon(u)$ is the small strain tensor; (4.2) represents the equilibrium equation in which f are the given body forces; (4.3) and (4.4) are the linear boundary conditions discussed in Section 3 and, finally, (4.5) represents the initial conditions.

In the study of the problem (4.1) – (4.5) we consider the following assumptions:

$$
(4.6)\quad \begin{cases} \mathcal{E} : \Omega \times S_N \to S_N \text{ is a bounded, symmetric and positively definite tensor}, \\ \text{i.e.}, \\ \text{(a) } \mathcal{E}_{ijkh} \in L^\infty(\Omega) \text{ for all } i,\, j,\, k,\, h = \overline{1,N}, \\ \text{(b) } \mathcal{E}\sigma \cdot \tau = \sigma \cdot \mathcal{E}\tau \text{ for all } \sigma,\, \tau \in S_N \text{ a.e. in } \Omega, \\ \text{(c) there exists } \alpha > 0 \text{ such that } \mathcal{E}\sigma \cdot \sigma \geq \alpha\, |\sigma|^2 \text{ for all } \sigma \in S_N, \end{cases}
$$

$$
(4.7)\quad \begin{cases} G : \Omega \times S_N \times S_N \to S_N \text{ and} \\ \text{(a) there exists } L > 0 \text{ such that} \\ \qquad |G(x,\sigma_1,\varepsilon_1) - G(x,\sigma_2,\varepsilon_2)| \leq L\big(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2|\big) \\ \qquad \text{for all } \sigma_1,\, \sigma_2,\, \varepsilon_1,\, \varepsilon_2 \in S_N \text{ a.e. in } \Omega, \\ \text{(b) } x \mapsto G(x,\sigma,\varepsilon) \text{ is a measurable function with respect to the Lebesgue} \\ \qquad \text{measure on } \Omega \text{ for all } \sigma,\, \varepsilon \in S_N, \\ \text{(c) } x \mapsto G(x,0,0) \in \mathcal{H}, \end{cases}
$$

(4.8) $\quad f \in W^{1,\infty}(0,T,H_1)$,

(4.9) $\quad \hat{u} \in W^{1,\infty}(0,T,H_1), \quad \hat{\sigma} \in W^{1,\infty}(0,T,\mathcal{H}_1)$,

(4.10) $\quad V$ is a closed subspace of H_1 and $\mathcal{V}$ is defined by (3.3),

(4.11) $\quad u_0 \in H_1, \quad \sigma_0 \in \mathcal{H}_1$,

(4.12) $\quad \operatorname{Div}\sigma_0 + f(0) = 0 \text{ in } \Omega, \quad u_0 \in \hat{u}(0) + V, \quad \sigma_0 \in \hat{\sigma}(0) + \mathcal{V}$.

Moreover, everywhere in this section (as well as in Section 5), we denote by C strictely positive constants which may depend on Ω, V, $\mathcal{E}$, G, T and do not depend on time or input data.

The main result of this section is the following:

Theorem 4.1. *Let* (4.6) – (4.12) *and* (2.1) *hold. Then, there exists a unique solution* $u \in W^{1,\infty}(0,T,H_1)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H}_1)$ *of the problem* (4.1) – (4.5).

In order to prove Theorem 4.1 we need some preliminary results. For this, let us suppose in the sequel that the assumptions of Theorem 4.1 are fulfilled and let $\eta \in L^\infty(0,T,\mathcal{H})$. Let also $z_\eta \in W^{1,\infty}(0,T,\mathcal{H})$ be the function defined by

$$
(4.13)\qquad z_\eta(t) \;=\; \int_0^t \eta(s)\,ds + z_0 \quad \text{for all} \quad t \in [0,T],
$$

where

$$
(4.14)\qquad z_0 \;=\; \sigma_0 - \mathcal{E}\varepsilon(u_0).
$$

Lemma 4.2. *There exists a unique couple of functions* $u_\eta \in W^{1,\infty}(0,T,H_1)$, $\sigma_\eta \in W^{1,\infty}(0,T,\mathcal{H}_1)$ *such that*

$$
(4.15)\qquad \sigma_\eta \;=\; \mathcal{E}\varepsilon(u_\eta) + z_\eta,
$$

$$\operatorname{Div} \sigma_\eta + f = 0, \tag{4.16}$$

$$u_\eta \in \hat{u} + V, \tag{4.17}$$

$$\sigma_\eta \in \hat{\sigma} + \mathcal{V} \tag{4.18}$$

for all $t \in [0,T]$. *Moreover,*

$$u_\eta(0) = u_0, \quad \sigma_\eta(0) = \sigma_0. \tag{4.19}$$

Proof. Let $t \in [0,T]$. Using (4.6), (2.2) and Lax-Milgram theorem, we obtain the existence and uniqueness of an element $\tilde{u}_\eta(t)$ such that

$$\begin{aligned} &\tilde{u}_\eta(t) \in V, \\ &\langle \mathcal{E}\varepsilon(\tilde{u}_\eta(t)), \varepsilon(v)\rangle_{\mathcal{H}} = \langle f(t), v\rangle_H + \langle \hat{\sigma}(t)\nu, \gamma v\rangle_\Gamma \\ &\qquad - \langle \mathcal{E}\varepsilon(\hat{u}), \varepsilon(v)\rangle_{\mathcal{H}} - \langle z_\eta(t), \varepsilon(v)\rangle_{\mathcal{H}} \end{aligned} \tag{4.20}$$

for all $v \in V$. Taking now

$$u_\eta(t) = \tilde{u}_\eta(t) + \hat{u}(t) \tag{4.21}$$

and $\sigma_\eta(t)$ defined by (4.15), we obtain that $(u_\eta(t), \sigma_\eta(t)) \in H_1 \times \mathcal{H}_1$ and (4.15) - (4.18) hold. Moreover, by standard arguments it follows that

$$\begin{aligned} &|u_\eta(t_1) - u_\eta(t_2)|_{H_1} + |\sigma_\eta(t_1) - \sigma_\eta(t_2)|_{\mathcal{H}_1} \\ &\le C\big(|f(t_1) - f(t_2)|_{H_1} + |\hat{u}(t_1) - \hat{u}(t_2)|_{H_1} \\ &\quad + |\hat{\sigma}(t_1) - \hat{\sigma}(t_2)|_{\mathcal{H}_1} + |z_\eta(t_1) - z_\eta(t_2)|_{\mathcal{H}_1}\big) \end{aligned} \tag{4.22}$$

for all $t_1, t_2 \in [0,T]$. Using now (4.22), (4.8) and (4.9) we obtain the regularity $u_\eta \in W^{1,\infty}(0,T,H_1)$, $\sigma_\eta \in W^{1,\infty}(0,T,\mathcal{H}_1)$.

The uniqueness part in Lemma 4.2 follows from the uniqueness of the element $\tilde{u}_\eta(t)$ solution of (4.20).

In order to prove (4.19) let us remark that from (4.12) - (4.14) we have that (u_0, σ_0) is a solution of the problem (4.15) - (4.18) at $t = 0$. Since for all $t \in [0,T]$ the problem (4.15) - (4.18) has a unique solution $(u_\eta(t), \sigma_\eta(t))$, we obtain (4.19). □

Let us now remark that from (4.7) $t \mapsto G(\sigma_\eta(t), \varepsilon(u_\eta(t)))$ is a Lipschitz continuous function on $[0,T]$ with values in $\mathcal{H}$. This property allows us to consider the operator $\Lambda : L^\infty(0,T,\mathcal{H}) \to W^{1,\infty}(0,T,\mathcal{H})$ defined by

$$\Lambda\eta(t) = G(\sigma_\eta(t), \varepsilon(u_\eta(t))) \quad \text{for all} \quad t \in [0,T], \ \eta \in L^\infty(0,T,\mathcal{H}). \tag{4.23}$$

Lemma 4.3. *The operator* Λ *has a unique fixed point* $\eta^* \in L^\infty(0,T,\mathcal{H})$.

Proof. Let $\eta_1, \eta_2 \in L^\infty(0,T,\mathcal{H})$. For simplicity, we denote $z_{\eta_1} = z_1$, $z_{\eta_2} = z_2$, $u_{\eta_1} = u_1$, $u_{\eta_2} = u_2$, $\sigma_{\eta_1} = \sigma_1$, $\sigma_{\eta_2} = \sigma_2$ and let $t \in [0,T]$. Using (4.21), (4.20), (4.15) and (4.16) we obtain the inequality

$$|u_1(t) - u_2(t)|_{H_1} + |\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}_1} \le C\,|z_1(t) - z_2(t)|_{\mathcal{H}}$$

and, from (4.23), (4.7) it results

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \leq C\,|z_1(t) - z_2(t)|_{\mathcal{H}}\,.$$

Having now in mind (4.13), the previous inequality becomes

$$(4.24)\qquad |\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \leq C\int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}}\,ds\,.$$

By recurrence, denoting by Λ^p the powers of the operator Λ, (4.24) implies

$$\left|\Lambda^p\eta_1(t) - \Lambda^p\eta_2(t)\right|_{\mathcal{H}} \leq C^p \underbrace{\int_0^t\int_0^s\dots\int_0^q}_{p \text{ integrals}} |\eta_1(r) - \eta_2(r)|_{\mathcal{H}}\,dr\dots ds$$

for all $t \in [0,T]$ and $p \in \mathbb{N}$. It results

$$(4.25)\qquad \left|\Lambda^p\eta_1 - \Lambda^p\eta_2\right|_{\infty,\mathcal{H}} \leq \frac{C^p}{p!}\,|\eta_1 - \eta_2|_{\infty,\mathcal{H}} \quad \text{for all} \quad p \in \mathbb{N}\,,$$

and, since $\lim_p \frac{C^p}{p!} = 0$, (4.25) implies that for p large enough the operator Λ^p is a contraction in $L^\infty(0,T,\mathcal{H})$. Then there exists a unique $\eta^* \in L^\infty(0,T,\mathcal{H})$ such that $\Lambda^p\eta^* = \eta^*$. Moreover, η^* is the unique fixed point of Λ. □

Proof of Theorem 4.1. The existence part. Let $\eta^* \in L^\infty(0,T,\mathcal{H})$ be the fixed point of Λ and let $u_{\eta^*} \in W^{1,\infty}(0,T,H_1)$, $\sigma_{\eta^*} \in W^{1,\infty}(0,T,\mathcal{H}_1)$ be the functions given by Lemma 4.2 for $\eta = \eta^*$. Using (4.15) and (4.13) we have

$$\dot\sigma_{\eta^*} = \mathcal{E}\varepsilon(\dot u_{\eta^*}) + \eta^*$$

and from (4.23) it follows

$$\eta^* = \Lambda\eta^* = G\big(\sigma_{\eta^*}, \varepsilon(u_{\eta^*})\big)\,.$$

So, we obtain

$$\dot\sigma_{\eta^*} = \mathcal{E}\varepsilon(\dot u_{\eta^*}) + G\big(\sigma_{\eta^*}, \varepsilon(u_{\eta^*})\big)$$

and from Lemma 4.2 it results that $(u_{\eta^*}, \sigma_{\eta^*})$ is a solution of (4.1) – (4.5).

The uniqueness part. Let (u,σ) be a solution of (4.1) – (4.5) having the regularity $u \in W^{1,\infty}(0,T,H_1)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H}_1)$. Denoting by $\eta \in L^\infty(0,T,\mathcal{H})$ the function given by

$$(4.26)\qquad \eta(t) = G\big(\sigma(t), \varepsilon(u(t))\big) \quad \text{for all} \quad t \in [0,T]\,,$$

from (4.13) and (4.14) we obtain that (u,σ) is a solution of (4.15) – (4.19). Since this problem has a unique solution denoted previously by (u_η, σ_η), it results

$$(4.27)\qquad u = u_\eta\,, \quad \sigma = \sigma_\eta\,.$$

Using now (4.23), (4.27) and (4.26) we get $\Lambda\eta = \eta$ and by the uniqueness of the fixed point of Λ it results

$$\eta = \eta^* . \tag{4.28}$$

The uniqueness part in Theorem 4.1 is now a consequence of (4.27) and (4.28). □

As an application of Theorem 4.1 let us consider a quasistatic viscoplastic problem involving the boundary conditions presented in Example 3.1, Section 2. For this, let us suppose that

$$\begin{cases} \text{there exists } g \in W^{1,\infty}(0,T,H_\Gamma) \text{ such that} \\ g = g_1 \text{ on } \Gamma_1 \times (0,T),\ g_\nu = g_3 \text{ on } \Gamma_3 \times (0,T),\ g_\tau = g_4 \text{ on } \Gamma_4 \times (0,T), \end{cases} \tag{4.29}$$

$$\begin{cases} \text{there exists } h \in W^{1,\infty}(0,T,L^2(\Gamma)^N) \text{ such that} \\ h = h_2 \text{ on } \Gamma_2 \times (0,T),\ h_\tau = h_3 \text{ on } \Gamma_3 \times (0,T),\ h_\nu = h_4 \text{ on } \Gamma_4 \times (0,T), \end{cases} \tag{4.30}$$

$$\begin{cases} u_0 \in H_1, \quad \sigma_0 \in \mathcal{H}_1, \\ \operatorname{Div}\sigma_0 + f(0) = 0 \quad \text{in} \quad \Omega, \\ u_0 = g_1(0) \quad \text{on} \quad \Gamma_1, \\ \sigma_0\nu = h_2(0) \quad \text{on} \quad \Gamma_2, \\ u_{0\nu} = g_3(0), \quad \sigma_{0\tau} = h_3(0) \quad \text{on} \quad \Gamma_3, \\ u_{0\tau} = g_4(0), \quad \sigma_{0\nu} = h_4(0) \quad \text{on} \quad \Gamma_4 . \end{cases} \tag{4.31}$$

We have the following result:

Corollary 4.4. *Let* (4.6) – (4.8), (4.29) – (4.31) *hold and let* $meas\,\Gamma_1 > 0$. *Then there exists a solution* $u \in W^{1,\infty}(0,T,H_1)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H}_1)$ *of the problem*

$$\begin{cases} \dot\sigma = \mathcal{E}\varepsilon(\dot u) + G(\sigma,\varepsilon(u)) \quad in \quad \Omega \times (0,T), \\ \operatorname{Div}\sigma + f = 0 \quad in \quad \Omega \times (0,T), \\ u = g_1 \quad on \quad \Gamma_1 \times (0,T), \\ \sigma\nu = h_2 \quad on \quad \Gamma_2 \times (0,T), \\ u_\nu = g_3, \quad \sigma_\tau = h_3 \quad on \quad \Gamma_3 \times (0,T), \\ u_\tau = g_4, \quad \sigma_\nu = h_4 \quad on \quad \Gamma_4 \times (0,T), \\ u(0) = u_0, \quad \sigma(0) = \sigma_0 \quad in \quad \Omega . \end{cases} \tag{4.32}$$

Proof. Let $\hat u$ and $\hat\sigma$ be defined by $\hat u(t) = z\,g(t)$, $\hat\sigma(t) = \bar z\,h(t)$ for all $t \in [0,T]$ and let V, $\mathcal{V}$ be defined by (3.9), (3.3). Using (4.29) – (4.31) we obtain that (4.9) – (4.12) are satisfied and since $meas\,\Gamma_1 > 0$ it results that (2.1) also holds. Corollary 4.4 follows now from Theorem 4.1 and Lemma 3.2. □

Remark 4.5. Using again Lemma 3.2 and Theorem 4.1 it follows that if $(u_i,\sigma_i) \in W^{1,\infty}(0,T,H_1\times\mathcal{H}_1)$ are two solutions of the problem (4.32) such that $u_i(t) \in C^1(\overline{\Omega})^N$, $\sigma_i(t) \in C^1(\overline{\Omega})_s^{N\times N}$ for all $t \in [0,T]$, $i = 1,2$, then $u_1 = u_2$ and $\sigma_1 = \sigma_2$.

5. Dynamic processes for rate−type viscoplastic materials

In this section, we consider the dynamic evolution of a viscoplastic body submitted to linear boundary conditions. This evolution is given by the following mixed problem: find the displacement field $u : \Omega \times [0,T] \to \mathbb{R}^N$ and the stress field $\sigma : \Omega \times [0,T] \to S_N$ such that:

$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u)) \quad \text{in} \quad \Omega \times (0,T), \tag{5.1}$$

$$\rho \ddot{u} = \operatorname{Div} \sigma + f \quad \text{in} \quad \Omega \times (0,T), \tag{5.2}$$

$$u \in \hat{u} + V \quad \text{on} \quad (0,T), \tag{5.3}$$

$$\sigma \in \hat{\sigma} + \mathcal{V} \quad \text{on} \quad (0,T), \tag{5.4}$$

$$u(0) = u_0, \quad \dot{u}(0) = v_0, \quad \sigma(0) = \sigma_0 \quad \text{in} \quad \Omega. \tag{5.5}$$

Problem (5.1) – (5.5) is similar to the problem (4.1) – (4.5) except for the fact that the equilibrium equation (4.2) was replaced by the motion equation (5.2) in which ρ is the density function and an initial condition for $\dot{u}$ was added in (5.5).

In the study of the problem (5.1) – (5.5) we consider the following assumptions:

$$\rho \in L^\infty(\Omega) \text{ and there exists } \beta > 0 \text{ such that } \rho(x) \geq \beta \text{ a.e. in } \Omega, \tag{5.6}$$

$$f \in W^{1,1}(0,T,H), \tag{5.7}$$

$$\hat{u} \in W^{3,1}(0,T,H_1), \quad \hat{\sigma} \in W^{2,1}(0,T,\mathcal{H}_1), \tag{5.8}$$

$$u_0 \in H_1, \quad v_0 \in H_1, \quad \sigma_0 \in \mathcal{H}_1, \tag{5.9}$$

$$u_0 \in \hat{u}(0) + V, \quad v_0 \in \dot{\hat{u}}(0) + V, \quad \sigma_0 \in \hat{\sigma}(0) + \mathcal{V}. \tag{5.10}$$

The main result of this section is the following:

Theorem 5.1. *Let* (4.6), (4.7), (4.10), (5.6)−(5.10) *hold. Then there exists a unique solution* $u \in W^{2,\infty}(0,T,H) \cap W^{1,\infty}(0,T,H_1)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H}) \cap L^\infty(0,T,\mathcal{H}_1)$ *of the problem* (5.1) – (5.5).

In order to prove Theorem 5.1 we retake, with some adjustements, the proof given by I. R. Ionescu [IONES] in the particular case of the boundary conditions presented in Example 3.3 of Section 3. We start by homogenizing the boundary conditions (5.3) and (5.4). Let $u^* = u - \hat{u}$, $\sigma^* = \sigma - \hat{\sigma}$, $v^* = \dot{u} - \dot{\hat{u}}$, $u_0^* = u_0 - \hat{u}(0)$, $v_0^* = v_0 - \dot{\hat{u}}_0(0)$, $\sigma_0^* = \sigma_0 - \hat{\sigma}(0)$. From (5.10) we notice that $u_0^* \in V$, $v_0^* \in V$, $\sigma_0^* \in \mathcal{V}$. Hence we can easily deduce the following lemma:

Lemma 5.2. *The couple* (u, σ) *is a solution of* (5.1) – (5.5) *having the regularity of Theorem* 5.1 *iff* $u^* \in W^{1,\infty}(0,T,V)$, $v^* \in W^{1,\infty}(0,T,H) \cap L^\infty(0,T,V)$ *and* $\sigma^* \in W^{1,\infty}(0,T,\mathcal{H}) \cap L^\infty(0,T,\mathcal{V})$ *is the solution of the problem*

$$\dot{u}^* = v^*, \tag{5.11}$$

$$\dot{v}^* = \rho^{-1} \operatorname{Div} \sigma^* + f^*, \tag{5.12}$$

$$\dot{\sigma}^* = \mathcal{E}\varepsilon(v^*) + G^*(\sigma^*, \varepsilon^*(u^*)) \tag{5.13}$$

in $\Omega \times (0,T)$,

$$u^*(0) = u_0^*, \quad v^*(0) = v_0^*, \quad \sigma^*(0) = \sigma_0^* \quad in \quad \Omega, \tag{5.14}$$

where $f^* \in W^{1,1}(0,T,H)$ *and* $G^* : [0,T] \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ *are given by*

$$f^*(t) = \rho^{-1} f(t) - \ddot{\hat{u}}(t) + \rho^{-1} \mathrm{Div}\, \hat{\sigma}(t), \quad for\ all \quad t \in [0,T], \tag{5.15}$$

$$G^*(t,\sigma,\varepsilon) = G\big(\hat{\sigma}(t) + \sigma,\, \varepsilon(\hat{u}(t)) + \varepsilon\big) + \mathcal{E}\varepsilon\big(\dot{\hat{u}}(t)\big) - \dot{\hat{\sigma}}(t), \quad for\ all \quad t \in [0,T], \quad \sigma, \varepsilon \in \mathcal{H}. \tag{5.16}$$

Let us now consider the real Hilbert space $Y = H \times \mathcal{H}$ with the inner product

$$((v_1,\tau_1),(v_2,\tau_2))_Y = \langle \rho v_1, v_2 \rangle_H + \langle \mathcal{E}^{-1}\tau_1, \tau_2 \rangle_{\mathcal{H}} \tag{5.17}$$

which generates an equivalent norm on Y (see (4.6) and (5.6)). In addition, let $D(B) = V \times \mathcal{V} \subset Y$ and let $B : D(B) \to Y$ be the linear operator defined by

$$B(v,\sigma) = \big(\rho^{-1}\mathrm{Div}\,\sigma,\, \mathcal{E}\varepsilon(v)\big) \quad \text{for all} \quad (v,\sigma) \in D(B). \tag{5.18}$$

With the above notations, we have:

Lemma 5.3. *$B^* = -B$ where B^* denotes the adjoint operator of B.*

Proof. Let $(v,\sigma) \in D(B^*)$ and $(\overline{v},\overline{\sigma}) = B^*(v,\sigma)$. For all $(u,\tau) \in D(B)$ it follows from (5.17) and (5.18)

$$\langle \mathrm{Div}\,\tau, v\rangle_H + \langle \varepsilon(u), \sigma\rangle_{\mathcal{H}} = \langle \rho u, \overline{v}\rangle_H + \langle \mathcal{E}^{-1}\tau, \overline{\sigma}\rangle_{\mathcal{H}}. \tag{5.19}$$

Putting $\tau = 0$ in (5.19), we obtain

$$\mathrm{Div}\,\sigma = -\rho\overline{v} \tag{5.20}$$

which implies $\sigma \in \mathcal{H}_1$, and from (5.19), (5.20) and (2.9) we deduce $\langle \sigma\nu, \gamma u\rangle_\Gamma = 0$ for all $u \in V$. Hence $\sigma \in \mathcal{V}$.

Putting now $u = 0$ in (5.19), we obtain

$$\varepsilon(v) = -\mathcal{E}^{-1}\overline{\sigma} \tag{5.21}$$

which implies $v \in H_1$, and from (5.19), (5.21) and (2.9) we deduce

$$\langle \tau\nu, \gamma v\rangle_\Gamma = 0 \quad \text{for all} \quad \tau \in \mathcal{V}. \tag{5.22}$$

We have now to prove that $v \in V$. Indeed, let us suppose that $v \notin V$. Then, there exists $F : H_1 \to \mathbb{R}$ such that $F(v) \neq 0$ and $F(w) = 0$ for all $w \in V$. Riesz's representation theorem implies that there exists $g \in H_1$ such that

$$F(v) = \langle g, v\rangle_{H_1} \neq 0, \quad F(w) = \langle g, w\rangle_{H_1} = 0 \quad \text{for all} \quad w \in V. \tag{5.23}$$

But $\langle g,w\rangle_{H_1} = \langle g,w\rangle_H + \langle\varepsilon(g),\varepsilon(w)\rangle_{\mathcal{H}}$ for all $w\in H_1$. So, taking $\tau^*=\varepsilon(g)\in\mathcal{H}$ and using (5.23) we deduce $\langle g,w\rangle_H + \langle\tau^*,\varepsilon(w)\rangle_{\mathcal{H}} = 0$ for all $w\in V$. It results $\operatorname{Div}\tau^* = g$ hence $\tau^*\in\mathcal{H}_1$; moreover, using (2.9) we get $\langle\tau^*\nu,\gamma w\rangle_\Gamma = 0$ for all $w\in V$. Hence $\tau^*\in\mathcal{V}$. Applying now (5.22) for $\tau=\tau^*$ and using again (2.9) we have

$$\langle\operatorname{Div}\tau^*,v\rangle_H + \langle\tau^*,\varepsilon(v)\rangle_{\mathcal{H}} = 0. \tag{5.24}$$

But $\tau^*=\varepsilon(g)$ and $\operatorname{Div}\tau^* = g$. So, from (5.24) we deduce that $\langle g,v\rangle_{H_1} = 0$ which is in contradiction with (5.23). So, we proved that $v\in V$.

It results now from (5.20) and (5.21) that $B^*(v,\sigma) = -\big(\rho^{-1}\operatorname{Div}\sigma,\mathcal{E}\varepsilon(v)\big) = -B(v,\sigma)$. This last equality implies that $D(B^*)\subset D(B)$ and $B^*=-B$ on $D(B^*)$.

Finally, if $(v,\sigma)\in D(B)$, using (3.3), after a simple calculation it follows that $\langle B(u,\tau),(v,\sigma)\rangle_Y = \langle(u,\tau),-B(v,\sigma)\rangle_Y$ for all $(u,\tau)\in D(B)$. Hence $D(B)\subset D(B^*)$ and Lemma 5.3 is proved. □

Let us now consider the real Hilbert space $X=V\times H\times\mathcal{H}$, $D(A)=V\times V\times\mathcal{V}\subset X$ and $A: D(A)\subset X\to X$ the linear operator given by

$$A(u,v,\sigma) = \big(v\,,\rho^{-1}\operatorname{Div}\sigma\,,\mathcal{E}\varepsilon(v)\big) \quad\text{for all}\quad (u,v,\sigma)\in D(A). \tag{5.25}$$

With the above notations, we have the following result:

Lemma 5.4. *The operator A is the infinitesimal generator of a C_0 semigroup $\big(S(t)\big)_{t\ge0}$ of linear continuous operators in X.*

Proof. Using Lemma 5.3 it follows that B is the infinitesimal generator of a C_0 semigroup $\big(T(t)\big)_{t\ge0}$ of linear continuous operators in Y (see for example [PAZY], Ch. I). For all $t\ge0$, let $S(t)$ be the operator defined by

$$S(t)(u,v,\sigma) = \left(u+\int_0^t T_1(s)(v,\sigma)\,ds,\, T(t)(v,\sigma)\right), \quad\text{for all}\quad (u,v,\sigma)\in V\times H\times\mathcal{H}, \tag{5.26}$$

where $T_1(t)\in\mathcal{L}(Y,H)$, $T_2(t)\in\mathcal{L}(Y,\mathcal{H})$ are the two operators defined by $T(t)(v,\sigma) = \big(T_1(t)(v,\sigma),T_2(t)(v,\sigma)\big)\in H\times\mathcal{H}$. Let, in the sequel, $[D(B)]$ denote the linear space $D(B)$ endowed with the graph norm of B. Using (5.17), (5.6) and (4.6) we obtain that $[D(B)]$ is isomorphic to $V\times\mathcal{V}$. Moreover, since the mapping $y\mapsto\int_0^t T(s)y\,ds$ belongs to $\mathcal{L}\big(Y,[D(B)]\big)$ for all $t\ge0$ (see for example [IONSO2] p. 225), it results that the mapping $y\mapsto\int_0^t T_1(s)\,ds$ belongs to $\mathcal{L}(Y,V)$. Hence $S(t)$ is a linear continuous operator in X for all $t\ge0$. Now, having in mind (5.25), (5.26) and the fact that B is the infinitesimal generator of $\big(T(t)\big)_{t\ge0}$ we can deduce that A is the infinitesimal generator of $\big(S(t)\big)_{t\ge0}$. □

Proof of Theorem 5.1. Let $x(t) = \big(u^*(t),v^*(t),\sigma^*(t)\big)$, $x_0 = \big(u_0^*,v_0^*,\sigma_0^*\big)$ and let $h:[0,T]\times X\to X$ be defined by

$$h\big(t,(u,v,\sigma)\big) = \big(0\,,f^*(t)\,,G^*\big(t,\sigma,\varepsilon(u)\big)\big) \tag{5.27}$$

where f^* and G^* are given by (5.15), (5.16). Using (5.25) and (5.27), it results that the problem (5.11) – (5.14) can be written as

$$\dot{x}(t) = Ax(t) + h(t, x(t)) \quad \text{on} \quad (0,T), \tag{5.28}$$

$$x(0) = x_0 . \tag{5.29}$$

We shall now use a well known result (see for instance [IONSO2] p. 231) in order to prove the existence and uniqueness of the solution of (5.28) – (5.29). For this, let us remark that from (5.10) we obtain $x_0 \in D(A)$. Moreover, if $(u_i, v_i, \sigma_i) \in X$, $t_i \in [0,T]$, $i = 1,2$, from (5.16), (5.27), (4.6), (4.7) and (5.7), after some algebra it results

$$\begin{aligned}
&\left|h(t_1, (u_1, v_1, \sigma_1)) - h(t_2, (u_2, v_2, \sigma_2))\right|_X \\
\le\ & C\Big[\left|f^*(t_1) - f^*(t_2)\right|_H + \left|\dot{\hat{\sigma}}(t_1) - \dot{\hat{\sigma}}(t_2)\right|_{\mathcal{H}} + \left|\dot{\hat{u}}(t_1) - \dot{\hat{u}}(t_2)\right|_{H_1} + \left|\hat{\sigma}(t_1) - \hat{\sigma}(t_2)\right|_{\mathcal{H}} \\
&+ \left|\hat{u}(t_1) - \hat{u}(t_2)\right|_{H_1} + \left|u_1 - u_2\right|_{H_1} + \left|\sigma_1 - \sigma_2\right|_{\mathcal{H}}\Big] \\
\le\ & C \int_{t_1}^{t_2} \Big[\left|\dot{f}^*(s)\right|_H + \left|\ddot{\hat{\sigma}}(s)\right|_{\mathcal{H}} + \left|\ddot{\hat{u}}(s)\right|_{H_1} + \left|\dot{\hat{\sigma}}(s)\right|_{\mathcal{H}} + \left|\dot{\hat{u}}(s)\right|_{H_1}\Big]\, ds \\
&+ C\Big[\left|u_1 - u_2\right|_{H_1} + \left|v_1 - v_2\right|_{H_1} + \left|\sigma_1 - \sigma_2\right|_{\mathcal{H}}\Big] .
\end{aligned}$$

Using now (5.7), (5.8) and (5.15) it results that the function defined by

$$s \longmapsto \left|\dot{f}^*(s)\right|_H + \left|\ddot{\hat{\sigma}}(s)\right|_{\mathcal{H}} + \left|\ddot{\hat{u}}(s)\right|_{H_1} + \left|\dot{\hat{\sigma}}(s)\right|_{\mathcal{H}} + \left|\dot{\hat{u}}(s)\right|_{H_1}$$

belongs to $L^1(0,T,\mathbb{R})$. So, using the above quotated existence result, it results that the problem (5.28), (5.29) has a unique solution (u, v, σ) such that $(u(t), v(t), \sigma(t)) \in D(A)$ for all $t \in [0,T]$ and $(u, v, \sigma) \in W^{1,\infty}(0,T,X) \cap L^\infty(0,T,[D(A)])$. Here $[D(A)]$ represents the space $D(A)$ endowed with the graph norm of A, which is in fact isomorphic to $V \times V \times \mathcal{V}$ (see (4.6) and (5.6)). Hence we deduce that there exists a unique solution (u^*, v^*, σ^*) of the problem (5.11) – (5.14) such that $u^* \in W^{1,\infty}(0,T,V)$, $v^* \in W^{1,\infty}(0,T,H) \cap L^\infty(0,T,V)$, $\sigma^* \in W^{1,\infty}(0,T,\mathcal{H}) \cap L^\infty(0,T,\mathcal{V})$.

Finally, Theorem 5.1 follows from Lemma 5.2. □

As an application of Theorem 5.1 let us consider a dynamic viscoplastic problem involving the boundary conditions presented in Example 3.1, Section 2. For this, we make the following assumptions:

$$\begin{cases} \text{there exists } g \in W^{3,1}(0,T,H_\Gamma) \text{ such that} \\ g = g_1 \text{ on } \Gamma_1 \times (0,T),\ g_\nu = g_3 \text{ on } \Gamma_3 \times (0,T),\ g_\tau = g_4 \text{ on } \Gamma_4 \times (0,T), \end{cases} \tag{5.30}$$

$$\begin{cases} \text{there exists } h \in W^{2,1}(0,T,L^2(\Gamma)^N) \text{ such that} \\ h = h_2 \text{ on } \Gamma_2 \times (0,T),\ h_\tau = h_3 \text{ on } \Gamma_3 \times (0,T),\ h_\nu = h_4 \text{ on } \Gamma_4 \times (0,T), \end{cases} \tag{5.31}$$

$$\begin{cases} u_0 \in H_1, \quad v_0 \in H_1, \quad \sigma_0 \in \mathcal{H}_1, \\ u_0 = g_1(0), \quad v_0 = \dot{g}_1(0) \quad \text{on} \quad \Gamma_1 \\ \sigma_0 \nu = h_2(0) \quad \text{on} \quad \Gamma_2, \\ u_{0\nu} = g_3(0), \quad v_{0\nu} = \dot{g}_3(0), \quad \sigma_{0\tau} = h_3(0) \quad \text{on} \quad \Gamma_3, \\ u_{0\tau} = g_4(0), \quad v_{0\tau} = \dot{g}_4(0), \quad \sigma_{0\nu} = h_4(0) \quad \text{on} \quad \Gamma_4 . \end{cases} \tag{5.32}$$

We have the following result:

Corollary 5.5. *Let* (4.6), (4.7), (5.6), (5.7) *and* (5.30) - (5.32) *hold. Then there exists a solution* $u \in W^{2,\infty}(0,T,H) \cap W^{1,\infty}(0,T,H_1)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H}) \cap L^{\infty}(0,T,\mathcal{H}_1)$ *of the problem*

$$(5.33) \qquad \begin{cases} \rho\ddot{u} &= \operatorname{Div}\sigma + f \quad in \quad \Omega \times (0,T), \\ \dot{\sigma} &= \mathcal{E}\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u)) \quad in \quad \Omega \times (0,T), \\ u &= g_1 \quad on \quad \Gamma_1 \times (0,T), \\ \sigma\nu &= h_2 \quad on \quad \Gamma_2 \times (0,T), \\ u_\nu &= g_3, \quad \sigma_\tau = h_3 \quad on \quad \Gamma_3 \times (0,T), \\ u_\tau &= g_4, \quad \sigma_\nu = h_4 \quad on \quad \Gamma_4 \times (0,T), \\ u(0) &= u_0, \quad \dot{u}(0) = v_0, \quad \sigma(0) = \sigma_0 \quad in \quad \Omega. \end{cases}$$

Proof. Let $\hat{u}$ and $\hat{\sigma}$ be defined by $\hat{u}(t) = zg(t)$, $\hat{\sigma}(t) = \tilde{z}\,h(t)$ for all $t \in [0,T]$ and let V, $\mathcal{V}$ be defined by (3.9), (3.3). Using (5.30) - (5.32) we obtain that (5.8) - (5.10) are satisfied. Corollary 5.5 follows now from Theorem 5.1 and Lemma 3.2. □

Remark 5.6. Using again Lemma 3.2 and Theorem 5.1 it follows that if (u_i, σ_i) are two solutions of the problem (5.33) having the regularity of Corollary 5.5 and such that $u_i(t) \in C^1(\overline{\Omega})^N$, $\sigma_i(t) \in C^1(\overline{\Omega})_s^{N\times N}$ for all $t \in [0,T]$, $i = 1,2$, then $u_1 = u_2$ and $\sigma_1 = \sigma_2$.

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On Signorini's Contact Problem in Rate-Type Viscoplasticity

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On Signorini's Contact Problem in Rate-Type Viscoplasticity

Mohamed ROCHDI

Descriptif

Il s'agit dans ce papier de l'étude du problème quasistatique de contact unilatéral sans frottement entre un matériau viscoplastique et une fondation rigide. Ce problème connu sous le nom de problème de Signorini a fait l'objet de plusieurs études pour des matériaux élastiques.

On considère un milieu continu viscoplastique dont les particules matérielles occupent un domaine Ω de $\mathbb{R}^N$ ($N = 1, 2, 3$) et dont la frontière Γ, supposée suffisamment régulière, est divisée en trois parties disjointes Γ_1, Γ_2 et Γ_3. On suppose que, pendant l'intervalle de temps $[0, T]$, le champ des déplacements s'annulle sur Γ_1, que des forces surfaciques g s'appliquent sur Γ_2 et que des forces volumiques f agissent dans Ω. On suppose aussi que le matériau peut rentrer en contact avec une fondation rigide par la partie Γ_3 de sa frontière et que ce contact s'effectue sans frottement. Sous ces considérations, le problème quasistatique de contact étudié ici se formule de la façon suivante :

Problème P : Trouver le champ des déplacements $u : \Omega \times [0, T] \longrightarrow \mathbb{R}^N$ et le champ des contraintes $\sigma : \Omega \times [0, T] \longrightarrow \mathbb{R}_s^{N\times N}$ tels que

$$\begin{aligned}
\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u)) \quad &\text{dans} \quad \Omega \times (0, T),\\
Div\, \sigma + f = 0 \quad &\text{dans} \quad \Omega \times (0, T),\\
u = 0 \quad &\text{sur} \quad \Gamma_1 \times (0, T),\\
\sigma\nu = g \quad &\text{sur} \quad \Gamma_2 \times (0, T),\\
u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\tau = 0, \quad \sigma_\nu u_\nu = 0 \quad &\text{sur} \quad \Gamma_3 \times (0, T),\\
u(0) = u_0, \quad \sigma(0) = \sigma_0 \quad &\text{dans} \quad \Omega.
\end{aligned}$$

On note par $\mathbb{R}_s^{N\times N}$ l'espace des tenseurs symétriques du second ordre sur $\mathbb{R}^N$ et par $\varepsilon(u)$ le tenseur des petites déformations linéarisé. Le point au dessus d'une quantité désigne sa dérivée temporelle, $Div\, \sigma$ désigne la divergence de la fonction tensorielle σ, le vecteur ν est la normale unitaire sortante à Ω, $\sigma\nu$ est le vecteur des contraintes de Cauchy, et u_ν,

σ_ν et σ_τ représentent respectivement le déplacement normal, les contraintes normales et tangentielles.

Une fois le problème mécanique P posé, on établit deux formulations variationnelles P_1 et P_2. Aussi bien la formulation P_1 que la formulation P_2 représentent un couplage entre la loi de comportement et une inéquation variationnelle elliptique qui englobe l'équation d'équilibre et les conditions aux limites. L'originalité de la formulaion faible P_2 est qu'elle est exprimée en contraintes alors que la formulation P_1 est exprimée en déplacements. On prouve un résultat d'existence et d'unicité pour chacune des formulations et on établit un résultat d'équivalence entre ces deux formulations faibles. 0n termine par donner une interprétation mécanique pour les résultats mathématiques obtenus.

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ON SIGNORINI'S CONTACT PROBLEM IN RATE-TYPE VISCOPLASTICITY

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Abstract

This paper deals with initial and boundary value problems describing the quasistatic evolution of a rate-type viscoplastic material submitted to frictionless contact boundary conditions. Two variational formulations of this problem are considered, followed by existence and uniqueness results. An equivalence result between the previous variational formulations is discussed.

1 Introduction

In this paper we propose to investigate a problem of unilateral contact between an elastic-viscoplastic body and a rigid frictionless foundation. Thus, the famous Signorini's problem in linear elasticity already studied in the case of elastic-plastic or viscoelastic bodies in [1]-[4] is investigated here for rate-type elastic-viscoplastic models.

Everywhere in this paper we consider the case of small deformations, we denote by $\varepsilon = (\varepsilon_{ij})$ the small strain tensor and by $\sigma = (\sigma_{ij})$ the stress tensor. The dot above will represent the derivative with respect to the time variable. We consider here constitutive laws of the form

$$\dot{\sigma} = \mathcal{E}\dot{\varepsilon} + G(\sigma, \varepsilon) \tag{1.1}$$

in which $\mathcal{E}$ and G are constitutive functions.

Rate-type viscoplastic models of the form (1.1) are used in order to describe the behavior of real materials like rubbers, metals, pastes, rocks and so on. Various results and mechanical interpretations concerning models of this form

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may be found for instance in [5] (see also the references quoted there). Existence and uniqueness results for initial and boundary value problems involving (1.1) were obtained for instance in [6]-[12].

The aim of this paper is to investigate a quasistatic problem for the elastic-viscoplastic models (1.1) involving unilateral contact condition. So, it is structured as follows : in section 2 the mechanical problem P is stated and some functional notations are presented. For the problem P, after recalling the variational formulation P_1 given in [10], we establish in section 3 another variational formulation P_2. Both P_1 and P_2 involve a coupling between the constitutive law (1.1) and a variational inequality including the equilibrium equation and the boundary conditions. The unknowns in the problems P_1 and P_2 are the displacement field u and the stress field σ. For the problem P_1, we recall in section 4 the existence and uniqueness result (Theorem 4.1) given in [10]. In this case the main unknown is the displacement field u. We also prove, in this section, a similar existence and uniqueness result for the problem P_2 (Theorem 4.2). Here the main unknown is the stress field σ. The last section deals with the study of an equivalence result between the problems P_1 and P_2 (Theorem 5.1).

2 Problem statement and preliminaries

Let us consider an elastic-viscoplastic body whose material particles fulfil a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 1, 2, 3$) and whose boundary Γ, assumed to be sufficiently smooth, is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3. Let *meas* $\Gamma_1 > 0$ and let $T > 0$ be a time interval. We shall assume that the displacement field vanishes on $\Gamma_1 \times (0, T)$, that surface tractions g act on $\Gamma_2 \times (0,T)$ and that body forces f act in $\Omega \times (0, T)$. We also suppose that the body rests on a rigid foundation S by the part Γ_3 of the boundary and that this contact is frictionless, i.e. the tangential movements are completely free. We shall finally assume the case of quasistatic processes and we shall use (1.1) as constitutive law. With these assumptions, the mechanical problem that we study here may be formulated as follows :

Problem P. P : Find the displacement field $u : \Omega \times [0, T] \to \mathbb{R}^N$ and the stress field $\sigma : \Omega \times [0, T] \to S_N$ such that

(2.1) $$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u)) \quad in \quad \Omega \times (0, T)$$

(2.2) $$Div\, \sigma + f = 0 \quad in \quad \Omega \times (0, T)$$

(2.3) $$u = 0 \quad on \quad \Gamma_1 \times (0, T)$$

(2.4) $$\sigma\nu = g \quad on \quad \Gamma_2 \times (0,T)$$

(2.5) $$u_\nu \leq 0 \ , \ \sigma_\nu \leq 0 \ , \ \sigma_\tau = 0 \ , \ \sigma_\nu u_\nu = 0 \quad on \quad \Gamma_3 \times (0,T)$$

(2.6) $$u(0) = u_0 \quad , \quad \sigma(0) = \sigma_0 \quad in \quad \Omega$$

where S_N denotes the set of second order symmetric tensors on $\mathbb{R}^N$. In (2.1)-(2.6), $Div\ \sigma$ represents the divergence of the tensor-valued function σ, $\nu = (\nu_i)$ is the unit outward normal to Ω , $\sigma\nu$ is the stress vector, u_ν, σ_ν and σ_τ are given by

$$u_\nu = u_i\nu_i \quad , \quad \sigma_\nu = \sigma_{ij}\nu_j\nu_i \quad , \quad \sigma_{\tau i}\sigma_{ij}\nu_j - \sigma_\nu\nu_i \quad (i = \overline{1,N})$$

and finally u_0 and σ_0 are the initial data.

We denote in the sequel by " · " the inner product on the spaces $\mathbb{R}^N$ and S_N and by $|\cdot|$ the Euclidean norms on these spaces. The following notations are also used :

$$H = \{ v = (v_i) \mid v_i \in L^2(\Omega) \ , \ i = \overline{1,N} \} ,$$
$$H_1 = \{ v = (v_i) \mid v_i \in H^1(\Omega) \ , \ i = \overline{1,N} \} ,$$
$$\mathcal{H} = \{ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega) \ , \ i,j = \overline{1,N} \} ,$$
$$\mathcal{H}_1 = \{ \tau \in \mathcal{H} \mid Div\ \tau \in H \} .$$

The spaces H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products denoted by $<\cdot,\cdot>_H$, $<\cdot,\cdot>_{H_1}$, $<\cdot,\cdot>_{\mathcal{H}}$ and $<\cdot,\cdot>_{\mathcal{H}_1}$ respectively. Let $H_\Gamma = [H^{\frac{1}{2}}(\Gamma)]^N$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. We denote by V the closed subspace of H_1 given by

(2.7) $$V = \{ u \in H_1 \mid \gamma u = 0 \ \text{ on } \Gamma_1 \} .$$

The deformation operator $\varepsilon : H_1 \to \mathcal{H}$ defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)) \quad , \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

is a linear and continuous operator. Moreover, since $meas\ \Gamma_1 > 0$, Korn's inequality holds :

(2.8) $$|\varepsilon(v)|_{\mathcal{H}} \geq C|v|_{H_1} \quad \text{for all} \quad v \in V$$

where C is a strictly positive constant which depends only on Ω and Γ_1 (everywhere in this paper C will represent strictly positive generic constants which may depend on Ω, Γ_1, $\mathcal{E}$, G, T and do not depend on time or on input data).

We now endow V with the inner product $<\cdot,\cdot>_V$ defined by

$$(2.9)\qquad <v,w>_V = <\varepsilon(v),\varepsilon(w)>_{\mathcal{H}} \qquad \forall v,w\in V.$$

We also denote by $|\cdot|_V$ the associated norm. So, having in mind (2.8), we deduce that $|\cdot|_V$ and $|\cdot|_{H_1}$ are equivalent norms on V. Therefore, V endowed with the inner product defined by (2.9) is a real Hilbert space.

Let $H'_\Gamma = [H^{-\frac{1}{2}}(\Gamma)]^N$ be the strong dual of the space H_Γ and let $<\cdot,\cdot>$ denote the duality between H'_Γ and H_Γ. If $\tau\in\mathcal{H}_1$ there exists an element $\gamma_\nu\tau\in H'_\Gamma$ such that

$$(2.10)\qquad <\gamma_\nu\tau,\gamma v> \;=\; <\tau,\varepsilon(v)>_{\mathcal{H}} + <Div\ \tau, v>_H \qquad \forall v\in H_1.$$

Moreover, if τ is a regular (say C^1) function, then

$$(2.11)\qquad <\gamma_\nu\tau,\gamma v> \;=\; \int_\Gamma \tau\nu\cdot v da, \qquad \forall v\in H_1.$$

Finally, for every real Hilbert space X we denote by $|\cdot|_X$ the norm on X and by $|\cdot|_{\infty,X}$ the norm on the space $L^\infty(0,T,X)$.

3 Variational formulations

In this section we give two variational formulations for the mechanical problem P. For this, let us suppose that

$$(3.1)\qquad \begin{cases} \mathcal{E}:\Omega\times S_N\mapsto S_N \text{ is a symmetric and positively definite tensor i.e.:}\\ (a)\ \mathcal{E}_{khlm}\in L^\infty(\Omega) \ \text{ for all } k,h,l,m=\overline{1,N}\\ (b)\ \mathcal{E}\sigma\cdot\tau=\sigma\cdot\mathcal{E}\tau \ \text{ for all } \sigma,\tau\in S_N \text{ a.e. in } \Omega\\ (c)\ \text{there exists } \alpha>0 \ \text{ such that } \mathcal{E}\sigma\cdot\sigma\geq\alpha|\sigma| \text{ for all } \sigma\in S_N \end{cases}$$

$$(3.2)\qquad \begin{cases} G:\Omega\times S_N\times S_N\mapsto S_N \text{ and}\\ (a)\ \text{there exists } L>0 \text{ such that}\\ \qquad |G(x,\sigma_1,\varepsilon_1)-G(x,\sigma_2,\varepsilon_2)|\leq L\Big(|\sigma_1-\sigma_2|+|\varepsilon_1-\varepsilon_2|\Big)\\ \qquad \text{for all } \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2\in S_N, \text{ a.e. in } \Omega\\ (b)\ x\mapsto G(x,\sigma,\varepsilon) \text{ is a measurable function with respect to}\\ \qquad \text{the Lebesgue measure on } \Omega, \text{ for all } \sigma,\varepsilon\in S_N\\ (c)\ x\mapsto G(x,0,0)\in\mathcal{H} \end{cases}$$

(3.3) $$f \in W^{1,\infty}(0,T,H)$$

(3.4) $$g \in W^{1,\infty}(0,T,L^2(\Gamma_2)^N).$$

We denote now by $F(t)$ the element of V given by

(3.5) $$< F(t), v >_V = < f(t), v >_H + < g(t), \gamma v >_{L^2(\Gamma_2)^N} \quad \forall v \in V,\ t \in [0,T].$$

So, using (3.3), (3.4) and (3.5), it follows

(3.6) $$F \in W^{1,\infty}(0,T,V).$$

Finally, let U_{ad} and $\Sigma_{ad}(t)$ be the sets given by

(3.7) $$U_{ad} = \{v \in H_1 \mid v = 0 \text{ on } \Gamma_1,\ v_\nu \leq 0 \text{ on } \Gamma_3\}$$

(3.8) $$\Sigma_{ad}(t) = \{\tau \in \mathcal{H} \mid < \tau, \varepsilon(v) >_{\mathcal{H}} \ \geq \ < F(t), v >_V \ \forall v \in U_{ad}\} \ \forall t \in [0,T]$$

and let us suppose that

(3.9) $$u_0 \in U_{ad}\ ,\quad \sigma_0 \in \Sigma_{ad}(0)\ ,\quad < \sigma_0, \varepsilon(u_0) >_{\mathcal{H}} = < F(0), u_0 >_V .$$

Lemma 3.1. *If the couple of functions (u,σ) is a regular solution of the mechanical problem P then :*

(3.10) $$u(t) \in U_{ad}\ ,\quad < \sigma(t), \varepsilon(v) - \varepsilon(u(t)) >_{\mathcal{H}} \geq < F(t), v - u(t) >_V \quad \forall v \in U_{ad}$$

(3.11) $$\sigma(t) \in \Sigma_{ad}(t)\ ,\quad < \tau - \sigma(t)\ ,\ \varepsilon(u(t)) >_{\mathcal{H}} \ \geq\ 0 \quad \forall \tau \in \Sigma_{ad}(t)$$

for all $t \in [0,T]$.

Proof. Let $v \in U_{ad}$ and $t \in [0,T]$. Using (2.2)-(2.5), (2.10), (2.11) and (3.5) we have

(3.12) $$< \sigma(t), \varepsilon(v) - \varepsilon(u(t)) >_{\mathcal{H}} = < F(t), v - u(t) >_V + \int_{\Gamma_3} \sigma_\nu(t) v_\nu da .$$

The inequality in (3.10) follows from (3.12), (3.7) and (2.5). Moreover, (2.3), (2.5) and (3.7) imply $u(t) \in U_{ad}$.
Taking now $v = 2u(t)$ and $v = 0$ in (3.10) we obtain

(3.13) $$< \sigma(t), \varepsilon(u(t)) >_{\mathcal{H}} = < F(t), u(t) >_V,$$

and using (3.13), (3.10) and (3.8) it follows $\sigma(t) \in \Sigma_{ad}(t)$. The inequality in (3.11) follows now from (3.8) and (3.13). □

The previous results lead us to consider two weak formulations of the problem P :

Problem P_1. Find the displacement field $u : [0,T] \to H_1$ and the stress field $\sigma : [0,T] \to \mathcal{H}_1$ such that

$$\dot{\sigma}(t) = \mathcal{E}\varepsilon(\dot{u}(t)) + G(\sigma(t), \varepsilon(u(t))) \quad a.e.\ t \in (0,T) \tag{3.14}$$

$$\begin{cases} u(t) \in U_{ad}, & <\sigma(t), \varepsilon(v) - \varepsilon(u(t))>_{\mathcal{H}} \ \geq \ <F(t), v - u(t)>_V \\ & \forall v \in U_{ad},\ t \in [0,T] \end{cases} \tag{3.15}$$

$$u(0) = u_0 \ , \ \sigma(0) = \sigma_0 \,. \tag{3.16}$$

Problem P_2. Find the displacement field $u : [0,T] \to H_1$ and the stress field $\sigma : [0,T] \to \mathcal{H}_1$ such that

$$\dot{\sigma}(t) = \mathcal{E}\varepsilon(\dot{u}(t)) + G(\sigma(t), \varepsilon(u(t))) \quad a.e.\ t \in (0,T) \tag{3.17}$$

$$\sigma(t) \in \Sigma_{ad}(t)\ , \ <\tau - \sigma(t), \varepsilon(u(t))>_{\mathcal{H}} \ \geq 0 \quad \forall \tau \in \Sigma_{ad}(t),\ t \in [0,T] \tag{3.18}$$

$$u(0) = u_0 \ , \ \sigma(0) = \sigma_0 \,. \tag{3.19}$$

Remark 3.1. The variational formulation P_1 was already given in [10] and we shall see in the last section of this paper the link between the problems P_1 and P_2. Let us also remark that these two problems are formally equivalent to problem P. Indeed, if (u, σ) represents a regular solution of the variational problems P_1 or P_2, using the arguments of [6] it follows that (u, σ) is a solution of the mechanical problem P.

Under the assumptions (3.1)-(3.4) and (3.9), in the next section, we give existence and uniqueness results for the variational problems P_1 and P_2.

4 Existence and uniqueness results

We start this section by recalling the existence and uniqueness result concerning the problem P_1, given in [10].

Theorem 4.1. *Let (3.1) – (3.4) and (3.9) hold. Then there exists a unique solution of the problem P_1 having the regularity $u \in W^{1,\infty}(0,T,H_1)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H}_1)$.*

Proof. See for instance [10]. □

The main result of this section is given by

Theorem 4.2. *Let (3.1) – (3.4) and (3.9) hold. Then there exists a unique solution of the problem P_2 having the regularity $u \in W^{1,\infty}(0,T,V)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H})$.*

In order to prove Theorem 4.2, let us firstly remark that (3.18) is equivalent to the nonlinear evolution equation

(4.1) $$\varepsilon(u(t)) + \partial\psi_{\Sigma_{ad}(t)}(\sigma(t)) \ni 0 \qquad \forall t \in [0,T]$$

where $\partial\psi_{\Sigma_{ad}(t)}$ denotes the subdifferential of the indicator function $\psi_{\Sigma_{ad}(t)}$. Since the set $\Sigma_{ad}(t)$ depends on time, we shall replace (4.1) by a nonlinear evolution equation associated to a fixed convex set. For this, let us introduce the following notations :

(4.2) $$\Sigma_0 = \{\tau \in \mathcal{H} \mid <\tau, \varepsilon(v)>_{\mathcal{H}} \ \geq 0 \ \ \forall v \in U_{ad}\}$$

(4.3) $$\widetilde{\sigma} = \varepsilon(F)$$

(4.4) $$\overline{\sigma} = \sigma - \widetilde{\sigma} \ , \ \overline{\sigma}_0 = \sigma_0 - \widetilde{\sigma}(0)\,.$$

We can easily verify now that the couple of functions (u,σ) is a solution of P_2 having the regularity $u \in W^{1,\infty}(0,T,V), \sigma \in W^{1,\infty}(0,T,\mathcal{H})$ if and only if $u \in W^{1,\infty}(0,T,V)$, $\overline{\sigma} \in W^{1,\infty}(0,T,\mathcal{H})$ and

(4.5) $$\varepsilon(\dot{u}) = \mathcal{E}^{-1}\dot{\overline{\sigma}} - \mathcal{E}^{-1}G(\overline{\sigma} + \widetilde{\sigma}, \varepsilon(u)) \ + \ \mathcal{E}^{-1}\dot{\widetilde{\sigma}} \quad a.e. \ on \ (0,T)$$

(4.6) $$\overline{\sigma}(t) \in \Sigma_0 \ , \ <\tau - \overline{\sigma}(t), \varepsilon(u(t))>_{\mathcal{H}} \ \geq 0 \qquad \forall \tau \in \Sigma_0, \ t \in [0,T]$$

(4.7) $$u(0) = u_0 \ , \ \overline{\sigma}(0) = \overline{\sigma}_0\,.$$

Proof of Theorem 4.2. In order to solve (4.5)-(4.7) we shall use again a fixed point method as follows :

i) For every $\eta \in L^\infty(0,T,\mathcal{H})$, let $z_\eta \in W^{1,\infty}(0,T,\mathcal{H})$ be the function defined by

(4.8) $$z_\eta(t) = \int_0^t \eta(s)ds + \varepsilon(u_0) - \mathcal{E}^{-1}\sigma_0 \qquad \forall t \in [0,T]\,.$$

□

We consider the following elliptic problem :

Problem $P_{2\eta}$. Find $u_\eta : [0,T] \to H_1$ and $\sigma_\eta : [0,T] \to \mathcal{H}_1$ such that

$$\varepsilon(u_\eta(t)) = \mathcal{E}^{-1}\sigma_\eta(t) + z_\eta(t) + \mathcal{E}^{-1}\widetilde{\sigma}(t) \tag{4.9}$$

$$\sigma_\eta(t) \in \Sigma_0 \ , \quad < \tau - \sigma_\eta(t), \varepsilon(u_\eta(t)) >_{\mathcal{H}} \ \geq 0 \quad \forall \tau \in \Sigma_0 \tag{4.10}$$

for all $t \in [0,T]$.

Using standard arguments of elliptic variational inequalities followed by orthogonality arguments in the Hilbert space $\mathcal{H}$, it results that the problem $P_{2\eta}$ has a unique solution $(u_\eta(t), \sigma_\eta(t))$. Moreover,

$$u_\eta(0) = u_0 \ , \quad \sigma_\eta(0) = \overline{\sigma}_0 . \tag{4.11}$$

Having now in mind (4.9), (4.10), (3.1) and (2.8) we obtain

$$|u_\eta(t_1) - u_\eta(t_2)|_{H_1} \ \leq \ C\Big(|\widetilde{\sigma}(t_1) - \widetilde{\sigma}(t_2)|_V + |z_\eta(t_1) - z_\eta(t_2)|_{\mathcal{H}}\Big)$$

$$|\sigma_\eta(t_1) - \sigma_\eta(t_2)|_{\mathcal{H}_1} \ \leq \ C\Big(|\widetilde{\sigma}(t_1) - \widetilde{\sigma}(t_2)|_V + |z_\eta(t_1) - z_\eta(t_2)|_{\mathcal{H}}\Big)$$

for all $t_1, t_2 \in [0,T]$; and, using (3.6), (4.3) and the time regularity $z_\eta \in W^{1,\infty}(0,T,\mathcal{H})$, we deduce $u_\eta \in W^{1,\infty}(0,T,V), \sigma_\eta \in W^{1,\infty}(0,T,\mathcal{H}_1)$.

ii) This last time regularity and the assumption (3.2) allow us to consider the operator $\Lambda : L^\infty(0,T,\mathcal{H}) \longrightarrow L^\infty(0,T,\mathcal{H})$ defined by

$$\Lambda\eta = -\mathcal{E}^{-1}G(\sigma_\eta + \widetilde{\sigma}, \varepsilon(u_\eta)) \tag{4.12}$$

and, computing the difference between two solutions of $P_{2\eta}$ for two elements $\eta_1, \eta_2 \in L^\infty(0,T,\mathcal{H})$, it results

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \ \leq \ C|z_{\eta_1}(t) - z_{\eta_2}(t)|_{\mathcal{H}} \ \leq \ \int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}}$$

which implies, by recurrence, that for p large enough, the power Λ^p of the operator Λ is a contraction. Hence, we deduce that Λ^p has a unique fixed point $\eta^* \in L^\infty(0,T,\mathcal{H})$. Moreover, we can verify that $\eta^* \in L^\infty(0,T,\mathcal{H})$ is the unique fixed point of Λ.

iii) Finally, it can verified that the couple $(u_{\eta^*}, \sigma_{\eta^*})$ is a solution of the problem (4.5)-(4.7). Indeed, the equality (4.6) follows from (4.8), (4.9) and (4.12) since

$$\varepsilon(\dot{u}_{\eta^*}(t)) = \mathcal{E}^{-1}\dot{\sigma}_{\eta^*}(t) + \dot{z}_{\eta^*}(t) + \mathcal{E}^{-1}\dot{\tilde{\sigma}}(t)$$

$$\dot{z}_{\eta^*}(t) = \dot{\eta}^*(t) = \Lambda\eta^*(t) = -\mathcal{E}^{-1}G(\sigma_{\eta^*}(t) + \tilde{\sigma}(t), \varepsilon(u_{\eta^*}(t)))$$

a.e. $t \in (0,T)$. Moreover, the uniqueness part in Theorem 4.2 is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.12). It can be also proved directly from (3.17)-(3.19), (3.1) and (3.2), using a Gronwall-type inequality.

5 An equivalence result

We give in this section, a result concerning the link between the solutions of the variational problems P_1 and P_2 :

Theorem 5.1. *Let* (3.1) – (3.4) *and* (3.9) *hold and let* $u \in W^{1,\infty}(0,T,V)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H}_1)$. *Then* (u,σ) *is a solution of the variational problem* P_1 *if and only if* (u,σ) *is a solution of the variational problem* P_2.

Proof. Let $t \in [0,T]$ and let us suppose that (u,σ) is a solution of P_1. Taking $v = 2u(t)$ and $v = 0$ in (3.15) we obtain

$$(5.1) \qquad <\sigma(t), \varepsilon(u(t))>_{\mathcal{H}} \;=\; <F(t), u(t)>_V .$$

Using now (3.15) and (5.1) it follows $\sigma(t) \in \Sigma_{ad}(t)$. The inequality in (3.18) follows from (3.8) and (5.1). So, we conclude that (u,σ) is a solution of the problem P_2.

Conversely, let (u,σ) be a solution of P_2. We shall firstly prove that $u(t) \in U_{ad}$. Indeed, let us suppose in the sequel that $u(t) \notin U_{ad}$ and let us denote by $Pu(t)$ the projection of $u(t)$ on the closed convex set $U_{ad} \subset V$. We have

$$<Pu(t)-u(t), v>_V \;\geq\; <Pu(t)-u(t), Pu(t)>_V \;>\; <Pu(t)-u(t), u(t)>_V$$

for all $v \in U_{ad}$. From this inequalities we obtain that there exists $\alpha \in \mathbb{R}$ such that

$$(5.2) \quad <Pu(t)-u(t), v>_V \;>\alpha>\; <Pu(t)-u(t), u(t)>_V \quad \forall v \in U_{ad}.$$

Taking now $v = 0$ in (5.2), we deduce

$$(5.3) \qquad \alpha < 0.$$

Let now $\tilde{\tau}$ be the function defined by

$$(5.4) \qquad \tilde{\tau}(t) \;=\; \varepsilon(Pu(t) - u(t)) \in \mathcal{H}.$$

Supposing that there exists $w \in U_{ad}$ such that $< \tilde{\tau}(t), \varepsilon(w) >_{\mathcal{H}} < 0$, it follows from (5.2) and (5.4) since $\lambda w \in U_{ad}$ $\forall \lambda \geq 0$, $\lambda < \tilde{\tau}(t), \varepsilon(w) >_{\mathcal{H}} > \alpha$ $\forall \lambda \geq 0$. Hence, passing to the limit when $\lambda \to +\infty$, we obtain $\alpha \leq -\infty$ which is in contradiction with $\alpha \in \mathbb{R}$. So, it results $< \tilde{\tau}(t), \varepsilon(w) >_{\mathcal{H}} \geq 0$ $\forall w \in U_{ad}$, which implies $\tilde{\tau}(t) \in \Sigma_0$ (see (4.2)). Using (3.8) et (4.2) we obtain $\tilde{\tau}(t) + \tilde{\sigma}(t) \in \Sigma_{ad}(t)$ and, from (5.2), (5.4) and (3.18) it follows

$$(5.5) \qquad < \sigma(t) - \tilde{\sigma}(t), \varepsilon(u(t)) >_{\mathcal{H}} \; < 0.$$

Moreover, as $\sigma(t) - \tilde{\sigma}(t) \in \Sigma_0$, from (4.4) and (4.6) for $\tau = 2(\sigma(t) - \tilde{\sigma}(t))$ it results

$$(5.6) \qquad < \sigma(t) - \tilde{\sigma}(t), \varepsilon(u(t)) >_{\mathcal{H}} \; \geq 0.$$

It is clear that (5.5) and (5.6) are in contradiction. Therefore, $u(t) \in U_{ad}$. Using now (2.9) and (4.3) it follows $\tilde{\sigma}(t) \in \Sigma_{ad}(t)$ where $\tilde{\sigma}$ is given by (5.3). Hence, having in mind (3.18) and (3.8) we obtain

$$(5.7) \qquad < \sigma(t), \varepsilon(u(t)) >_{\mathcal{H}} \; = \; < F(t), u(t) >_V .$$

Finally, the inequality in (3.15) is a consequence of (5.7) and (3.10). So, it results that (u, σ) is a solution of the problem P_1. □

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On the Existence and Behaviour of the Solution for a Class of Nonlinear Evolution Systems

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On the Existence and Behaviour of the Solution for a Class of Nonlinear Evolution Systems

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Descriptif

Cette publication est dédiée à l'étude d'un problème d'évolution abstrait non linéaire généralisant une classe de problèmes de contact avec ou sans frottement entre un corps viscoplastique et une fondation rigide.

La formulation du problème étudié est la suivante : soit H un espace de Hilbert et $[0,T]$ un intervalle de temps. On se donne aussi les opérateurs $A : H \longrightarrow H$ et $B : [0,T] \times H \times H \longrightarrow H$ ainsi que la fonctionnelle $\varphi : [0,T] \times H \longrightarrow]-\infty, +\infty]$. On considère les problèmes d'évolution de la forme

$$\dot{y}(t) = A\dot{x}(t) + B\big(t, x(t), y(t)\big), \tag{1}$$

$$y(t) + \partial\varphi\big(t, x(t)\big) \ni f(t), \tag{2}$$

$$x(0) = x_0, \quad y(0) = y_0, \tag{3}$$

où les inconnues sont les fonction $x : [0,T] \longrightarrow H$ et $y : [0,T] \longrightarrow H$. Dans (1)–(3), le point au dessus d'une quantité désigne sa dérivée temporelle et $\partial\varphi$ est le sous-différentiel classique de la fonction φ.

Le problème abstrait (1)–(3) est présenté ici pour unifier et généraliser un ensemble de techniques utilisées pour l'analyse d'une classe de problèmes quasistatiques de contact entre un milieu continu viscoplastique et une fondation rigide. En effet, les inconnues x et y représentent respectivement le tenseur des petites déformations et celui des contraintes, l'équation (1) désigne la loi de comportement viscoplastique, (2) est une inéquation variationnelle elliptique englobant l'équation d'équilibre ainsi que les conditions aux limites. La fonctionnelle φ est entièrement déterminée par le type de conditions aux limites de contact considérées. Ainsi, aussi bien des conditions classiques de déplacement-traction que des conditions de contact avec ou sans frottement peuvent être mises sous la forme (2) avec un choix approprié de la fonctionnelle φ.

Cet article est structuré de la façon suivante : on prouve un résultat d'existence et d'unicité pour le problème (1)–(3) en utilisant des arguments sur les inéquations variationnelles elliptiques suivis d'une technique de point fixe. On poursuit avec un résultat de dépendance continue de la solution par rapport aux données ainsi qu'un résultat de stabilité. On étudie ensuite la dépendance de la solution par rapport à une petite perturbation de l'opérateur B et on prouve alors un résultat de convergence uniforme de la solution. Finalement, on considère le problème (1)–(3) dans le cas particulier d'une fonction indicatrice $\varphi = \psi_K$ où K est un convexe fermé et non vide de H. La solution est obtenue, dans ce cas, comme la limite d'une suite de solutions régulières de problèmes pénalisés de la forme (1)–(3).

ON THE EXISTENCE AND BEHAVIOUR OF THE SOLUTION FOR A CLASS OF NONLINEAR EVOLUTION SYSTEMS

MOHAMED ROCHDI and MIRCEA SOFONEA

This paper deals with the study of a nonlinear evolution system arising from rate-type viscoplasticity. The existence and uniqueness of the solution are obtained by using standard arguments of the elliptic variational inequalities followed by a fixed point method. The continuous dependence of the solution upon the input data as well as the behaviour of the solution with respect to various parameters are also investigated.

1. INTRODUCTION

Let H be a real Hilbert space, $A : H \to H$ and $T > 0$. Let also $B : [0, T] \times H \times H \to H$ and $\varphi : [0, T] \times H \to]-\infty, +\infty]$. We consider evolution systems of the form

$$\dot{y}(t) = A\,\dot{x}(t) + B(t, x(t), y(t)) \quad \text{a.e. } t \in (0, T) \tag{1.1}$$

$$y(t) + \partial\varphi(t, x(t)) \ni f(t) \quad \text{for all } t \in [0, T] \tag{1.2}$$

$$x(0) = x_0, \quad y(0) = y_0 \tag{1.3}$$

in which the unknowns are the functions $x : [0, T] \to H$ and $y : [0, T] \to H$. In (1.1)–(1.3) and everywhere in this paper, the dot above represents the derivative with respect to the time variable and $\partial\varphi$ is the subdifferential of the function φ.

Evolution problems of the form (1.1)–(1.3) arise in the study of quasi-static processes for elastic-viscoplastic materials in the case of the linearized theory. In this case, the unknowns x and y are, respectively, the small deformation tensor and the stress tensor, (1.1) represents the constitutive law in which the operators A and B are given, (1.2) involves the equilibrium equation as well as the boundary conditions, and, finally, (1.3) represents the initial conditions. The function φ in (1.2) is determined by the type of the boundary conditions. So, classical displacement-traction conditions, unilateral boundary conditions as well as contact conditions with or without friction may be modelled by (1.2) with an appropriate choice of φ (see, for example, [1], [2] and [3]).

The purpose of this paper is to investigate abstract systems of the form (1.1)–(1.3) in order to unify various results already obtained in [2], [4]–[6] in the study of quasis-static rate-type viscoplastic problems. So, in Section 2 we prove an existence and uniqueness result for (1.1)–(1.3), using standard arguments of elliptic variational inequalities followed by a fixed point technique (Theorem 2.1). In Section 3 we obtain an estimation of the distance between two solutions of (1.1)–(1.3) for two different sets of input data

(Theorem 3.1). This estimation involves, in particular, the continuous dependence of the solution with respect to the data as well as a stability result. In Section 4 the behaviour of the solution with respect to a small perturbation of the nonlinear operator B is studied and a uniform convergence result is obtained (Theorem 4.1). Finally, in Section 5 we consider (1.1)–(1.3) in the particular case of an indicator function $\varphi = \psi_K$ and we obtain the solution of this problem as the limit of a sequence of regular solutions associated to penalized problems (Theorem 5.1).

2. AN EXISTENCE AND UNIQUENESS RESULT

Everywhere in this paper we denote by $<\cdot,\cdot>_H$ the inner product of H and by $|\cdot|_H$ the associated norm. We also denote by "$\xrightarrow{H}$", "$\xrightarrow{H}$", respectively, the weak and strong convergences in H and by $\mathcal{C}(0,T,H)$ the space of continuous functions on $[0,T]$ with values in H, endowed with the canonical norm denoted by $|\cdot|_{\infty,H}$.

In the study of the problem (1.1)–(1.3) we denote by $D\varphi(t,\cdot)$ the effective domain of the function $\varphi(t,\cdot): H \to]-\infty,+\infty]$ defined by $D\varphi(t,\cdot) = \{x \in H \mid \varphi(t,x) < +\infty\}$ and we consider the following assumptions:

$$(2.1)\quad \begin{cases} A: H \to H \text{ is a positive definite and symmetric operator, i.e:} \\ \text{(a) there exists } m > 0 \text{ such that } <Ax,x>_H \geq m|x|_H^2 \text{ for all } x \in H \\ \text{(b) } <Ax,y>_H = <x,Ay>_H \text{ for all } x, y \in H \end{cases}$$

$$(2.2)\quad \begin{cases} B: [0,T] \times H \times H \to H \text{ and} \\ \text{(a) there exists } L > 0 \text{ such that } |B(t,x_1,y_1) - B(t,x_2,y_2)| \leq \\ \quad \leq L(|x_1 - x_2|_H + |y_1 - y_2|_H) \text{ for all } t \in [0,T],\ x_1, x_2, y_1, y_2 \in H \\ \text{(b) } t \mapsto B(t,x,y) \text{ is a mesurable function for all } x, y \in H \\ \text{(c) } t \mapsto B(t,0,0) \in L^\infty(0,T,H) \end{cases}$$

$$(2.3)\quad \begin{cases} \varphi: [0,T] \times H \to]-\infty,+\infty] \text{ and} \\ \text{(a) } D\varphi(t,\cdot) = K \subset H \text{ is independent on } t \in [0,T] \\ \text{(b) } K \neq \emptyset \text{ and for all } t \in [0,T]\ \varphi(t,\cdot) \text{ is a convex and lower} \\ \quad \text{semicontinuous function on } H \\ \text{(c) there exists } \tilde{L} > 0 \text{ such that} \\ \quad |\varphi(t_1,z_2) - \varphi(t_1,z_1) + \varphi(t_2,z_1) - \varphi(t_2,z_2)| \leq \\ \quad \leq \tilde{L}|t_1 - t_2||z_1 - z_2|_H \text{ for all } t_1, t_2 \in [0,T],\ z_1, z_2 \in K \end{cases}$$

$$(2.4)\quad f \in W^{1,\infty}(0,T,H)$$

$$(2.5)\quad x_0, y_0 \in H$$

$$y_0+\partial\varphi(0,x_0)\ni f(0). \tag{2.6}$$

The main result of this section is as follows.

THEOREM 2.1. *Let* (2.1)–(2.6) *hold. Then the problem* (1.1)–(1.3) *has a unique solution* $x\in W^{1,\infty}(0,T,H)$, $y\in W^{1,\infty}(0,T,H)$.

Theorem 2.1. may be obtained by using similar arguments than in [2] and [6]. However, for the convenience of the reader and in order to prepare the next sections of this paper, we summarize here the main ideas of the proof. For this, let us suppose in the sequel that the assumptions of Theorem 2.1. are fulfilled and let $\eta\in L^\infty(0,T,H)$. Let also $z_\eta\in W^{1,\infty}(0,T,H)$ be the function defined by

$$z_\eta(t) = \int_0^t \eta(s)ds+z_0 \qquad \forall t\in[0,T], \tag{2.7}$$

where

$$z_0=y_0-Ax_0. \tag{2.8}$$

LEMMA 2.1. *There exists a unique couple of functions* $x_\eta\in W^{1,\infty}(0,T,H)$, $y_\eta\in W^{1,\infty}(0,T,H)$ *such that*

$$y_\eta(t)=Ax_\eta(t)+z_\eta(t) \tag{2.9}$$

$$y_\eta(t)+\partial\varphi(t,x_\eta(t))\ni f(t) \tag{2.10}$$

for all $t\in[0,T]$. *Moreover,*

$$x_\eta(0)=x_0, \quad y_\eta(0)=y_0. \tag{2.11}$$

Proof. For all $t\in[0,T]$, using (2.1) and standard elliptic variational arguments, we obtain the existence and uniqueness of the elements $x_\eta(t)\in H$, $y_\eta(t)\in H$ such that (2.9)–(2.10) hold. Using now (2.3), it results that for all $t_1, t_2\in[0,T]$

$$\begin{cases} |x_\eta(t_1)-x_\eta(t_2)|_H+|y_\eta(t_1)-y_\eta(t_2)|_H\leq \\ \leq C(|z_\eta(t_1)-z_\eta(t_2)|_H+|f(t_1)-f(t_2)|_H+|t_1-t_2|) \end{cases} \tag{2.12}$$

and, having in mind (2.7) and (2.4), we deduce the regularity $x_\eta\in W^{1,\infty}(0,T,H)$, $y_\eta\in W^{1,\infty}(0,T,H)$. (In (2.12) and everywhere in this paper C represents a generic strictly positive constant, which may depend on A, B, φ and T and is independent of time and of the input data.)

Finally, (2.11) follows from (2.5)–(2.8) and the uniqueness of the solution for the elliptic problem (2.9), (2.10) at $t=0$.

Let us now remark that by hypothesis (2.2) we may consider the operator $\Lambda: L^\infty(0,T,H)\to L^\infty(0,T,H)$ defined by

$$\Lambda\eta(t)=B(t,x_\eta(t),y_\eta(t)) \qquad \forall\eta\in L^\infty(0,T,H),\ t\in[0,T]. \tag{2.13}$$

We have

LEMMA 2.2. *The operator* Λ *has a unique fixed point* $\eta^*\in L^\infty(0,T,H)$.

Proof. Let $\eta_1,\eta_2\in L^\infty(0,T,H)$. Using (2.9), (2.10), (2.1) and (2.2), after some algebra, it results

$$|\Lambda^p\eta_1-\Lambda^p\eta_2|_{\infty,H}\leq\frac{C^p}{p!}|\eta_1-\eta_2|_{\infty,H}\quad\forall p\in\mathbb{N},$$

where Λ^p denotes the powers of the operator Λ. Since $C^p/p!\to 0$ when $p\to 0$, it follows that for p large enough the operator Λ^p is a contraction in $L^\infty(0,T,H)$. Hence there exists a unique element $\eta^*\in L^\infty(0,T,H)$ such that $\Lambda\eta^*=\eta^*$.

Proof. of Theorem 2.1. Using Lemmas 2.1 and 2.2, it is easy to see that the couple of functions $x=x_{\eta^*}$, $y=y_{\eta^*}$ given by (2.9), (2.10) for $\eta=\eta^*$, represents a solution of the problem (1.1)–(1.3). Moreover, the uniqueness part Theorem 2.1 follows from the uniqueness of the fixed point of the operator Λ defined by (2.13).

3. THE CONTINUOUS DEPENDENCE OF THE SOLUTION UPON THE INPUT DATA

In this section two solutions of the variational problem (1.1)–(1.3) for two different data are considered. An estimation of the difference between these solutions is obtained that gives the continuous dependence of the solution upon the input data.

THEOREM 3.1. *Let* (2.1)–(2.3) *hold and let* (x_i, y_i) *be the solution of the evolution problem* (1.1)–(1.3) *for the data* f_i, x_{0i}, y_{0i} $(i=1,2)$ *such that* (2.4)–(2.6) *hold. Then there exists* $C>0$ *which depends on* A, B, φ *and* T *such that*

$$|x_1-x_2|_{\infty,H}+|y_1-y_2|_{\infty,H}\leq C(|x_{01}-x_{02}|_H+|y_{01}-y_{02}|_H+|f_1-f_2|_{\infty,H}). \tag{3.1}$$

Proof. Let $t\in[0,T]$ and $i\in\{1,2\}$. Using (1.2), we obtain

$$\langle y_i(t),z-x_i(t)\rangle_H+\varphi(t,z)-\varphi(t,x_i(t))\geq\langle f_i(t),z-x_i(t)\rangle_H\quad\forall\, z\in H \tag{3.2}$$

and, from (1.1) and (1.3), it results

$$y_i(t)=A\,x_i(t)+z_i(t), \tag{3.3}$$

where z_i is defined by

$$z_i(s)=\int_0^s B(u,x_i(u),y_i(u))du+z_{0i}\quad\forall s\in[0,T], \tag{3.4}$$

$$z_{0i}=y_{0i}-A\,x_{0i}. \tag{3.5}$$

Using now (3.2), (3.3) and (2.1), we obtain

$$|x_1(t)-x_2(t)|_H+|y_1(t)-y_2(t)|_H\leq C(|f_1(t)-f_2(t)|_H+|z_1(t)-z_2(t)|_H). \tag{3.6}$$

Moreover, from (3.4), (3.5), (2.1) and (2.2) it results

$$\begin{cases}|z_1(t)-z_2(t)|_H\leq C\Big[|x_{01}-x_{02}|_H+|y_{01}-y_{02}|_H+\\ +\int_0^t\big(|x_1(s)-x_2(s)|_H+|y_1(s)-y_2(s)|_H\big)ds\Big]\end{cases} \tag{3.7}$$

The inequality (3.1) follows now from (3.6) and (3.7).

4. A UNIFORM CONVERGENCE RESULT

In this section we study the dependence of the solution of the evolution problem (1.1)–(1.3) with respect to a perturbation of the operator B. For this, let us suppose that (2.1)–(2.6) hold. For every $\mu > 0$ we consider a perturbation B_μ of the operator B which satisfies (2.2). Using Theorem 2.1, we obtain the existence and uniqueness of the functions $x_\mu \in W^{1,\infty}(0,T,H)$, $y_\mu \in W^{1,\infty}(0,T,H)$, such that

(4.1) $$\dot{y}_\mu(t) = A\dot{x}_\mu(t) + B_\mu(t, x_\mu(t), y_\mu(t)) \quad \text{a.e. } t \in (0,T)$$

(4.2) $$y_\mu(t) + \partial\varphi(t, x_\mu(t)) \ni f(t) \quad \text{for all } t \in [0,T]$$

(4.3) $$x_\mu(0) = x_0, \quad y_\mu(0) = y_0.$$

Let us now consider the following assumption:

(4.4) $$\begin{cases} \text{There exists } g:\mathbb{R}_+ \to \mathbb{R}_+ \text{ such that} \\ (a)\ |B_\mu(t,x,y) - B(t,x,y)|_H \leqslant g(\mu) \quad \text{for all } t \in [0,T],\ x,y \in H \\ (b)\ \lim\limits_{\mu \to 0} g(\mu) = 0. \end{cases}$$

The main result of this section is the following

THEOREM 4.1. *Let (4.4) hold. Then the solution (x_μ, y_μ) of the problem (4.1)–(4.3) uniformly converges to the solution (x,y) of the problem* (1.1)–(1.3)

$$x_\mu \to x \text{ in } \mathcal{C}(0,T,H),\ y_\mu \to y \text{ in } \mathcal{C}(0,T,H) \text{ when } \mu \to 0.$$

Proof. Let $t \in [0,T]$ and $\mu > 0$. As it results from the proof of Theorem 2.1, (x,y) is the solution of the elliptic problem (2.9), (2.10) for $\eta = \eta^*$, where η^* is the fixed point of the operator $\Lambda : L^\infty(0,T,H) \to L^\infty(0,T,H)$ defined by (2.13), while (x_μ, y_μ) is the solution of the same problem (2.9), (2.10) for $\eta = \eta^*_\mu$, where η^*_μ is the fixed point of the operator $\Lambda_\mu : L^\infty(0,T,H) \to L^\infty(0,T,H)$ defined by

(4.5) $$\Lambda_\mu \eta(t) = B_\mu(t, x_\eta(t), y_\eta(t)) \quad \text{for all } \eta \in L^\infty(0,T,H),\ t \in [0,T].$$

Using this remark, from (2.9), (2.10) and (2.1) we obtain

$$|x_\mu(t) - x(t)|_H + |y_\mu(t) - y(t)|_H \leqslant C\,|z_{\eta^*_\mu}(t) - z_{\eta^*}(t)|_H$$

and using now (2.7) and (2.8), the previous inequality becomes

(4.6) $$|x_\mu(t) - x(t)|_H + |y_\mu(t) - y(t)|_H \leqslant C\int_0^t |\eta^*_\mu(s) - \eta^*(s)|_H\, ds.$$

Since $\eta^*_\mu = \Lambda_\mu \eta^*_\mu$ and $\eta^* = \Lambda\eta^*$, from (4.5) and (2.13) we deduce

$$|\eta^*_\mu(s) - \eta^*(s)|_H = B_\mu(s, x_\mu(s), y_\mu(s)) - B(s, x(s), y(s))|_H \quad \forall\ s \in [0,T].$$

Using now (4.4) and (2.2) in the previous equality, it follows

(4.7) $$|\eta^*_\mu(s) - \eta^*(s)|_H \leqslant g(\mu) + L(|x_\mu(s) - x(s)|_H + |y_\mu(s) - y(s)|_H) \quad \forall\ s \in [0,T].$$

So, from (4.6) and (4.7) we obtain

$$
(4.8)\qquad \begin{cases} |x_\mu(t) - x(t)|_H + |y_\mu(t) - y(t)|_H \leqslant C\tilde{g}(\mu) + \\ + C\int_0^t \left(|x_\mu(s) - x(s)|_H + |y_\mu(s) - y(s)|_H\right)\mathrm{d}s. \end{cases}
$$

Theorem 4.1 follows now from (4.8) and (4.4).

5. A PENALIZED METHOD

In this section we consider the problem (1.1)–(1.3) in the case when φ is the indicator function of a set K and we study this problem by using a penalized method. So, let K be a nonempty closed convex subset of H and, for simplification, let us suppose that $0_H \in K$. We denote by ψ_K the indicator of K and let $\partial\psi_K$ be the subdifferential of ψ_K.

Let (2.1), (2.2), (2.4) and (2.5) hold and let us suppose that

$$
(5.1)\qquad y_0 + \partial\psi_K(x_0) \ni f(0).
$$

Using Theorem 2.1, there exists a unique couple of functions $x \in W^{1,\infty}(0,T,H)$, $y \in W^{1,\infty}(0,T,H)$ such that

$$
(5.2)\qquad \dot{y}(t) = A\dot{x}(t) + B(t,x(t),y(t)) \quad \text{a.e } t \in (0,T)
$$

$$
(5.3)\qquad y(t) + \partial\psi_K(x(t)) \ni f(t) \quad \text{for all } t \in [0,T]
$$

$$
(5.4)\qquad x(0) = x_0, \qquad y(0) = y_0.
$$

In order to approach the solution of (5.2)–(5.4) by the solution of a penalized problem, let us consider $\varphi : H \to \mathbb{R}$ such that

$$
(5.5)\qquad \begin{cases} \text{(a)}\ \varphi \text{ is a convex and lower semicontinuous function on } H \\ \text{(b)}\ \varphi(z) = 0 \quad \text{iff } z \in K \\ \text{(c)}\ \varphi(z) \geqslant 0 \quad \text{for all } z \in H. \end{cases}
$$

Moreover, for every $h > 0$ let x_{0h} and y_{0h} be such that

$$
(5.6)\qquad x_{0h} \in H,\ y_{0h} \in H
$$

$$
(5.7)\qquad y_{0h} + \frac{1}{h}\partial\varphi(x_{0h}) \ni f(0).
$$

Using again Theorem 2.1, we obtain that if (2.1), (2.2), (2.4), (5.5)–(5.7) hold, then there exists a unique couple of functions $x_h \in W^{1,\infty}(0,T,H)$, $y_h \in W^{1,\infty}(0,T,H)$ such that

$$
(5.8)\qquad \dot{y}_h(t) = A\dot{x}_h(t) + B(t,x_h(t),y_h(t)) \quad \text{a.e } t \in (0,T)
$$

$$
(5.9)\qquad y_h + \frac{1}{h}\partial\varphi(x_h(t)) \ni f(t) \quad \text{for all } t \in [0,T]
$$

$$
(5.10)\qquad x_h(0) = x_{0h}, \qquad y_h(0) = y_{0h}.
$$

The solution of the evolution problem (5.8)–(5.10) depends on $h > 0$. The behaviour of this solution when $h \to 0$ is given by the following.

THEOREM 5.1. *Let* (2.1), (2.2), (2.4), (2.5), (5.1), (5.5)–(5.7) *hold. Let* (x_h, y_h) *be the solution of* (5.8)–(5.10) *for* $h > 0$ *and let* (x, y) *be the solution of* (5.2)–(5.4). *If*

(5.11) $$x_{0h} \xrightarrow{H} x_0, \quad y_{0h} \xrightarrow{H} y_0 \quad \text{when} \quad h \to 0,$$

then, for all $t \in [0,T]$ *we have*

(5.12) $$x_h(t) \xrightarrow{H} x(t), \quad y_h(t) \xrightarrow{H} y(t) \quad \text{when} \quad h \to 0.$$

In order to prove Theorem 5.1, we need some preliminary results. For this, let us suppose in the sequel that the assumptions of Theorem 5.1 are fulfilled and let us consider $\eta \in L^\infty(0,T,H)$. We denote by z_η the function defined by (2.7), (2.8) and, for every $h>0$, let $z_{h\eta}$ be given by

(5.13) $$z_{h\eta}(t) = \int_0^t \eta(s)ds + z_{0h} \quad \forall t \in [0,T],$$

where

(5.14) $$z_{0h} = y_{0h} - A\,x_{0h}.$$

From (5.13), (5.14) and (5.11) it results that

(5.15) $$z_{h\eta}(t) \xrightarrow{H} z_\eta(t) \quad \forall t \in [0,T] \quad \text{when} \quad h \to 0.$$

Using now Lemma 2.1, we obtain the existence and uniqueness of the functions $x_{h\eta} \in W^{1,\infty}(0,T,H)$ and $y_{h\eta} \in W^{1,\infty}(0,T,H)$ such that

(5.16) $$y_{h\eta}(t) = A x_{h\eta}(t) + z_{h\eta}(t)$$

(5.17) $$y_{h\eta}(t) + \frac{1}{h}\partial\varphi(x_{h\eta}(t)) \ni f(t)$$

for all $t \in [0,T]$

(5.18) $$x_{h\eta}(0) = x_{0h}, \quad y_{h\eta}(0) = y_{0h}.$$

Using again Lemma 2.1, we obtain the existence and uniqueness of the functions $x_\eta \in W^{1,\infty}(0,T,H)$, and $y_\eta \in W^{1,\infty}(0,T,H)$ such that

(5.19) $$y_\eta(t) = Ax_\eta(t) + z_\eta(t)$$

(5.20) $$y_\eta(t) + \partial\psi_K(x_\eta(t)) \ni f(t)$$

for all $t \in [0,T]$

(5.21) $$x_\eta(0) = x_0, \quad y_\eta(0) = y_0.$$

We have

LEMMA 5.1. *If* (5.11) *holds, then for all* $t \in [0,T]$ *we have*

(5.22) $$x_{h\eta}(t) \xrightarrow{H} x_\eta(t), \quad y_{h\eta}(t) \xrightarrow{H} y_\eta(t) \quad \text{when} \quad h \to 0.$$

Proof. Let $t \in [0,T]$. Using (5.17), (5.16), (5.5(b)) and (2.1), we obtain

(5.23) $$C|x_{h\eta}(t)|_H^2 + \frac{1}{h}\varphi(x_{h\eta}(t)) \leq \langle\langle f(t) - z_{h\eta}(t), x_{h\eta}(t)\rangle\rangle_H.$$

Using now (5.5(c)) and (5.15) in (5.23), it results that $(x_{h_\eta}(t))_h$ is a bounded sequence in H. Therefore, there exists an element $\tilde{x}_\eta(t) \in H$ and a subsequence $(x_{h'\eta}(t))_{h'} \subset (x_{h\eta}(t))_h$ such that

(5.24) $$x_{h'\eta}(t) \xrightarrow{H} \tilde{x}_\eta(t) \quad \text{when} \quad h' \to 0.$$

Moreover, since $(x_{h'\eta}(t))_{h'}$ and $(z_{h'\eta}(t))_{h'}$ are bounded in H, from (5.23) we obtain

$$\varphi(x_{h'\eta}(t)) \leq Ch' \quad \text{for all } h'>0. \tag{5.25}$$

Passing to the limit when $h'\to 0$, from (5.5), (5.24) and (5.25) it results

$$\tilde{x}_\eta(t) \in K. \tag{5.26}$$

Using now (5.16), (5.17), (5.24) and (5.15), by standard lower semi-continuity arguments we deduce

$$A\tilde{x}_\eta(t) + z_\eta(t) + \partial\psi_K(\tilde{x}_\eta(t)) \ni f(t). \tag{5.27}$$

Moreover, by the uniqueness of the solution of the problem (5.19), (5.20), from (5.26) and (5.27) it results $\tilde{x}_\eta(t) = x_\eta(t)$. So, having in mind (5.24), it follows that

$$x_{h\eta}(t) \xrightarrow{H-} x_\eta(t) \quad \text{when} \quad h\to 0. \tag{5.28}$$

In order to obtain the strong convergence, let us remark that from (2.1) it results

$$\begin{cases} C|x_{h\eta}(t) - x_\eta(t)|_H^2 \leq \langle Ax_{h\eta}(t), x_{h\eta}(t) - x_\eta(t)\rangle_H - \\ \qquad - \langle Ax_\eta(t), x_{h\eta}(t) - x_\eta(t)\rangle_H \end{cases} \tag{5.29}$$

and, from (5.16), (5.17), (5.5), we deduce

$$\begin{cases} \langle Ax_{h\eta}(t),\ x_{h\eta}(t) - x_\eta(t)\rangle_H \leq \langle f(t), x_{h\eta}(t) - x_\eta(t)\rangle_H + \\ \qquad + \langle z_{h\eta}(t), x_\eta(t) - x_{h\eta}(t)\rangle_H . \end{cases} \tag{5.30}$$

Lemma 5.2 follows now from (5.29), (5.30), (5.15), (5,28) and (5.16).

Proof of Theorem 5.1. Let $h > 0$ and $t \in [0,T]$. As it results from the proof of theorem 2.1, (x,y) is the solution of the elliptic problem (5.19), (5.20) for $\eta = \eta^*$, where η^* is the fixed point of the operator $\Lambda : L^\infty(0,T,H) \to L^\infty(0,T,H)$ defined by (2.13). Similarly, (x_h, y_h) is the solution of the problem (5.16), (5.17) for $\eta = \eta_h^*$, where η_h^* is the fixed point of the operator Λ_h: $L^\infty(0,T,H) \to L^\infty(0,T,H)$ defined by

$$\Lambda_{h\eta}(t) = B(t, x_{h\eta}(t), y_{h\eta}(t)) \quad \forall \eta \in L^\infty(0,T,H),\ \ t\in[0,T]. \tag{5.31}$$

So, we have

$$\begin{cases} |x_h(t) - x(t)|_H + |y_h(t) - y(t)|_H \leq \left|x_{h\eta_h^*}(t) - x_{h\eta^*}(t)\right|_H + \left|y_{h\eta_h^*}(t) - y_{h\eta^*}(t)\right|_H \\ \qquad + \left|x_{h\eta^*}(t) - x_{\eta^*}(t)\right|_H + \left|y_{h\eta^*}(t) - y_{\eta^*}(t)\right|_H . \end{cases} \tag{5.32}$$

Using standard arguments, from (5.16) and (5.17) it follows

$$\left|x_{h\eta_h^*}(t) - x_{h\eta^*}(t)\right|_H + \left|y_{h\eta_h^*}(t) - y_{h\eta^*}(t)\right|_H \leq C\int_0^t \left|\eta_h^*(s) - \eta^*(s)\right|_H ds$$

and, having in mind (5.31), (2.13) and (2.2), we deduce

(5.33)
$$\begin{cases} \left|x_{h\eta_h^*}(t)-x_{h\eta^*}(t)\right|_H+\left|y_{h\eta_h^*}(t)-y_{h\eta^*}(t)\right|_H\leqslant \\ \leqslant C\int_0^t\left(|x_h(s)-x(s)|_H+|y_h(s)-y(s)|_H\right)\mathrm{d}s. \end{cases}$$

Let us now consider $\varepsilon>0$. Using Lemma 5.1, we obtain that there exists $\delta>0$ which depends on ε, t and η^* such that for $0<h<\delta$ we have

(5.34)
$$\left|x_{h\eta^*}(t)-x_{\eta^*}(t)\right|_H+\left|y_{h\eta^*}(t)-y_{\eta^*}(t)\right|_H\leqslant\varepsilon.$$

So, if $0<h<\delta$, from (5.32)–(5.34) it results

(5.35)
$$\begin{cases} |x_h(t)-x(t)|_H+|y_h(t)-y(t)|_H\leqslant \\ \leqslant C\int_0^t\left(|x_h(s)-x(s)|_H+|y_h(s)-x(t)|_H\right)\mathrm{d}s+\varepsilon. \end{cases}$$

Using now a Gronwall-type inequality, (5.35) implies

$$|x_h(t)-x(t)|_H+|y_h(t)-y(t)|_H\leqslant C\varepsilon$$

for all h such that $0<h<\delta$, which proves (5.12).

Remark 5.1. Concrete examples, applications and mechanical interpretation of the strong convergence result given by Theorem 5.1 in the study of contact problems for elastic-viscoplastic bodies may be found in [2] and [3].

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On Frictionless Contact between two Elastic-Viscoplastic Bodies

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Descriptif

On se propose, dans cette publication, d'étudier un problème quasistatique de contact sans frottement entre deux matériaux viscoplastiques. Les conditions aux limites de contact considérées ici sont du type Signorini.

Soient deux milieux continus viscoplastiques occupant deux domaines Ω^1 et Ω^2 de $\mathbb{R}^N$ $(N = 1, 2, 3)$ et dont les frontières respectives Γ^1 et Γ^2, supposées suffisamment régulières, sont divisées en trois parties disjointes Γ_1^i, Γ_2^i et Γ_3^i $(i = 1, 2)$. Pendant l'intervalle de temps $[0, T]$, on suppose que des forces volumiques φ_1^i agissent dans Ω^i, que les parties Γ_1^i sont encastrées dans des structures fixes et que des tractions φ_2^i s'appliquent sur Γ_2^i. On suppose en outre que les deux matériaux sont en contact permanent sans frottement le long de la partie Γ_3^i commune à leurs frontières, notée dans la suite par Γ_3. Ainsi, le problème quasistatique de contact sans frottement entre deux milieux continus viscoplastiques peut se formuler de la façon suivante :

Problème P : Pour $i = 1, 2$, trouver le champ des déplacements $u^i : \Omega^i \times [0, T] \longrightarrow \mathbb{R}^N$ et le champ des contraintes $\sigma^i : \Omega^i \times [0, T] \longrightarrow \mathbb{R}_s^{N\times N}$ tels que

$$\dot{\sigma}^i = \mathcal{E}^i \varepsilon(\dot{u}^i) + G^i(\sigma^i, \varepsilon(u^i)) \qquad \text{dans} \quad \Omega^i \times (0, T),$$

$$Div\, \sigma^i + \varphi_1^i = 0 \qquad \text{dans} \quad \Omega^i \times (0, T),$$

$$u^i = 0 \qquad \text{sur} \quad \Gamma_1^i \times (0, T),$$

$$\sigma^i \nu = \varphi_2^i \qquad \text{sur} \quad \Gamma_2^i \times (0, T),$$

$$u_\nu^1 + u_\nu^2 \leq 0, \quad \sigma_\nu^1 = \sigma_\nu^2 \leq 0, \quad \sigma_\tau^i = 0, \quad \sigma_\nu^1(u_\nu^1 + u_\nu^2) = 0 \qquad \text{sur} \quad \Gamma_3 \times (0, T),$$

$$u^i(0) = u_0^i, \quad \sigma^i(0) = \sigma_0^i \qquad \text{dans} \quad \Omega^i.$$

On note par $\mathbb{R}_s^{N\times N}$ l'espace des tenseurs symétriques du second ordre sur $\mathbb{R}^N$ et par $\varepsilon(u^i)$ le tenseur des petites déformations linéarisé. Le point au dessus d'une quantité désigne sa dérivée temporelle, $Div\, \sigma^i$ désigne la divergence de la fonction tensorielle σ^i, le vecteur ν est la normale unitaire sortante à Ω^1 ou à Ω^2, $\sigma^i \nu$ est le vecteur des contraintes de Cauchy, u_ν^i, σ_ν^i et σ_τ^i représentent respectivement le déplacement normal, les contraintes normales et tangentielles.

Il s'agit dans cet article de l'analyse variationnelle du problème P. On commence par établir deux formulations faibles P_1 et P_2. La première est définie en déplacements alors que la seconde est définie en contraintes. On poursuit avec un résultat d'existence et d'unicité de la solution pour chacune de ces formulations. On termine par l'étude de quelques propriétés de la solution. On prouve ainsi une équivalence entre les formulations P_1 et P_2 et la dépendance continue de la solution par rapport aux données ainsi qu'un résultat de concergence par rapport à un coefficient de viscosité.

ON FRICTIONLESS CONTACT BETWEEN TWO ELASTIC–VISCOPLASTIC BODIES

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SUMMARY

A quasistatic contact problem between two elastic–viscoplastic bodies involving frictionless boundary conditions is considered. Two variational formulations of this problem are proposed, followed by existence and uniqueness results. An equivalence result between the previous variational formulations, the continuous dependence of the solution with respect to the data as well as a convergence result with respect to a viscosity parameter are also obtained.

1. Introduction

IN THIS paper we are interested in the study of a frictionless contact problem for materials whose behaviour may be modelled by a constitutive law of the form

$$\dot{\sigma} = \mathscr{E}\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u)) \tag{1.1}$$

in which $\sigma = (\sigma_{kh})$ denotes the stress tensor, $u = (u_k)$ is the displacement field, $\varepsilon(u) = (\varepsilon_{kh}(u))$ represents the small strain tensor, and $\mathscr{E}$ and G are constitutive functions. In (1.1) as well as everywhere in this paper the dot above represents the derivative with respect to the time variable.

Rate-type viscoplastic models of the form (1.1) are used in order to describe the behaviour of real materials like rubbers, metals, pastes, rocks and so on. Various results and mechanical interpretations concerning models of this form may be found for instance in (**1**) (see also the references quoted there). Existence and uniqueness results for initial- and boundary-value problems involving (1.1) were obtained in (**2, 3**) in the case of classical displacement-traction boundary conditions, in (**4**) in the case of 'linear boundary conditions' and in (**5**) in the case of frictionless contact with a rigid foundation.

The purpose of this work is to investigate a quasistatic problem involving a frictionless contact between two elastic–viscoplastic bodies, each having a constitutive law of the form (1.1). So, some results already obtained in (**6** to **9**) in the case of two elastic bodies are extended here to the case of rate-type elastic–viscoplastic models.

This paper is structured as follows: in section 2 the mechanical problem P is stated and some functional notations are presented. For the problem P we establish in section 3 two variational formulations P_1 and P_2. Both P_1 and P_2 involve a coupling between the constitutive law (1.1) and a variational inequality including the equilibrium equation and the boundary conditions. The unknowns in the variational problems P_1 and P_2 are the displacement field u and the stress field σ.

For the problem P_1 we prove in section 4 an existence and uniqueness result (Theorem 1). In this case the main unknown is u and the existence of the solution results from classical elliptic variational inequalities arguments followed by a fixed-point method. In section 5 we also prove a similar existence and uniqueness result for the problem P_2 (Theorem 2). Here the main unknown is σ. The last section deals with the study of some properties of the solution. So, we prove an equivalence result between the problems P_1 and P_2, the continuous dependence of the solution with respect to the input data, and a convergence result with respect to a viscosity parameter (Theorems 3 to 5).

2. Problem statement and preliminaries

Let us consider two elastic–viscoplastic bodies whose material particles occupy the bounded domains Ω^1 and Ω^2 of $\mathbb{R}^N$ $(N = 1, 2, 3)$ and whose boundaries Γ^1 and Γ^2, assumed to be sufficiently smooth, are partitioned into three disjoint measurable parts Γ_1^i, Γ_2^i and Γ_3^i $(i = 1, 2)$. Let $\operatorname{meas}\Gamma_1^i > 0$ and let $T > 0$ be a time interval. We shall assume that body forces φ_1^i act in $\Omega^i \times (0, T)$, that the displacement fields vanish on $\Gamma_1^i \times (0, T)$ and that surface tractions φ_2^i act on $\Gamma_2^i \times (0, T)$. We also suppose that, in the time interval $[0, T]$, the bodies are in contact along the common part Γ_3^i of their boundary which we denote in the sequel by Γ_3. Moreover, we suppose contact boundary conditions of the Signorini type on $\Gamma_3 \times (0, T)$ in the form with a zero gap function. We shall finally assume the case of quasistatic processes and we shall use (1.1) as the constitutive law. With these assumptions, the mechanical problem we study here may be formulated as follows.

PROBLEM P For $i = 1, 2$ find the displacement field $u^i : \Omega^i \times [0, T] \to \mathbb{R}^N$ and the stress field $\sigma^i : \Omega^i \times [0, T] \to S_N$ such that

$$\dot{\sigma}^i = \mathscr{E}^i \varepsilon(\dot{u}^i) + G^i(\sigma^i, \varepsilon(u^i)) \quad \text{in } \Omega^i \times (0, T), \tag{2.1}$$

$$\operatorname{Div} \sigma^i + \varphi_1^i = 0 \quad \text{in } \Omega^i \times (0, T), \tag{2.2}$$

$$u^i = 0 \quad \text{on } \Gamma_1^i \times (0, T), \tag{2.3}$$

$$\sigma^i \nu = \varphi_2^i \quad \text{on } \Gamma_2^i \times (0, T), \tag{2.4}$$

$$u_\nu^1 + u_\nu^2 \leqslant 0,\ \sigma_\nu^1 = \sigma_\nu^2 \leqslant 0,\ \sigma_{\tau k}^i = 0,\ \sigma_\nu^1(u_\nu^1 + u_\nu^2) = 0 \quad \text{on } \Gamma_3 \times (0, T) \tag{2.5}$$

$$u^i(0) = u_0^i, \quad \sigma^i(0) = \sigma_0^i \quad \text{in} \quad \Omega^i, \tag{2.6}$$

where S_N denotes the set of second-order symmetric tensors on $\mathbb{R}^N$. In (2.1) to (2.6) $\operatorname{Div}\sigma^i$ represents the divergence of the tensor-valued function σ^i, $\nu = (\nu_k)$ denotes the unit outward normal vector to both Ω^1 and Ω^2, $\sigma^i\nu$ is the stress vector, u_ν^i, σ_ν^i and $\sigma_{\tau k}^i$ are given by

$$u_\nu^i = u_k^i \nu_k, \qquad \sigma_\nu^i = \sigma_{kh}^i \nu_h \nu_k, \qquad \sigma_{\tau k}^i = \sigma_{kh}^i \nu_h - \sigma_\nu^i \nu_k$$

and finally u_0^i and σ_0^i are the initial data.

We denote in the sequel by $\cdot$ the inner product on the spaces $\mathbb{R}^N$ and S_N and by $|\cdot|$ the Euclidean norms on these spaces. The following notation is also used, for $i = 1, 2$:

$$H^i = \{v = (v_k) \mid v_k \in L^2(\Omega^i),\ k = \overline{1, N}\} = L^2(\Omega^i)^N,$$

$$H_1^i = \{v = (v_k) \mid v_k \in H^1(\Omega^i),\ k = \overline{1, N}\} = H^1(\Omega^i)^N,$$

$$\mathcal{H}^i = \{\tau = (\tau_{kh}) \mid \tau_{kh} = \tau_{hk} \in L^2(\Omega^i),\ k, h = \overline{1, N}\} = L^2(\Omega^i)_s^{N\times N},$$

$$\mathcal{H}_1^i = \{\tau \in \mathcal{H}^i \mid \operatorname{Div}\tau \in H\}.$$

The spaces H^i, H_1^i, $\mathcal{H}^i$ and $\mathcal{H}_1^i$ are real Hilbert spaces endowed with the canonical inner products denoted respectively by $\langle\cdot,\cdot\rangle_X$, where X is one of these spaces.

Let $H_{\Gamma^i} = [H^{\frac{1}{2}}(\Gamma^i)]^N$ and let $\gamma_i : H_1^i \longrightarrow H_{\Gamma^i}$ be the trace map. We denote by V^i the closed subspace of H_1^i defined by

$$V^i = \{u \in H_1^i \mid \gamma_i u = 0 \quad \text{on} \quad \Gamma_1^i\}.$$

The deformation operator $\varepsilon : H_1^i \longrightarrow \mathcal{H}^i$ given by

$$\varepsilon(u) = (\varepsilon_{kh}(u)), \quad \varepsilon_{kh}(u) = \tfrac{1}{2}(u_{k,h} + u_{h,k})$$

is a linear and continuous operator. Moreover, since $\operatorname{meas}\Gamma_1^i > 0$, Korn's inequality holds (see for instance (10, p. 79)):

$$|\varepsilon(v)|_{\mathcal{H}^i} \geqslant C|v|_{H_1^i} \quad \text{for all} \quad v \in V^i, \quad i = 1, 2. \tag{2.7}$$

Here C is a strictly positive constant which depends only on Ω^i and Γ_1^i (everywhere in this paper C will represent strictly positive generic constants which may depend on Ω^i, Γ_1^i, Γ_2^i, Γ_3, $\mathscr{E}^i$, G^i, T and do not depend on time or on input data φ_1^i, φ_2^i, u_0^i, σ_0^i, $i = 1, 2$).

We now endow V^i with the inner product $\langle\cdot,\cdot\rangle_{V^i}$ defined by

$$\langle v, w\rangle_{V^i} = \langle\varepsilon(v), \varepsilon(w)\rangle_{\mathcal{H}^i} \qquad \text{for all } v, w \in V^i. \tag{2.8}$$

We also denote by $|\cdot|_{V^i}$ the associated norm. So, having in mind (2.7), we deduce that $|\cdot|_{V^i}$ and $|\cdot|_{H_1^i}$ are equivalent norms on V^i. Therefore, V^i endowed with the inner product defined by (2.8) is a real Hilbert space.

Let now $H'_{\Gamma^i} = [H^{-\frac{1}{2}}(\Gamma^i)]^N$ be the strong dual of the space H_{Γ^i} and let $\langle\cdot,\cdot\rangle_i$ denote the duality between H'_{Γ^i} and H_{Γ^i}. For all $\tau \in \mathcal{H}_1^i$ there exists an element $\gamma_\nu^i \tau \in H'_{\Gamma^i}$ such that

$$\langle \gamma_\nu^i \tau, \gamma_i v\rangle_i \;=\; \langle \tau, \varepsilon(v)\rangle_{\mathcal{H}^i} + \langle \mathrm{Div}\,\tau, v\rangle_{H^i} \qquad \text{for all } v \in H_1^i. \tag{2.9}$$

Moreover, if τ is a regular (say C^1) function, then

$$\langle \gamma_\nu^i \tau, \gamma_i v\rangle_i \;=\; \int_{\Gamma^i} \tau\nu\, v\, da \qquad \text{for all } v \in H_1^i. \tag{2.10}$$

We now define the spaces $H_1 = H_1^1 \times H_1^2$, $\mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2$, $\mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2$ and $V = V^1 \times V^2$ each one endowed with the canonical inner product and, for simplicity, we shall use the notation

$$\varepsilon(v) = (\varepsilon(v^1), \varepsilon(v^2))$$

for all $v = (v^1, v^2) \in H_1$.

Finally, for every real Hilbert space X we denote by $|\cdot|_X$ the norm on X and by $|\cdot|_{\infty,X}$ the norm on the space $L^\infty(0,T,X)$. Moreover, $C(0,T,X)$ will denote the space of continuous functions on $[0,T]$ into X.

3. Variational formulations

In this section we give two variational formulations for the mechanical problem P. For this, let us suppose that for $i = 1,2$

$$\left.\begin{array}{l} \mathcal{E}^i : \Omega^i \times S_N \to S_N \ \text{ is a symmetric and positive definite tensor:} \\ \text{(a) } \mathcal{E}^i_{khlm} \in L^\infty(\Omega^i) \quad \text{for all } k,h,l,m = \overline{1,N}, \\ \text{(b) } \mathcal{E}^i\sigma\cdot\tau = \sigma\cdot\mathcal{E}^i\tau \ \text{ for all } \ \sigma,\tau \in S_N, \ \text{a.e. in } \Omega^i, \\ \text{(c) there exists } \ \alpha^i > 0 \ \text{ such that } \mathcal{E}^i\sigma\cdot\sigma \geqslant \alpha^i|\sigma|^2 \\ \qquad \text{for all } \ \sigma \in S_N, \ \text{a.e. in } \Omega^i, \end{array}\right\} \tag{3.1}$$

$$\left.\begin{array}{l} G^i : \Omega^i \times S_N \times S_N \to S_N \ \text{ and} \\ \text{(a) there exists } \ L^i > 0 \ \text{ such that} \\ \qquad |G^i(x,\sigma_1,\varepsilon_1) - G^i(x,\sigma_2,\varepsilon_2)| \leqslant L^i(|\sigma_1-\sigma_2| + |\varepsilon_1-\varepsilon_2|) \\ \qquad \text{for all } \ \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2 \in S_N, \ \text{a.e. in } \Omega^i, \\ \text{(b) } x \mapsto G^i(x,\sigma,\varepsilon) \ \text{ is a measurable function with respect to} \\ \qquad \text{the Lebesgue measure on } \Omega^i, \text{for all } \ \sigma,\varepsilon \in S_N, \\ \text{(c) } x \mapsto G^i(x,0,0) \in \mathcal{H}^i, \end{array}\right\} \tag{3.2}$$

$$\varphi_1^i \in W^{1,\infty}(0,T,H^i), \tag{3.3}$$

$$\varphi_2^i \in W^{1,\infty}(0,T,L^2(\Gamma_2^i)^N). \tag{3.4}$$

The assumptions (3.1) and (3.2) allow us to consider two operators denoted again by $\mathscr{E}^i$ and G^i such that $\mathscr{E}^i : \mathscr{H}^i \longrightarrow \mathscr{H}^i$, $G^i : \mathscr{H}^i \times \mathscr{H}^i \longrightarrow \mathscr{H}^i$ and

$$(\mathscr{E}^i\sigma)(\cdot) = \mathscr{E}^i(\cdot, \sigma(\cdot)) = (\mathscr{E}^i_{khlm}(\cdot)\sigma_{lm}(\cdot)) \quad \forall \sigma \in \mathscr{H}^i, \text{ a.e. in } \Omega^i,$$

$$G^i(\sigma, \varepsilon)(\cdot) = G^i(\cdot, \sigma(\cdot), \varepsilon(\cdot)) \quad \forall \sigma, \varepsilon \in \mathscr{H}^i, \text{ a.e. in } \Omega^i.$$

Hence, we may also consider the operators $\mathscr{E}$ and G defined by

$$\mathscr{E}\varepsilon = (\mathscr{E}^1\varepsilon^1, \mathscr{E}^2\varepsilon^2) \qquad \forall \varepsilon = (\varepsilon^1, \varepsilon^2) \in \mathscr{H},$$

$$G(\sigma, \varepsilon) = (G^1(\sigma^1, \varepsilon^1), G^2(\sigma^2, \varepsilon^2)) \qquad \forall \sigma = (\sigma^1, \sigma^2), \ \varepsilon = (\varepsilon^1, \varepsilon^2) \in \mathscr{H}.$$

We also denote by $f(t)$ the element of $V = V^1 \times V^2$ given by

$$\begin{aligned} \langle f(t), v\rangle_V = {} & \langle \varphi_1^1(t), v^1\rangle_{H^1} + \langle \varphi_1^2(t), v^2\rangle_{H^2} \\ & + \langle \varphi_2^1(t), \gamma_1 v^1\rangle_{L^2(\Gamma_2^1)^N} + \langle \varphi_2^2(t), \gamma_2 v^2\rangle_{L^2(\Gamma_2^2)^N} \\ & \forall v = (v^1, v^2) \in V, \ t \in [0, T]. \end{aligned} \tag{3.5}$$

So, using (3.3), (3.4) and (3.5), it follows that

$$f \in W^{1,\infty}(0, T, V). \tag{3.6}$$

Finally, let U_{ad} and $\Sigma_{ad}(t)$ be the sets given by

$$\begin{aligned} U_{ad} = \{ & v = (v^1, v^2) \in H_1 \mid \\ & v^1 = 0 \text{ on } \Gamma_1^1, \ v^2 = 0 \text{ on } \Gamma_1^2, \ v_\nu^1 + v_\nu^2 \leqslant 0 \text{ on } \Gamma_3\}, \end{aligned} \tag{3.7}$$

$$\Sigma_{ad}(t) = \{\tau \in \mathscr{H} \mid \langle \tau, \varepsilon(v)\rangle_{\mathscr{H}} \geqslant \langle f(t), v\rangle_V \ \ \forall v \in U_{ad}\} \quad \forall t \in [0, T], \tag{3.8}$$

and let us suppose that

$$u_0 = (u_0^1, u_0^2) \in U_{ad}, \ \sigma_0 = (\sigma_0^1, \sigma_0^2) \in \Sigma_{ad}(0), \ \langle \sigma_0, \varepsilon(u_0)\rangle_{\mathscr{H}} = \langle f(0), u_0\rangle_V. \tag{3.9}$$

LEMMA 1 *If the couple of functions $u = (u^1, u^2)$, $\sigma = (\sigma^1, \sigma^2)$ is a regular solution of the mechanical problem P then*

$$u(t) \in U_{ad}, \ \langle \sigma(t), \varepsilon(v) - \varepsilon(u(t))\rangle_{\mathscr{H}} \geqslant \langle f(t), v - u(t)\rangle_V \quad \forall v \in U_{ad} \tag{3.10}$$

$$\sigma(t) \in \Sigma_{ad}(t), \quad \langle \tau - \sigma(t), \ \varepsilon(u(t))\rangle_{\mathscr{H}} \geqslant 0 \ \ \forall \tau \in \Sigma_{ad}(t) \tag{3.11}$$

for all $t \in [0, T]$.

Proof. Let $v = (v^1, v^2) \in U_{ad}$ and $t \in [0, T]$. Using (2.2), (2.9) and (2.10) we have

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(u(t))\rangle_{\mathscr{H}} = \sum_{i=1}^{2}\Big[\langle \varphi_1^i(t), v^i - u^i(t)\rangle_{H^i} + \int_{\Gamma^i} \sigma^i(t)\nu \ (v^i - u^i(t))\, da\Big]$$

and, having in mind (2.3), (2.4) and (3.5), we obtain

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(u(t)) \rangle_{\mathcal{H}} = \langle f(t), v - u(t) \rangle_V + \sum_{i=1}^{2} \int_{\Gamma_3} \sigma^i(t)\nu \cdot (v^i - u^i(t))\, da.$$

Using now (2.5), the previous equality leads to

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(u(t)) \rangle_{\mathcal{H}} = \langle f(t), v - u(t) \rangle_V + \int_{\Gamma_3} \sigma^1_\nu(t)(v^1_\nu + v^2_\nu)\, da. \quad (3.12)$$

The inequality in (3.10) follows from (3.12), (3.7) and (2.5). Moreover, (2.3), (2.5) and (3.7) imply that $u(t) \in U_{ad}$.

Taking now $v = 2u(t)$ and $v = 0$ in (3.10) we obtain

$$\langle \sigma(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} = \langle f(t), u(t) \rangle_V. \quad (3.13)$$

Using now (3.13), (3.10) and (3.8) it follows that $\sigma(t) \in \Sigma_{ad}(t)$. The inequality in (3.11) follows now from (3.8) and (3.13).

The previous results lead us to consider two weak formulations of the problem P.

PROBLEM P_1 Find the displacement field $u : [0, T] \to H_1$ and the stress field $\sigma : [0, T] \to \mathcal{H}_1$ such that

$$\dot\sigma(t) = \mathcal{E}\varepsilon(\dot u(t)) + G(\sigma(t), \varepsilon(u(t))) \quad \text{a.e. } t \in (0, T) \quad (3.14)$$

$$u(t) \in U_{ad}, \langle \sigma(t), \varepsilon(v) - \varepsilon(u(t)) \rangle_{\mathcal{H}} \geqslant \langle f(t), v - u(t) \rangle_V \qquad \forall v \in U_{ad},\ t \in [0, T] \quad (3.15)$$

$$u(0) = u_0, \quad \sigma(0) = \sigma_0. \quad (3.16)$$

PROBLEM P_2 Find the displacement field $u : [0, T] \to H_1$ and the stress field $\sigma : [0, T] \to \mathcal{H}_1$ such that (3.14) and (3.16), and

$$\sigma(t) \in \Sigma_{ad}(t), \ \langle \tau - \sigma(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} \geqslant 0 \quad \forall \tau \in \Sigma_{ad}(t),\ t \in [0, T] \quad (3.17)$$

Let us now remark that the problems P_1 and P_2 are formally equivalent to problem P. Indeed, if (u, σ) represents a regular solution of the variational problems P_1 or P_2, using the arguments of (11) it follows that (u, σ) is a solution of the mechanical problem P.

Under the assumptions (3.1) to (3.4) and (3.9), in the next two sections we give existence and uniqueness results for the variational problems P_1 and P_2.

4. The first existence and uniqueness result

The main result of this section is the following.

THEOREM 1 *Let* (3.1) *to* (3.4) *and* (3.9) *hold. Then there exists a unique solution of the problem* P_1 *having the regularity*

$$u \in W^{1,\infty}(0,T,H_1), \quad \sigma \in W^{1,\infty}(0,T,\mathcal{H}_1).$$

In order to prove Theorem 1 we need some preliminary results. For this, let us suppose in the sequel that the assumptions (3.1) to (3.4) and (3.9) are satisfied and let $\eta \in L^\infty(0,T,\mathcal{H})$. Let also $z_\eta \in W^{1,\infty}(0,T,\mathcal{H})$ be the function defined by

$$z_\eta(t) = \int_0^t \eta(s)ds + \sigma_0 - \mathcal{E}\varepsilon(u_0) \qquad \forall t \in [0,T]. \tag{4.1}$$

We consider the following elliptic problem.

PROBLEM $P_{1\eta}$ Find $u_\eta : [0,T] \to H_1$ and $\sigma_\eta : [0,T] \to \mathcal{H}_1$ such that

$$\sigma_\eta(t) = \mathcal{E}\varepsilon(u_\eta(t)) + z_\eta(t) \tag{4.2}$$

$$u_\eta(t) \in U_{ad}, \ \langle \sigma_\eta(t), \varepsilon(v) - \varepsilon(u_\eta(t))\rangle_{\mathcal{H}} \geqslant \langle f(t), v - u_\eta(t)\rangle_V \quad \forall v \in U_{ad} \tag{4.3}$$

for all $t \in [0,T]$.

LEMMA 2 *There exists a unique solution of the problem* $P_{1\eta}$ *having the regularity* $u_\eta \in W^{1,\infty}(0,T,H_1)$, $\sigma_\eta \in W^{1,\infty}(0,T,\mathcal{H}_1)$. *Moreover,*

$$u_\eta(0) = u_0, \quad \sigma_\eta(0) = \sigma_0. \tag{4.4}$$

Proof. Let $t \in [0,T]$. Using (3.6), (3.7), (3.1), (2.7) and standard arguments of elliptic variational inequalities theory, we obtain the existence and the uniqueness of an element $u_\eta(t) = (u_\eta^1(t), u_\eta^2(t))$ such that

$$\begin{aligned} u_\eta(t) \in U_{ad}, \ \ & \langle \mathcal{E}\varepsilon(u_\eta(t)), \varepsilon(v) - \varepsilon(u_\eta(t))\rangle_{\mathcal{H}} + \langle z_\eta(t), \varepsilon(v) - \varepsilon(u_\eta(t))\rangle_{\mathcal{H}} \\ & \geqslant \langle f(t), v - u_\eta(t)\rangle_V \quad \forall v \in U_{ad}. \end{aligned} \tag{4.5}$$

Taking now $\sigma_\eta(t) = (\sigma_\eta^1(t), \sigma_\eta^2(t)) \in \mathcal{H}$ defined by (4.2), we obtain (4.3).

Let us remark that taking the functions $v = (u_\eta^1(t), u_\eta^2(t) \pm \psi^2)$ and $v = (u_\eta^1(t) \pm \psi^1, u_\eta^2(t))$ where $\psi^i \in \mathfrak{D}(\Omega^i)^N$, from (4.3) and (3.5) it follows that

$$\operatorname{Div} \sigma_\eta^i(t) + \varphi_1^i(t) = 0, \tag{4.6}$$

hence $\sigma_\eta(t) \in \mathcal{H}_1$. So, we proved the existence of a couple $(u_\eta(t), \sigma_\eta(t))$ in $H_1 \times \mathcal{H}_1$ solving the problem $P_{1\eta}$. The initial condition (4.4) follows from (3.9) and the uniqueness of the solution of the problem $P_{1\eta}$ for $t = 0$.

Let now $t_1, t_2 \in [0,T]$; using (4.5), (3.1) and (2.7) we obtain

$$|u_\eta(t_1) - u_\eta(t_2)|_{H_1} \leqslant C(|f(t_1) - f(t_2)|_V + |z_\eta(t_1) - z_\eta(t_2)|_{\mathcal{H}}) \tag{4.7}$$

and, using (4.2), (4.6) and (4.7), it follows that

$$|\sigma_\eta(t_1) - \sigma_\eta(t_2)|_{\mathcal{H}_1} \leqslant C(|f(t_1) - f(t_2)|_H + |z_\eta(t_1) - z_\eta(t_2)|_{\mathcal{H}}). \tag{4.8}$$

So, having in mind (3.6) and the time regularity $z_\eta \in W^{1,\infty}(0,T,\mathcal{H})$, from (4.7) and (4.8) we deduce that $u_\eta \in W^{1,\infty}(0,T,H_1)$, $\sigma_\eta \in W^{1,\infty}(0,T,\mathcal{H}_1)$.

Let us now remark that by the assumption (3.2) we may consider the operator $\Lambda : L^\infty(0,T,\mathcal{H}) \to L^\infty(0,T,\mathcal{H})$ defined by

$$\Lambda\eta(t) = G(\sigma_\eta(t), \varepsilon(u_\eta(t))) \qquad \forall \eta \in L^\infty(0,T,\mathcal{H}),\ t \in [0,T] \tag{4.9}$$

where, for every $\eta \in L^\infty(0,T,\mathcal{H})$, (u_η, σ_η) is the solution of the variational problem $P_{1\eta}$. Moreover, we have the following.

LEMMA 3 *The operator Λ has a unique fixed point $\eta^* \in L^\infty(0,T,\mathcal{H})$.*

Proof. Let $\eta_1, \eta_2 \in L^\infty(0,T,\mathcal{H})$ and $t \in [0,T]$. Using (4.9), (3.2), (4.2) and (4.3) we obtain

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \leq C|z_{\eta_1}(t) - z_{\eta_2}(t)|_{\mathcal{H}} \leq C\int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}}, \tag{4.10}$$

using (4.1). By recurrence, denoting by Λ^p the powers of the operator Λ, (4.10) implies that

$$|\Lambda^p\eta_1(t) - \Lambda^p\eta_2(t)|_{\mathcal{H}} \leq C^p \underbrace{\int_0^t \int_0^s \ldots \int_0^q}_{p \text{ integrals}} |\eta_1(r) - \eta_2(r)|_{\mathcal{H}} dr \ldots ds$$

for all $t \in [0,T]$ and $p \in \mathbb{N}$. Hence

$$|\Lambda^p\eta_1 - \Lambda^p\eta_2|_{\infty,\mathcal{H}} \leq \frac{C^p}{p!}|\eta_1 - \eta_2|_{\infty,\mathcal{H}} \qquad \forall p \in \mathbb{N} \tag{4.11}$$

and, since $\lim_p C^p/p! = 0$, (4.11) implies that for p large enough the operator Λ^p is a contraction in $L^\infty(0,T,\mathcal{H})$. Therefore, there exists a unique element $\eta^* \in L^\infty(0,T,\mathcal{H})$ such that $\Lambda^p\eta^* = \eta^*$. Moreover η^* is the unique fixed point of Λ.

Proof of Theorem 1. Let $\eta^* \in L^\infty(0,T,\mathcal{H})$ be the fixed point of Λ and let $u_{\eta^*} \in W^{1,\infty}(0,T,H_1)$, $\sigma_{\eta^*} \in W^{1,\infty}(0,T,\mathcal{H}_1)$ denote the solution of the problem $P_{1\eta^*}$. We shall prove that $(u_{\eta^*}, \sigma_{\eta^*})$ is the unique solution for the problem P_1. For this, we have to prove (3.14) and (3.16). The equality (3.16) follows from (4.2), (4.1) and (4.9) since

$$\dot\sigma_{\eta^*}(t) = \mathcal{E}\varepsilon(\dot u_{\eta^*}(t)) + \dot z_{\eta^*}(t),\ \ \dot z_{\eta^*}(t) = \eta^*(t) = \Lambda\eta^*(t) = G(\sigma_{\eta^*}(t), \varepsilon(u_{\eta^*}(t)))$$

a.e. on $(0,T)$. Let us also remark that (3.16) follows from (4.4) used for $\eta = \eta^*$.

Let now (u,σ) be another solution of problem P_1 having the regularity $u \in W^{1,\infty}(0,T,H_1)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H}_1)$. It is easy to verify that (u,σ) is a solution of the problem $P_{1\eta}$ for $\eta \in L^\infty(0,T,\mathcal{H})$ defined by

$$\eta(t) = G(\sigma(t), \varepsilon(u(t))) \qquad \forall t \in [0,T]. \tag{4.12}$$

Using Lemma 2 it follows that

$$u = u_\eta, \quad \sigma = \sigma_\eta. \tag{4.13}$$

Using now (4.9), (4.13) and (4.12) we deduce $\Lambda\eta = \eta$ and, by the uniqueness of the fixed point of Λ,

$$\eta = \eta^*. \tag{4.14}$$

The uniqueness part of Theorem 1 is now a consequence of (4.13) and (4.14).

5. The second existence and uniqueness result

The main result of this section now follows.

THEOREM 2 *Let* (3.1) *to* (3.4) *and* (3.9) *hold. Then there exists a unique solution of the problem* P_2 *having the regularity*

$$u \in W^{1,\infty}(0,T,V), \quad \sigma \in W^{1,\infty}(0,T,\mathcal{H}_1).$$

In order to prove Theorem 2, let us first remark that (3.20) is equivalent to the nonlinear evolution equation

$$\varepsilon(u(t)) + \partial\psi_{\Sigma_{ad}(t)}(\sigma(t)) \ni 0 \qquad \forall t \in [0,T] \tag{5.1}$$

where $\partial\psi_{\Sigma_{ad}(t)}$ denotes the subdifferential of the indicator function $\psi_{\Sigma_{ad}(t)}$. Since the set $\Sigma_{ad}(t)$ depends on time, we shall replace (5.1) by a nonlinear evolution equation associated to a fixed convex set. For this, let us introduce the following notation:

$$\Sigma_0 = \{\tau \in \mathcal{H} \mid \langle \tau, \varepsilon(v)\rangle_{\mathcal{H}} \geqslant 0 \ \ \forall v \in U_{ad}\} \tag{5.2}$$

$$\tilde{\sigma} = \varepsilon(f) \tag{5.3}$$

$$\overline{\sigma} = \sigma - \tilde{\sigma}, \quad \overline{\sigma}_0 = \sigma_0 - \tilde{\sigma}(0). \tag{5.4}$$

We have the following result.

LEMMA 4 *The couple of functions* (u,σ) *is a solution of problem* P_2 *having the regularity* $u \in W^{1,\infty}(0,T,V)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H}_1)$ *if and only if* $u \in W^{1,\infty}(0,T,V)$, $\overline{\sigma} \in W^{1,\infty}(0,T,\mathcal{H}_1)$ *and*

$$\varepsilon(\dot{u}) = \mathcal{E}^{-1}\dot{\overline{\sigma}} - \mathcal{E}^{-1}G(\overline{\sigma} + \tilde{\sigma}, \varepsilon(u)) + \mathcal{E}^{-1}\dot{\tilde{\sigma}} \quad \text{a.e. on } (0,T) \tag{5.5}$$

$$\overline{\sigma}(t) \in \Sigma_0, \quad \langle \tau - \overline{\sigma}(t), \varepsilon(u(t))\rangle_{\mathcal{H}} \geqslant 0 \quad \forall \tau \in \Sigma_0, \ t \in [0,T] \tag{5.6}$$

$$u(0) = u_0, \quad \overline{\sigma}(0) = \overline{\sigma}_0. \tag{5.7}$$

Proof. Let us remark that from (5.3) and (2.8) we have

$$\langle \tilde{\sigma}(t), \varepsilon(v)\rangle_{\mathcal{H}} = \langle f(t), v\rangle_V \quad \forall v \in U_{ad}, \ t \in [0,T]. \tag{5.8}$$

Moreover, using (3.5), $\mathrm{Div}\,\widetilde{\sigma}^i(t) = -\varphi_1^i(t) \in \mathcal{H}^i$ where $\widetilde{\sigma}(t) = (\widetilde{\sigma}^1(t), \widetilde{\sigma}^2(t))$ and from (3.6) and (3.3) we obtain

$$\widetilde{\sigma} \in W^{1,\infty}(0, T, \mathcal{H}_1). \tag{5.9}$$

Let us also remark that from (3.8), (5.2) and (5.8) we have

$$\tau \in \Sigma_{ad}(t) \iff \tau - \widetilde{\sigma}(t) \in \Sigma_0 \tag{5.10}$$

for all $t \in [0, T]$. Lemma 4 follows now from (5.4), (5.9) and (5.10).

In order to solve (5.5) to (5.7) we shall use again a fixed-point method. For this, let $\eta \in L^\infty(0, T, \mathcal{H})$ and let $z_\eta \in W^{1,\infty}(0, T, \mathcal{H})$ be the function defined by

$$z_\eta(t) = \int_0^t \eta(s)ds + \varepsilon(u_0) - \mathcal{E}^{-1}\sigma_0 \qquad \forall t \in [0, T]. \tag{5.11}$$

Let us consider the following variational problem.

PROBLEM $P_{2\eta}$ Find $u_\eta : [0, T] \to H_1$ and $\sigma_\eta : [0, T] \to \mathcal{H}_1$ such that

$$\varepsilon(u_\eta(t)) = \mathcal{E}^{-1}\sigma_\eta(t) + z_\eta(t) + \mathcal{E}^{-1}\widetilde{\sigma}(t) \tag{5.12}$$

$$\sigma_\eta(t) \in \Sigma_0, \quad (\tau - \sigma_\eta(t), \varepsilon(u_\eta(t)))_{\mathcal{H}} \geq 0 \qquad \forall \tau \in \Sigma_0 \tag{5.13}$$

for all $t \in [0, T]$.

We have the following result.

LEMMA 5 *There exists a unique solution of the problem $P_{2\eta}$ having the regularity $u_\eta \in W^{1,\infty}(0, T, V)$, $\sigma_\eta \in W^{1,\infty}(0, T, \mathcal{H}_1)$. Moreover,*

$$u_\eta(0) = u_0, \quad \sigma_\eta(0) = \overline{\sigma}_0. \tag{5.14}$$

Proof. Let $t \in [0, T]$ and let us remark that Σ_0 is a closed convex set in $\mathcal{H}$. Using (3.1), (5.9) and classical results of elliptic variational inequalities, we obtain the existence and uniqueness of a function $\sigma_\eta(t)$ such that

$$\sigma_\eta(t) \in \Sigma_0, \quad (\tau - \sigma_\eta(t), \mathcal{E}^{-1}\sigma_\eta(t) + z_\eta(t) + \mathcal{E}^{-1}\widetilde{\sigma}(t))_{\mathcal{H}} \geq 0 \qquad \forall \tau \in \Sigma_0. \tag{5.15}$$

Moreover, using (5.15) and (3.1) we obtain

$$|\sigma_\eta(t_1) - \sigma_\eta(t_2)|_{\mathcal{H}} \leq C\big(|z_\eta(t_1) - z_\eta(t_2)|_{\mathcal{H}} + |\widetilde{\sigma}(t_1) - \widetilde{\sigma}(t_2)|_{\mathcal{H}}\big) \tag{5.16}$$

for all $t_1, t_2 \in [0, T]$. Let us remark that since $\sigma_\eta(t) = (\sigma_\eta^1(t), \sigma_\eta^2(t)) \in \Sigma_0$, we deduce that

$$\mathrm{Div}\,\sigma_\eta^i(t) = 0 \qquad \forall t \in [0, T], \ i = 1, 2. \tag{5.17}$$

As $z_\eta \in W^{1,\infty}(0, T, \mathcal{H})$, using (5.9), (5.16) and (5.17), it follows that

$$\sigma_\eta \in W^{1,\infty}(0, T, \mathcal{H}_1).$$

Using again (5.15),

$$\langle \tau - \sigma_\eta(t), \varepsilon_\eta(t)\rangle_{\mathcal{H}} \geqslant 0 \qquad \forall \tau \in \Sigma_0, \tag{5.18}$$

where

$$\varepsilon_\eta(t) = \mathcal{E}^{-1}\sigma_\eta(t) + z_\eta(t) + \mathcal{E}^{-1}\widetilde{\sigma}(t) \qquad \forall t \in [0, T]. \tag{5.19}$$

Let us now introduce the space $\mathcal{V}^i$ $(i = 1, 2)$ defined by

$$\mathcal{V}^i = \{\tau \in \mathcal{H}^i \mid \operatorname{Div} \tau = 0 \ \text{ in } \Omega^i, \ \ \tau\nu = 0 \ \text{ on } \ \Gamma_2^i \cup \Gamma_3^i\},$$

where $\tau\nu$ denotes the normal trace of τ (see for instance (3 and 4)). For all $z \in \mathcal{V}^1 \times \mathcal{V}^2$, since $\sigma_\eta(t) \in \Sigma_0$, $\sigma_\eta(t) \pm z \in \Sigma_0$. Hence, taking $\tau = \sigma_\eta(t) \pm z$ in (5.18), we obtain $\langle z, \varepsilon_\eta(t)\rangle_{\mathcal{H}} = 0 \quad \forall z \in \mathcal{V}^1 \times \mathcal{V}^2$, $t \in [0, T]$ which implies that $\langle z^i, \varepsilon_\eta^i(t)\rangle_{\mathcal{H}} = 0 \ \forall z^i \in \mathcal{V}^i$, $t \in [0, T]$, $i = 1, 2$. Since the orthogonal complement of $\mathcal{V}^i$ in $\mathcal{H}^i$ is the space $\varepsilon(V^i)$, where V^i is defined in section 2 (see for instance (3, p. 34)), we obtain that there exists $u_\eta = (u_\eta^1, u_\eta^2)$ in $W^{1,\infty}(0, T, V)$ such that

$$\varepsilon_\eta(t) = \varepsilon(u_\eta(t)) \qquad \forall t \in [0, T]. \tag{5.20}$$

Using (5.18) to (5.20) it follows that (u_η, σ_η) is a solution of the problem $P_{2\eta}$. The uniqueness part in Lemma 5 follows from the uniqueness of the solution of (5.15) and from Korn's inequality (2.7). Finally, the initial condition (5.14) follows from (3.9) and the uniqueness of the solution of (5.12) to (5.13) for $t = 0$.

Lemma 5 and the assumption (3.2) allow us to consider the operator $\Lambda : L^\infty(0, T, \mathcal{H}) \longrightarrow L^\infty(0, T, \mathcal{H})$ defined by

$$\Lambda\eta = -\mathcal{E}^{-1}G(\sigma_\eta + \widetilde{\sigma}, \varepsilon(u_\eta)) \tag{5.21}$$

where, for every $\eta \in L^\infty(0, T, \mathcal{H})$, (u_η, σ_η) denotes the solution of the variational problem $P_{2\eta}$. We have the following.

LEMMA 6 *The operator Λ has a unique fixed point $\eta^* \in L^\infty(0, T, \mathcal{H})$.*

Proof. Let $\eta_1, \eta_2 \in L^\infty(0, T, \mathcal{H})$ and $t \in [0, T]$; using (5.21), (3.1), (3.2) and (5.11) to (5.13) we obtain

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \leqslant C \int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}}\, ds \qquad \forall t \in [0, T].$$

Lemma 6 follows now from the same arguments as in Lemma 3.

Proof of Theorem 2. Let $\eta^* \in L^\infty(0, T, \mathcal{H})$ be the fixed point of Λ and let $u_{\eta^*} \in W^{1,\infty}(0, T, V)$, $\sigma_{\eta^*} \in W^{1,\infty}(0, T, \mathcal{H}_1)$ be the functions given by Lemma 5 for $\eta = \eta^*$. It follows that $(u_{\eta^*}, \sigma_{\eta^*})$ is a solution of (5.5) to (5.7) and, using Lemma 4, we obtain the existence part in Theorem 2.

The uniqueness part follows from the uniqueness of the fixed point of the operator Λ defined by (5.21), using the same arguments as in the proof of

Theorem 1. It can also be proved directly from (3.14), (3.17), (3.21), (3.1) and (3.2), using a Gronwall-type inequality.

6. Some properties of the solution

We start this section with a result concerning the link between the solutions of the variational problems P_1 and P_2.

THEOREM 3 *Let* (3.1) *to* (3.4) *and* (3.9) *hold and, let* $u \in W^{1,\infty}(0, T, V)$, $\sigma \in W^{1,\infty}(0, T, \mathcal{H}_1)$. *Then* (u, σ) *is a solution of the variational problem* P_1 *if and only if* (u, σ) *is a solution of the variational problem* P_2.

Proof. Let $t \in [0, T]$ and let us suppose that (u, σ) is a solution of P_1. Taking $v = 2u(t)$ and $v = 0$ in (3.15) we obtain

$$\langle \sigma(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} = \langle f(t), u(t) \rangle_V. \tag{6.1}$$

Using now (3.15) and (6.1) it follows that $\sigma(t) \in \Sigma_{ad}(t)$. The inequality in (3.20) follows from (3.8) and (6.1). So, we conclude that (u, σ) is a solution of the problem P_2.

Conversely, let (u, σ) be a solution of problem P_2. We shall first prove that $u(t) \in U_{ad}$. Indeed, let us suppose in the sequel that $u(t) \notin U_{ad}$ and let us denote by $Pu(t)$ the projection of $u(t)$ on the closed convex set $U_{ad} \subset V$. We have

$$\langle Pu(t) - u(t), v \rangle_V \geqslant \langle Pu(t) - u(t), Pu(t) \rangle_V > \langle Pu(t) - u(t), u(t) \rangle_V$$

for all $v \in U_{ad}$. From these inequalities we obtain that there exists $\alpha \in \mathbb{R}$ such that

$$\langle Pu(t) - u(t), v \rangle_V > \alpha > \langle Pu(t) - u(t), u(t) \rangle_V \qquad \forall v \in U_{ad}. \tag{6.2}$$

Let $\tilde{\tau}$ be the function defined by

$$\tilde{\tau}(t) = \varepsilon(Pu(t) - u(t)) \in \mathcal{H}. \tag{6.3}$$

Using (6.2), (6.3) and (2.8) we deduce that

$$\langle \tilde{\tau}(t), \varepsilon(v) \rangle_{\mathcal{H}} > \alpha > \langle \tilde{\tau}(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} \qquad \forall v \in U_{ad} \tag{6.4}$$

and, taking $v = 0$ in (6.4), we obtain

$$\alpha < 0. \tag{6.5}$$

Let us now suppose that there exists $w \in U_{ad}$ such that

$$\langle \tilde{\tau}(t), \varepsilon(w) \rangle_{\mathcal{H}} < 0. \tag{6.6}$$

Using (6.4), since $\lambda w \in U_{ad}$ for $\lambda \geq 0$, it follows that

$$\lambda \langle \tilde{\tau}(t), \varepsilon(w) \rangle_{\mathcal{H}} > \alpha \qquad \forall \lambda \geqslant 0$$

and, passing to the limit when $\lambda \to +\infty$, from (6.6) we obtain $\alpha \leq -\infty$ which is in contradiction with $\alpha \in \mathbb{R}$. So, $\langle \tilde{\tau}(t), \varepsilon(w) \rangle_{\mathcal{H}} \geqslant 0$ for all $w \in U_{ad}$, which implies $\widetilde{\tau}(t) \in \Sigma_0$ (see (5.2)). Using (5.10) we obtain $\widetilde{\tau}(t) + \widetilde{\sigma}(t) \in \Sigma_{ad}(t)$ where $\widetilde{\sigma}$ is given by (5.3) and, from (3.20), (6.4) and (6.5) it follows that

$$0 > \langle \widetilde{\tau}(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} \geqslant \langle \sigma(t) - \widetilde{\sigma}(t), \varepsilon(u(t)) \rangle_{\mathcal{H}}$$

which implies that

$$\langle \sigma(t) - \widetilde{\sigma}(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} < 0. \tag{6.7}$$

Moreover, as $\sigma(t) - \widetilde{\sigma}(t) \in \Sigma_0$, from (5.4) and (5.6) for $\tau = 2(\sigma(t) - \widetilde{\sigma}(t))$ it follows that

$$\langle \sigma(t) - \widetilde{\sigma}(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} \geqslant 0. \tag{6.8}$$

We note that (6.7) and (6.8) are in contradiction. Therefore, $u(t) \in U_{ad}$. Using now (2.8) and (5.3) it follows that $\widetilde{\sigma}(t) \in \Sigma_{ad}(t)$. Hence, by (3.20) we obtain

$$\langle f(t), u(t) \rangle_V \geqslant \langle \sigma(t), \varepsilon(u(t)) \rangle_{\mathcal{H}}. \tag{6.9}$$

As $\sigma(t) \in \Sigma_{ad}(t)$ and $u(t) \in U_{ad}$, from (3.8) it follows that

$$\langle \sigma(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} \;\geqslant\; \langle f(t), u(t) \rangle_V. \tag{6.10}$$

So, from (6.9) and (6.10) we obtain

$$\langle \sigma(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} \;=\; \langle f(t), u(t) \rangle_V. \tag{6.11}$$

The inequality in (3.15) results now from (6.11) and (3.8). Hence (u, σ) is a solution of the problem P_1.

REMARK 1 Theorem 3 shows that the solutions (u, σ) obtained in Theorems 1 and 2 are the same. This result proves the equivalence between the variational problems P_1 and P_2.

In the sequel we consider two solutions of the problems P_1 and P_2 for two different data and we give an estimation of the difference of these solutions. More precisely, we have the following.

THEOREM 4 *Let* (3.1) *and* (3.2) *hold and let* (u_k, σ_k) *be the solution of the variational problem* P_1 *or* P_2 *for the data* f_k, u_{0k}, σ_{0k} *such that* (3.3), (3.4) *and* (3.9) *hold for* $k = 1, 2$. *Then, there exists* $C > 0$ *which depends on* Ω^i, Γ_1^i, $\mathscr{E}^i$, G^i *and* T $(i = 1, 2)$ *such that*

$$\begin{aligned} |u_1 - u_2|_{\infty, H_1} &+ |\sigma_1 - \sigma_2|_{\infty, \mathcal{H}_1} \\ &\leqslant C\Big(|u_{01} - u_{02}|_{H_1} + |\sigma_{01} - \sigma_{02}|_{\mathcal{H}_1} + |f_1 - f_2|_{\infty, V} \Big). \end{aligned} \tag{6.12}$$

Proof. Let $t \in [0, T]$ and $k \in \{1, 2\}$. Using (3.15) we obtain

$$u_k(t) \in U_{ad}, \;\; \langle \sigma_k(t), \varepsilon(v) - \varepsilon(u_k(t)) \rangle_{\mathcal{H}} \geqslant \langle f_k(t), v - u_k(t) \rangle_V \quad \forall v \in U_{ad} \tag{6.13}$$

and, from (3.14) and (3.16),

$$\sigma_k(t) = \mathscr{E}\varepsilon(u_k(t)) + \int_0^t G(\sigma_k(s), \varepsilon(u_k(s)))ds + \sigma_{0k} - \mathscr{E}\varepsilon(u_{0k}). \tag{6.14}$$

Using now (6.13), (6.14), (3.1), (3.2) and (2.7) we obtain

$$\begin{aligned} |u_1(t) - u_2(t)|_{H_1} + |\sigma_1(t) - \sigma_2(t)|_{\mathscr{H}_1} \leqslant C\Big[&|u_{01} - u_{02}|_{H_1} + |\sigma_{01} - \sigma_{02}|_{\mathscr{H}_1} + |f_1 - f_1|_{\infty, V} \\ &+ \int_0^t \left(|u_1(s) - u_2(s)|_{H_1} + |\sigma_1(s) - \sigma_2(s)|_{\mathscr{H}_1}\right) ds\Big]. \end{aligned} \tag{6.15}$$

The inequality (6.12) follows now from (6.15), using a Gronwall-type inequality.

REMARK 2 The estimate (6.12) gives a *continuous dependence result* for the solution of problems P_1 and P_2 with respect to the input data. Moreover, under the assumptions of Theorem 4, if $f_1 = f_2$,

$$|u_1 - u_2|_{\infty, H_1} + |\sigma_1 - \sigma_2|_{\infty, \mathscr{H}_1} \leqslant C\left(|u_{01} - u_{02}|_{H_1} + |\sigma_{01} - \sigma_{02}|_{\mathscr{H}_1}\right)$$

which represents a *finite-time stability result* for every solution of the variational problems P_1 and P_2.

We shall now study the dependence of the solution of the evolution problems P_1 and P_2 with respect to a perturbation of the constitutive functions G^1 and G^2. For this, let us suppose that (3.1) to (3.4) and (3.9) hold. For every $\mu^i \geqslant 0$ $(i = 1, 2)$, let $G^i_{\mu^i}$ be a perturbation of G^i which satisfies (3.2) with the Lipschitz constant $L^i_{\mu^i}$ and let $\mu = (\mu^1, \mu^2)$. Let us also denote

$$G_\mu(\sigma, \varepsilon) = \left(G^1_{\mu^1}(\sigma^1, \varepsilon^1), G^2_{\mu^2}(\sigma^2, \varepsilon^2)\right) \quad \forall \sigma = (\sigma^1, \sigma^2), \varepsilon = (\varepsilon^1, \varepsilon^2) \in \mathscr{H}.$$

We consider the following problems.

PROBLEM $P_{1\mu}$ Find the displacement field $u_\mu : [0, T] \to H_1$ and the stress field $\sigma_\mu : [0, T] \to \mathscr{H}_1$ such that

$$\dot{\sigma}_\mu(t) = \mathscr{E}\varepsilon(\dot{u}_\mu(t)) + G_\mu(\sigma_\mu(t), \varepsilon(u_\mu(t))) \quad \text{a.e. } t \in (0, T), \tag{6.16}$$

$$u_\mu(t) \in U_{ad}, \ \langle \sigma_\mu(t), \varepsilon(v) - \varepsilon(u_\mu(t))\rangle_{\mathscr{H}} \geqslant \langle f(t), v - u_\mu(t)\rangle_V \quad \forall v \in U_{ad}, \ t \in [0, T], \tag{6.17}$$

$$u_\mu(0) = u_0, \quad \sigma_\mu(0) = \sigma_0. \tag{6.18}$$

PROBLEM $P_{2\mu}$ Find the displacement field $u_\mu : [0, T] \to H_1$ and the stress field $\sigma_\mu : [0, T] \to \mathscr{H}_1$ such that (6.16) and (6.18) hold, and

$$\sigma_\mu(t) \in \Sigma_{ad}(t), \ \langle \tau - \sigma_\mu(t), \varepsilon(u_\mu(t))\rangle_{\mathscr{H}} \geqslant 0 \quad \forall \tau \in \Sigma_{ad}(t), \ t \in [0, T] \tag{6.19}$$

Using Theorems 1, 2 and 3 we obtain that the problems $P_{1\mu}$ and $P_{2\mu}$ have the same solution (u_μ, σ_μ) having the regularity $u_\mu \in W^{1,\infty}(0,T,H_1)$, $\sigma_\mu \in W^{1,\infty}(0,T,\mathcal{H}_1)$.

Let us now consider the following assumption.

$$\left.\begin{array}{l} \text{There exist } g^i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \text{ and } \beta^i \in \mathbb{R} \text{ such that} \\ \text{(a) } |G^i_{\mu^i}(\cdot,\sigma,\varepsilon) - G^i(\cdot,\sigma,\varepsilon)| \le g^i(\mu^i)\big(|\sigma| + |\varepsilon| + \beta^i\big) \\ \quad \forall \sigma, \varepsilon \in S_N, \quad \text{a.e. on } \Omega^i \\ \text{(b) } \lim\limits_{\mu^i \to 0} g^i(\mu^i) = 0. \\ \text{There exists } M^i > 0 \text{ such that } L^i_{\mu^i} \leqslant M^i. \end{array}\right\} \tag{6.20}$$

We have the following result.

THEOREM 5 *Let* (6.20) *hold for* $i = 1, 2$. *Then the solution* (u_μ, σ_μ) *of the problems* $P_{1\mu}$ *and* $P_{2\mu}$ *converges uniformly to the solution* (u, σ) *of the problems* P_1 *and* P_2:

$$u_\mu \longrightarrow u \quad \textit{in} \quad C(0,T,H_1), \quad \sigma_\mu \longrightarrow \sigma \quad \textit{in} \quad C(0,T,\mathcal{H}_1) \quad \textit{when} \quad \mu \to 0.$$

Proof. Let $t \in [0,T]$ and $\mu = (\mu^1, \mu^2) \in \mathbb{R}_+ \times \mathbb{R}_+$. From the proof of Theorem 1, (u, σ) is the solution of the elliptic problem $P_{1\eta}$ for $\eta = \eta^*$, where η^* is the fixed point of the operator $\Lambda : L^\infty(0,T,\mathcal{H}) \longrightarrow L^\infty(0,T,\mathcal{H})$ defined by (4.9) while (u_μ, σ_μ) is the solution of the same problem $P_{1\eta}$ for $\eta = \eta^*_\mu$, where η^*_μ is the fixed point of the operator $\Lambda_\mu : L^\infty(0,T,\mathcal{H}) \longrightarrow L^\infty(0,T,\mathcal{H})$ defined by

$$\Lambda_\mu \eta(t) = G_\mu(\sigma_\eta(t), \varepsilon(u_\eta(t))) \qquad \forall \eta \in L^\infty(0,T,\mathcal{H}), \ t \in [0,T]. \tag{6.21}$$

Using this remark, from (4.2), (4.3), (3.1) and (2.7) we obtain

$$\begin{aligned} |u_\mu(t) - u(t)|_{H_1} + |\sigma_\mu(t) - \sigma(t)|_{\mathcal{H}_1} &\le C|z_{\eta^*_\mu}(t) - z_{\eta^*}(t)|_{\mathcal{H}} \\ &\le C\int_0^t |\eta^*_\mu(s) - \eta^*(s)|_{\mathcal{H}}\, ds, \end{aligned} \tag{6.22}$$

using (4.1). Since $\eta^*_\mu = \Lambda_\mu \eta^*_\mu$ and $\eta^* = \Lambda\eta^*$, from (6.21) and (4.9) we deduce that

$$|\eta^*_\mu(s) - \eta^*(s)|_{\mathcal{H}} = \Big|G_\mu\big(\sigma_\mu(s), \varepsilon(u_\mu(s))\big) - G\big(\sigma(s), \varepsilon(u(s))\big)\Big|_{\mathcal{H}} \qquad \forall s \in [0,T].$$

Using (6.20) and (3.2), after some algebra, it follows that

$$\begin{aligned} |\eta^*_\mu(s) - \eta^*(s)|_{\mathcal{H}} &\le \max\Big(g^1(\mu^1), g^2(\mu^2)\Big)\Big(|u|_{\infty,H_1} + |\sigma|_{\infty,\mathcal{H}_1} + \beta^1 + \beta^2\Big) \\ &\quad + \max(M^1, M^2)\Big(|u_\mu(s) - u(s)|_{H_1} + |\sigma_\mu(s) - \sigma(s)|_{\mathcal{H}_1}\Big) \quad \forall s \in [0,T]. \end{aligned} \tag{6.23}$$

So, from (6.22) and (6.23) we obtain

$$|u_\mu(t) - u(t)|_{H_1} + |\sigma_\mu(t) - \sigma(t)|_{\mathcal{H}_1} \le C \max\Big(g^1(\mu^1), g^2(\mu^2)\Big) + \max(M^1, M^2) \int_0^t \Big(|u_\mu(s) - u(s)|_{H_1} + |\sigma_\mu(s) - \sigma(s)|_{\mathcal{H}_1}\Big)\, ds. \quad (6.24)$$

Theorem 5 follows now from (6.24) and (6.20).

REMARK 3 The mechanical interpretation of the previous convergence result is the following: under the assumptions of Theorem 5, for small viscosity coefficients μ_1 and μ_2, the study of the frictionless contact between two elastic–viscoplastic bodies having the constitutive laws given by

$$\dot\sigma = \mathscr{E}^i \varepsilon(\dot u) + G^i_{\mu_i}(\sigma, \varepsilon(u)) \qquad i = 1, 2$$

may be replaced by the study of the frictionless contact between two elastic–viscoplastic bodies having the constitutive law given by

$$\dot\sigma = \mathscr{E}^i \varepsilon(\dot u) + G^i(\sigma, \varepsilon(u)).$$

In particular, for small viscosity coefficients, the viscoplastic constitutive laws of the form $\dot\sigma = \mathscr{E}\varepsilon(\dot u) + \mu G(\sigma, \varepsilon(u))$ as well as the viscoelastic laws of the form $\dot\sigma = \mathscr{E}\varepsilon(\dot u) + \mu(\sigma - F(\varepsilon(u)))$ may be replaced by the elastic law $\sigma = \mathscr{E}\varepsilon(u)$, in the study of frictionless contact problems.

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A Frictionless Contact Problem for Elastic-Viscoplastic Materials with Internal State Variables

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Descriptif

Le but de ce travail est l'analyse variationnelle du problème quasistatique de contact unilatéral sans frottement d'un matériau ayant une loi de comportement viscoplastique à variable interne d'état avec une fondation rigide.

On considère un milieu continu viscoplastique occupant un domaine Ω de $\mathbb{R}^N$ ($N = 1, 2, 3$) et dont la frontière Γ, supposée suffisamment régulière, est divisée en trois parties disjointes Γ_1, Γ_2 et Γ_3. On suppose que, pendant l'intervalle de temps $[0, T]$, le champ des déplacements s'annulle sur Γ_1, que des forces surfaciques g s'appliquent sur Γ_2 et que des forces volumiques f agissent dans Ω. On suppose aussi que le matériau peut rentrer en contact sans frottement avec une fondation rigide le long de la partie Γ_3 de sa frontière. Les conditions de contact considérées sont celles de Signorini. Le problème quasistatique de contact étudié ici est le suivant :

Problème P : Trouver le champ des déplacements $u : \Omega \times [0, T] \longrightarrow \mathbb{R}^N$, le champ des contraintes $\sigma : \Omega \times [0, T] \longrightarrow \mathbb{R}_s^{N \times N}$ et la variable interne d'état $\kappa : \Omega \times [0, T] \longrightarrow \mathbb{R}^M$ tels que

$$\begin{aligned}
\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u), \kappa) \quad & \text{dans} \quad \Omega \times (0, T),\\
\dot{\kappa} = \varphi(\sigma, \varepsilon(u), \kappa) \quad & \text{dans} \quad \Omega \times (0, T),\\
Div\, \sigma + f = 0 \quad & \text{dans} \quad \Omega \times (0, T),\\
u = 0 \quad & \text{sur} \quad \Gamma_1 \times (0, T),\\
\sigma\nu = g \quad & \text{sur} \quad \Gamma_2 \times (0, T),\\
u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\tau = 0, \quad \sigma_\nu u_\nu = 0 \quad & \text{sur} \quad \Gamma_3 \times (0, T),\\
u(0) = u_0, \quad \sigma(0) = \sigma_0, \quad \kappa(0) = \kappa_0 \quad & \text{dans} \quad \Omega.
\end{aligned}$$

On note par $\mathbb{R}_s^{N \times N}$ l'espace des tenseurs symétriques du second ordre sur $\mathbb{R}^N$ et par $\varepsilon(u)$ le tenseur des petites déformations linéarisé. Le point au dessus d'une quantité désigne sa dérivée temporelle, $Div\,\sigma$ désigne la divergence de la fonction tensorielle σ, le vecteur ν est la normale unitaire sortante à Ω, $\sigma\nu$ est le vecteur des contraintes de Cauchy, et u_ν, σ_ν et σ_τ représentent respectivement le déplacement normal, les contraintes normales et tangentielles.

On établit pour le problème P deux formulations variationnelles P_1 et P_2. La formulaion faible P_1 admet comme inconnues le couple (déplacements, variable interne d'état) alors que la formulation faible P_2 admet comme inconnues le couple (contraintes, variable interne d'état). L'existence et l'unicité de la solution pour chacun des problèmes sont établies ainsi que l'équivalence entre ces deux formulations faibles. Un problème pénalisé est ensuite introduit et l'existence et l'unicité de sa solution ainsi qu'un résultat de convergence sont finalement prouvés.

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A Frictionless Contact Problem for Elastic-Viscoplastic Materials with Internal State Variables

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Abstract—This paper deals with an initial and boundary value problem describing the quasistatic evolution of a rate-type viscoplastic material with internal state variables which is in frictionless contact. Two variational formulations of the problem are proposed, and existence and uniqueness results established. The equivalence of the variational formulation is studied and a strong convergence result involving penalized problems is proved.

Keywords—Frictionless contact, Viscoplastic material, Internal state variable, Quasistatic process.

1. INTRODUCTION

The subject of this work is the study of the contact between a body and a rigid frictionless foundation. The contact boundary conditions considered here are the well-known Signorini conditions which have been studied in the case of elastic, elastic-plastic, viscoelastic, or viscoplastic bodies (see, for instance, [1–10]).

We consider here materials having an elastic-viscoplastic constitutive law of the form

$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u), \kappa), \tag{1.1}$$

$$\dot{\kappa} = \varphi(\sigma, \varepsilon(u), \kappa), \tag{1.2}$$

where $\mathcal{E}$, G, and φ are constitutive functions. In this paper, we consider the case of small deformations, we denote by $\varepsilon = (\varepsilon_{ij})$ the small strain tensor and by $\sigma = (\sigma_{ij})$ the stress tensor. The function κ may be interpreted as an internal state variable. A dot above a variable represents the time derivative.

Models of the form (1.1),(1.2) are used to describe the behaviour of real materials like rubbers, metals, rocks, etc., (see, for example, [11]). Existence and uniqueness results for processes involving models of this form with classical displacement-traction boundary conditions were obtained in [12] using a Cauchy-Lipschitz method and in [13] using a fixed-point technique.

The aim of this paper is to investigate a quasistatic problem for the elastic-viscoplastic models (1.1),(1.2) involving unilateral contact conditions. It is structured as follows. In Section 2, the mechanical Problem P is stated and some functional preliminaries are presented. We establish in Section 3 two variational formulations, P_1 and P_2, for the model. Both P_1 and P_2 involve the

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX coupling between the constitutive law (1.1) and (1.2) and a variational inequality including the equilibrium equation and the boundary conditions. The unknowns in Problems P_1 and P_2 are the displacement field u, the stress field σ, and the internal state variable κ. For Problem P_1, we prove in Section 4 an existence and uniqueness result. In this case, the main unknown is the displacement field u and the existence of the solution results from classical elliptic variational inequalities arguments followed by a fixed-point method. In Section 5, we prove a similar existence and uniqueness result for Problem P_2. In this case, the main unknown is the stress field σ. The last section deals with the study of some properties of the solution. In fact, we prove an equivalence result between Problems P_1 and P_2, and we consider a penalized problem, governed by a parameter $h > 0$, for which we prove the existence and uniqueness of the solution as well as a convergence result when $h \to \infty$.

2. PROBLEM STATEMENT-PRELIMINARIES

Let us consider an elastic-viscoplastic body whose material particles occupy a bounded domain $\Omega \subset \mathbf{R}^N$ ($N = 1,2,3$) and whose boundary Γ, assumed to be sufficiently smooth is partitioned into three disjoint measurable parts Γ_1, Γ_2, and Γ_3. Let $\operatorname{meas}\Gamma_1 > 0$, and let $T > 0$ be a time interval. We assume that the displacement field vanishes on Γ_1, that surface tractions g act on Γ_2 and that body forces f act in Ω. We also suppose that the body rests on a rigid foundation on the part Γ_3 of the boundary and that this contact is frictionless, i.e., the tangential movements are completely free. Finally, we assume the case of quasistatic processes and we use (1.1),(1.2) as the constitutive law. With these assumptions, the mechanical problem we study may be formulated as follows.

PROBLEM P. Find the displacement field $u : \Omega \times [0,T] \longrightarrow \mathbf{R}^N$, the stress field $\sigma : \Omega \times [0,T] \longrightarrow S_N$ and the internal state variable $\kappa : \Omega \times [0,T] \longrightarrow \mathbf{R}^M$ such that

$$\dot\sigma = \mathcal{E}\varepsilon(\dot u) + G(\sigma, \varepsilon(u), \kappa), \qquad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$\dot\kappa = \varphi(\sigma, \varepsilon(u), \kappa), \qquad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$\operatorname{Div}\sigma + f = 0, \qquad \text{in } \Omega \times (0,T), \tag{2.3}$$

$$u = 0, \qquad \text{on } \Gamma_1 \times (0,T), \tag{2.4}$$

$$\sigma\nu = g, \qquad \text{on } \Gamma_2 \times (0,T), \tag{2.5}$$

$$u_\nu \le 0, \quad \sigma_\nu \le 0, \quad \sigma_\tau = 0, \quad \sigma_\nu, u_\nu = 0, \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.6}$$

$$u(0) = u_0, \quad \sigma(0) = \sigma_0, \quad \kappa(0) = \kappa_0, \qquad \text{in } \Omega, \tag{2.7}$$

where S_N denotes the set of second-order symmetric tensors on $\mathbf{R}^N$ and $M \in \mathbf{N}$. In (2.1)–(2.7), $\operatorname{Div}\sigma$ represents the divergence of the tensor-valued function σ, $\nu = (\nu_i)$ is the unit outward normal to Ω, $\sigma\nu$ is the stress vector, u_ν, σ_ν, and σ_τ are given by

$$u_\nu = u_i\nu_i, \quad \sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = (\sigma_{\tau i}), \quad \sigma_{\tau i} = \sigma_{ij}\nu_j - \sigma_\nu\nu_i, \qquad \left(i = \overline{1,N}\right),$$

and finally u_0, σ_0, and κ_0 are the initial data.

We denote in the sequel by $\cdot$ the inner product on the spaces $\mathbf{R}^N$, $\mathbf{R}^M$, and S_N and by $|\cdot|$ the Euclidean norms on these spaces. The following notations are also used:

$$H = \left\{v = (v_i) \mid v_i \in L^2(\Omega),\ i = \overline{1,N}\right\} = L^2(\Omega)^N,$$

$$H_1 = \left\{v = (v_i) \mid v_i \in H^1(\Omega),\ i = \overline{1,N}\right\} = H^1(\Omega)^N,$$

$$\mathcal{H} = \left\{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ i,j = \overline{1,N}\right\} = L^2(\Omega)_s^{N\times N},$$

$$\mathcal{H}_1 = \{\tau \in \mathcal{H} \mid \operatorname{Div}\tau \in H\},$$

$$Y = \left\{\kappa = (\kappa_i) \mid \kappa_i \in L^2(\Omega),\ i = \overline{1,M}\right\} = L^2(\Omega)^M.$$

The spaces H, H_1, $\mathcal{H}$, $\mathcal{H}_1$, and Y are real Hilbert spaces endowed with the canonical inner products denoted by $\langle\cdot,\cdot\rangle_X$, where X is one of these spaces.

Let $H_\Gamma = H^{1/2}(\Gamma)^N$, and let $\gamma : H_1 \longrightarrow H_\Gamma$ be the trace map. We denote by V the closed subspace of H_1 given by

$$V = \{u \in H_1 \mid \gamma u = 0 \text{ on } \Gamma_1\}. \tag{2.8}$$

The deformation operator $\varepsilon : H_1 \longrightarrow \mathcal{H}$ defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \qquad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

is a linear and continuous operator. Moreover, since $\operatorname{meas}\Gamma_1 > 0$, Korn's inequality holds (see, for instance, [3, p. 79]):

$$|\varepsilon(v)|_{\mathcal{H}} \geq C|v|_{H_1}, \qquad \forall v \in V, \tag{2.9}$$

where C is a strictly positive constant which depends only on Ω and Γ_1 (everywhere in this paper C will represent strictly positive generic constants which may depend on Ω, Γ_1, Γ_2, Γ_3, $\mathcal{E}$, G, T, and do not depend on time or on input data).

We now endow V with the inner product $\langle\cdot,\cdot\rangle_V$ defined by

$$\langle v, w\rangle_V = \langle\varepsilon(v), \varepsilon(w)\rangle_{\mathcal{H}}, \qquad \forall v, w \in V, \tag{2.10}$$

and we denote by $|\cdot|_V$ the associated norm. So, from (2.9) we deduce that $|\cdot|_V$ and $|\cdot|_{H_1}$ are equivalent norms on V. Therefore, V endowed with the inner product defined by (2.10) is a real Hilbert space.

Let $H'_\Gamma = H^{-1/2}(\Gamma)^N$ be the strong dual of the space H_Γ, and let $\langle\cdot,\cdot\rangle$ denote the duality between H'_Γ and H_Γ. If $\tau \in \mathcal{H}_1$, there exists an element $\gamma_\nu\tau \in H'_\Gamma$ such that Green's formula holds:

$$\langle\gamma_\nu\tau, \gamma v\rangle = \langle\tau, \varepsilon(v)\rangle_{\mathcal{H}} + \langle \operatorname{Div}\tau, v\rangle_H, \qquad \forall v \in H_1. \tag{2.11}$$

Moreover, if τ is a regular (say C^1) function, then

$$\langle\gamma_\nu\tau, \gamma v\rangle = \int_\Gamma \tau\nu\cdot v\, da, \qquad \forall v \in H_1. \tag{2.12}$$

For simplicity, we shall use the notation $\tau\nu$ instead of $\gamma_\nu\tau$, and we say that $\tau\nu = 0$ on Γ_1 if $\langle\gamma_\nu\tau, \gamma v\rangle = 0$, $\forall v \in V$.

We also notice that for every real Hilbert spaces X_1 and X_2, we use the notation $X_1 \times X_2$ for the canonical spaces product.

Finally, for every real Hilbert space X, we denote by $|\cdot|_X$ the norm on X and by $|\cdot|_{\infty,X}$ the norm on the space $L^\infty(0,T,X)$.

3. VARIATIONAL FORMULATION

In this section, we present two variational formulations for the mechanical Problem P. We make the following assumptions on the problem data.

$\mathcal{E} : \Omega \times S_N \longrightarrow S_N$ is a symmetric and positive definite tensor, i.e.,

(a) $\mathcal{E}_{khlm} \in L^\infty(\Omega)$, $\forall k, h, l, m = 1, \cdot, N$;

(b) $\mathcal{E}\sigma\cdot\tau = \sigma\cdot\mathcal{E}\tau$, $\forall\sigma$, $\tau \in S_N$, a.e. in Ω; (3.1)

(c) there exists $\alpha > 0$ such that $\mathcal{E}\sigma\cdot\sigma \geq \alpha|\sigma|^2$, $\forall\sigma \in S_N$, a.e. in Ω.

$G : \Omega \times S_N \times S_N \times \mathbb{R}^M \longrightarrow S_N$ and

(a) there exists $L > 0$ such that $|G(\cdot,\sigma_1,\varepsilon_1,\kappa_1) - G(\cdot,\sigma_2,\varepsilon_2,\kappa_2)| \leq L(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\kappa_1 - \kappa_2|)$, $\forall\sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S_N$, $\kappa_1, \kappa_2 \in \mathbb{R}^M$, a.e. in Ω;

(b) $x \longmapsto G(x,\sigma,\varepsilon,\kappa)$ is a measurable function with respect to the Lebesgue measure on Ω, $\forall\sigma$, $\varepsilon \in S_N$, $\kappa \in \mathbb{R}^M$; (3.2)

(c) $x \longmapsto G(x,0,0,0) \in \mathcal{H}$.

$\varphi : \Omega \times S_N \times S_N \times \mathbf{R}^M \longrightarrow \mathbf{R}^M$ and

(a) there exists $L' > 0$ such that $|\varphi(\cdot, \sigma_1, \varepsilon_1, \kappa_1) - \varphi(\cdot, \sigma_2, \varepsilon_2, \kappa_2)| \le L'(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\kappa_1 - \kappa_2|)$, $\forall\, \sigma_1,\ \sigma_2, \varepsilon_1, \varepsilon_2 \in S_N,\ \kappa_1, \kappa_2 \in \mathbf{R}^M$, a.e. in Ω;

(b) $x \longmapsto \varphi(x, \sigma, \varepsilon, \kappa)$ is a measurable function with respect to the Lebesgue measure on Ω, $\forall \sigma,\ \varepsilon \in S_N,\ \kappa \in \mathbf{R}^M$; (3.3)

(c) $x \longmapsto \varphi(x, 0, 0, 0) \in Y$;

$$f \in W^{1,\infty}(0, T, H), \tag{3.4}$$

$$g \in W^{1,\infty}\left(0, T, L^2(\Gamma_2)^N\right). \tag{3.5}$$

The assumptions (3.1)–(3.3) allow us to consider three operators denoted again by $\mathcal{E}$, G, and φ such that $\mathcal{E} : \mathcal{H} \longrightarrow \mathcal{H}$, $G : \mathcal{H} \times \mathcal{H} \times Y \longrightarrow \mathcal{H}$, $\varphi : \mathcal{H} \times \mathcal{H} \times Y \longrightarrow Y$, and

$$\begin{aligned} (\mathcal{E}\sigma)(\cdot) &= \mathcal{E}(\cdot, \sigma(\cdot)) = (\mathcal{E}_{ijkh}(\cdot)\sigma_{kh}(\cdot)), && \forall \sigma \in \mathcal{H} \text{ a.e. in } \Omega, \\ G(\sigma, \varepsilon, \kappa)(\cdot) &= G(\cdot, \sigma(\cdot), \varepsilon(\cdot), \kappa(\cdot)), && \forall \sigma, \varepsilon \in \mathcal{H},\ \kappa \in Y \text{ a.e. in } \Omega, \\ \varphi(\sigma, \varepsilon, \kappa)(\cdot) &= \varphi(\cdot, \sigma(\cdot), \varepsilon(\cdot), \kappa(\cdot)), && \forall \sigma, \varepsilon \in \mathcal{H},\ \kappa \in Y \text{ a.e. in } \Omega. \end{aligned}$$

We also denote by $F(t)$ the element of V given by

$$\langle F(t), v \rangle_V = \langle f(t), v \rangle_H + \langle g(t), \gamma v \rangle_{L^2(\Gamma_2)^N}, \qquad \forall v \in V,\ t \in [0, T]. \tag{3.6}$$

So, using (3.4)–(3.6), it follows that

$$F \in W^{1,\infty}(0, T, V). \tag{3.7}$$

Finally, let U_{ad} and $\Sigma_{ad}(t)$ be the sets given by

$$U_{ad} = \{v \in H_1 \mid v = 0 \text{ on } \Gamma_1,\ v_\nu \le 0 \text{ on } \Gamma_3\}, \tag{3.8}$$

$$\Sigma_{ad}(t) = \{\tau \in \mathcal{H} \mid \langle \tau, \varepsilon(v) \rangle_{\mathcal{H}} \ge \langle F(t), v \rangle_V\, \forall v \in U_{ad}\}, \qquad \forall t \in [0, T], \tag{3.9}$$

and let us suppose that

$$u_0 \in U_{ad}, \quad \sigma_0 \in \Sigma_{ad}(0), \quad \kappa_0 \in Y, \qquad \langle \sigma_0, \varepsilon(u_0) \rangle_{\mathcal{H}} = \langle F(0), u_0 \rangle_V. \tag{3.10}$$

Let us remark that assumption (3.10) (which involve regularity conditions on the initial data u_0, σ_0, and κ_0 as well as a compatibility condition between u_0, κ_0, f, and g) are satisfied if $u_0 \in H_1$, $\sigma_0 \in \mathcal{H}$, $\kappa_0 \in Y$, and if u_0 and σ_0 verify (2.3)–(2.6) for $t = 0$.

LEMMA 3.1. *If (u, σ, κ) is a regular solution of the mechanical Problem P, then*

$$u(t) \in U_{ad}, \qquad \langle \sigma(t), \varepsilon(v) - \varepsilon(u(t)) \rangle_{\mathcal{H}} \ge \langle F(t), v - u(t) \rangle_V, \qquad \forall v \in U_{ad}, \tag{3.11}$$

$$\sigma(t) \in \Sigma_{ad}(t), \qquad \langle \tau - \sigma(t), \varepsilon(u(t)) \rangle_{\mathcal{H}} \ge 0, \qquad \forall \tau \in \Sigma_{ad}(t), \tag{3.12}$$

for all $t \in [0, T]$.

PROOF. Let $v \in U_{ad}$ and $t \in [0, T]$. Using (2.3), (2.11), and (2.12), we have

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(u(t)) \rangle_{\mathcal{H}} = \langle f(t), v - u(t) \rangle_H + \int_\Gamma \sigma(t)\nu \cdot (v - u(t))\, da,$$

and from (2.4), (2.5), and (3.6), we obtain that

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(u(t)) \rangle_{\mathcal{H}} = \langle F(t), v - u(t) \rangle_V + \int_{\Gamma_3} \sigma(t)\nu \cdot (v - u(t))\, da.$$

Using now (2.6), the previous equality leads to

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(u(t))\rangle_{\mathcal{H}} = \langle F(t), v - u(t)\rangle_V + \int_{\Gamma_3} \sigma_\nu(t) v_\nu \, da. \tag{3.13}$$

The inequality in (3.11) follows from (3.13), (3.8), and (2.6). Moreover, (2.4), (2.6), and (3.8) imply that $u(t) \in U_{\mathrm{ad}}$.

Taking now $v = 2u(t)$ and $v = 0$ in (3.11), we obtain that

$$\langle \sigma(t), \varepsilon(u(t))\rangle_{\mathcal{H}} = \langle F(t), u(t)\rangle_V, \tag{3.14}$$

and using (3.14), (3.11), and (3.9), it follows that $\sigma(t) \in \Sigma_{\mathrm{ad}}(t)$. The inequality in (3.12) is now a consequence of (3.9) and (3.14).

The previous lemma leads us to consider two weak formulations of Problem P.

PROBLEM P_1. Find the displacement field $u : [0,T] \longrightarrow H_1$, the stress field $\sigma : [0,T] \longrightarrow \mathcal{H}_1$, and the internal state variable $\kappa : [0,T] \longrightarrow Y$ such that

$$\dot\sigma(t) = \mathcal{E}\varepsilon(\dot u(t)) + G(\sigma(t), \varepsilon(u(t)), \kappa(t)), \qquad \text{a.e. } t \in (0,T), \tag{3.15}$$

$$\dot\kappa(t) = \varphi(\sigma(t), \varepsilon(u(t)), \kappa(t)), \qquad \text{a.e. } t \in (0,T), \tag{3.16}$$

$$u(t) \in U_{\mathrm{ad}}, \quad \langle \sigma(t), \varepsilon(v) - \varepsilon(u(t))\rangle_{\mathcal{H}} \geq \langle F(t), v - u(t)\rangle_V, \quad \forall v \in U_{\mathrm{ad}}, \quad t \in [0,T], \tag{3.17}$$

$$u(0) = u_0, \qquad \sigma(0) = \sigma_0, \qquad \kappa(0) = \kappa_0. \tag{3.18}$$

PROBLEM P_2. Find the displacement field $u : [0,T] \longrightarrow H_1$, the stress field $\sigma : [0,T] \longrightarrow \mathcal{H}_1$, and the internal state variable $\kappa : [0,T] \longrightarrow Y$ such that

$$\dot\sigma(t) = \mathcal{E}\varepsilon(\dot u(t)) + G(\sigma(t), \varepsilon(u(t)), \kappa(t)), \qquad \text{a.e. } t \in (0,T), \tag{3.19}$$

$$\dot\kappa(t) = \varphi(\sigma(t), \varepsilon(u(t)), \kappa(t)), \qquad \text{a.e. } t \in (0,T), \tag{3.20}$$

$$\sigma(t) \in \Sigma_{\mathrm{ad}}(t), \quad \langle \tau - \sigma(t), \varepsilon(u(t))\rangle_{\mathcal{H}} \geq 0, \qquad \forall \tau \in \Sigma_{\mathrm{ad}}(t), \quad t \in [0,T], \tag{3.21}$$

$$u(0) = u_0, \qquad \sigma(0) = \sigma_0, \qquad \kappa(0) = \kappa_0. \tag{3.22}$$

Let us remark that Problems P_1 and P_2 are formally equivalent to the mechanical Problem P. Indeed, if (u, σ, κ) represents a regular solution of the variational problems P_1 or P_2, using the arguments of [14], it follows that (u, σ, κ) is a solution of Problem P.

Under the assumptions (3.1)–(3.5) and (3.10), in the next two sections we give existence and uniqueness results for the variational Problems P_1 and P_2.

4. FIRST EXISTENCE AND UNIQUENESS RESULT

The main result of this section is the following.

THEOREM 4.1. *Let (3.1)–(3.5) and (3.10) hold. Then there exists a unique solution of Problem P_1 having the regularity*

$$u \in W^{1,\infty}(0,T,H_1), \qquad \sigma \in W^{1,\infty}(0,T,\mathcal{H}_1), \qquad \kappa \in W^{1,\infty}(0,T,Y).$$

In order to prove this theorem, we shall use a fixed-point method. For this, we suppose in the sequel that the assumptions of Theorem 4.1 are satisfied, and for each $\eta = (\eta^1, \eta^2) \in L^\infty(0,T,\mathcal{H} \times Y)$ we introduce the function $z_\eta = (z_\eta^1, z_\eta^2) \in W^{1,\infty}(0,T,\mathcal{H} \times Y)$ defined by

$$z_\eta(t) = \int_0^t \eta(s)\, ds + z_0, \qquad \forall t \in [0,T], \tag{4.1}$$

where

$$z_0 = (\sigma_0 - \mathcal{E}\varepsilon(u_0),\, \kappa_0). \tag{4.2}$$

We consider now the following elliptic problem.

Problem $P_{1\eta}$. Find $u_\eta : [0,T] \longrightarrow H_1$ and $\sigma_\eta : [0,T] \longrightarrow \mathcal{H}_1$, such that

$$\sigma_\eta(t) = \mathcal{E}\varepsilon\left(u_\eta(t)\right) + z^1_\eta(t), \tag{4.3}$$

$$u(t) \in U_{\text{ad}}, \quad \langle \sigma_\eta(t), \varepsilon(v) - \varepsilon\left(u_\eta(t)\right)\rangle_{\mathcal{H}} \geq \langle F(t), v - u_\eta(t)\rangle_V, \qquad \forall\, v \in U_{\text{ad}}, \tag{4.4}$$

for all $t \in [0,T]$.

Lemma 4.2. *There exists a unique solution of the variational Problem $P_{1\eta}$ having the regularity $u_\eta \in W^{1,\infty}(0,T,H_1)$, $\sigma_\eta \in W^{1,\infty}(0,T,\mathcal{H}_1)$. Moreover,*

$$u_\eta(0) = u_0, \qquad \sigma_\eta(0) = \sigma_0. \tag{4.5}$$

Proof. Let $t \in [0,T]$. Using (3.7), (3.8), (3.1), (2.9), and standard arguments of elliptic variational inequalities theory, we obtain the existence and the uniqueness of an element $u_\eta(t)$ such that

$$\begin{gathered} u_\eta(t) \in U_{\text{ad}}, \\ \langle \mathcal{E}\varepsilon\left(u_\eta(t)\right), \varepsilon(v) - \varepsilon\left(u_\eta(t)\right)\rangle_{\mathcal{H}} + \left\langle z^1_\eta(t), \varepsilon(v) - \varepsilon\left(u_\eta(t)\right)\right\rangle_{\mathcal{H}} \geq \langle F(t), v - u_\eta(t)\rangle_V, \\ \forall\, v \in U_{\text{ad}}. \end{gathered} \tag{4.6}$$

Taking now $\sigma_\eta(t)$ defined by (4.3), we deduce (4.4).

Let us remark that for $v = u_\eta(t) \pm \psi$, where $\psi \in \mathcal{D}(\Omega)^N$ from (4.4) and (3.6), it follows that

$$\operatorname{Div} \sigma_\eta(t) + f(t) = 0, \tag{4.7}$$

hence, $\sigma_\eta(t) \in \mathcal{H}_1$. Therefore, the existence and uniqueness of $(u_\eta(t), \sigma_\eta(t)) \in H_1 \times \mathcal{H}_1$, solution of Problem $P_{1\eta}$ is established. The initial conditions (4.5) follows from (3.10), (4.1), (4.2), and the uniqueness of the solution of Problem $P_{1\eta}$ for $t = 0$.

Let now $t_1, t_2 \in [0,T]$. Using (4.6), (3.1), and (2.9), we obtain that

$$|u_\eta(t_1) - u_\eta(t_2)|_{H_1} \leq C\left(|F(t_1) - F(t_2)|_V + \left|z^1_\eta(t_1) - z^1_\eta(t_2)\right|_{\mathcal{H}}\right), \tag{4.8}$$

and from (4.3), (4.7), and (4.8), it results that

$$|\sigma_\eta(t_1) - \sigma_\eta(t_2)|_{\mathcal{H}_1} \leq C\left(|F(t_1) - F(t_2)|_V + \left|z^1_\eta(t_1) - z^1_\eta(t_2)\right|_{\mathcal{H}}\right). \tag{4.9}$$

So, considering (3.7) and the time regularity $z^1_\eta \in W^{1,\infty}(0,T,\mathcal{H})$, from (4.8) and (4.9), we deduce that $u_\eta \in W^{1,\infty}(0,T,H_1)$, $\sigma_\eta \in W^{1,\infty}(0,T,\mathcal{H}_1)$.

We denote now by $\kappa_\eta \in W^{1,\infty}(0,T,Y)$ the function defined by

$$\kappa_\eta = z^2_\eta. \tag{4.10}$$

Using the assumptions (3.2), (3.3), (4.1), (4.2), and (4.10), we may consider the operator $\Lambda : L^\infty(0,T,\mathcal{H} \times Y) \longrightarrow L^\infty(0,T,\mathcal{H} \times Y)$ defined by

$$\Lambda\eta = \left(G\left(\sigma_\eta, \varepsilon(u_\eta), \kappa_\eta\right),\, \varphi\left(\sigma_\eta, \varepsilon(u_\eta), \kappa_\eta\right)\right), \qquad \forall\, \eta \in L^\infty(0,T,\mathcal{H} \times Y), \tag{4.11}$$

where (u_η, σ_η) is the solution of the variational Problem $P_{1\eta}$.

LEMMA 4.3. *The operator* Λ *has a unique fixed-point* $\eta^* \in L^\infty(0,T,\mathcal{H}\times Y)$.

PROOF. Let $\eta_1 = (\eta_1^1, \eta_1^2)$ and $\eta_2 = (\eta_2^1, \eta_2^2) \in L^\infty(0,T,\mathcal{H}\times Y)$, and let $t \in [0,T]$. Using (4.3), (4.4), (3.2), (3.3), and (4.10), we obtain that

$$|u_{\eta_1} - u_{\eta_2}|_{H_1} + |\sigma_{\eta_1} - \sigma_{\eta_2}|_{\mathcal{H}} + |\kappa_{\eta_1} - \kappa_{\eta_2}|_Y \le |z_{\eta_1} - z_{\eta_2}|_{\mathcal{H}\times Y}.$$

From (4.11), (3.2), (3.3), (4.1), and the last inequality, it results that

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}\times Y} \le C\int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}\times Y}. \tag{4.12}$$

Denoting now by Λ^p the powers of the operator Λ, (4.12) implies by recurrence that

$$|\Lambda^p\eta_1(t) - \Lambda^p\eta_2(t)|_{\mathcal{H}\times Y} \le C^p \underbrace{\int_0^t\int_0^s\cdots\int_0^q}_{p\,\text{integrals}} |\eta_1(r) - \eta_2(r)|_{\mathcal{H}\times Y}\, dr\ldots ds,$$

for all $t \in [0,T]$ and $p \in \mathbb{N}$. Hence, it follows that

$$|\Lambda^p\eta_1 - \Lambda^p\eta_2|_{\infty,\mathcal{H}\times Y} \le \frac{C^p}{p!}|\eta_1 - \eta_2|_{\infty,\mathcal{H}\times Y}, \qquad \forall p \in \mathbb{N}, \tag{4.13}$$

and since $\lim_p C^p/p! = 0$, (4.13) implies that for p large enough, the operator Λ^p is a contraction in $L^\infty(0,T,\mathcal{H}\times Y)$. Therefore, there exists a unique element $\eta^* \in L^\infty(0,T,\mathcal{H}\times Y)$ such that $\Lambda^p\eta^* = \eta^*$. Moreover, it can be easily verified that η^* is the unique fixed-point of Λ. Hence, from (4.11) it results that for all $t \in [0,T]$,

$$\eta^*(t) = \left(\eta^{*1}(t), \eta^{*2}(t)\right) = \left(G\left(\sigma_{\eta^*}(t), \varepsilon\left(u_{\eta^*}(t)\right), \kappa_{\eta^*}(t)\right), \varphi\left(\sigma_{\eta^*}(t), \varepsilon\left(u_{\eta^*}(t)\right), \kappa_{\eta^*}(t)\right)\right), \tag{4.14}$$

for all $t \in [0,T]$.

PROOF OF THEOREM 4.1. Let $\eta^* \in L^\infty(0,T,\mathcal{H}\times Y)$ be the fixed-point of the operator Λ, and let $u_{\eta^*} \in W^{1,\infty}(0,T,H_1)$ and $\sigma_{\eta^*} \in W^{1,\infty}(0,T,\mathcal{H}_1)$ denote the solution of Problem $P_{1\eta^*}$. Let, also, $\kappa_{\eta^*} \in W^{1,\infty}(0,T,Y)$ be the function given by (4.10) for $\eta = \eta^*$. We shall prove that $(u_{\eta^*}, \sigma_{\eta^*}, \kappa_{\eta^*})$ is the unique solution of Problem P_1.

The initial conditions (3.18) follow from (4.5), (4.2), and (4.10) for $\eta = \eta^*$. Moreover, the equalities (3.15) and (3.16) follow from (4.1), (4.3), and (4.14) since

$$\dot\sigma_{\eta^*}(t) = \mathcal{E}\varepsilon\left(\dot u_{\eta^*}(t)\right) + \dot z^1_{\eta^*}(t), \qquad \dot z^1_{\eta^*}(t) = \eta^{*1}(t) = G\left(\sigma_{\eta^*}(t), \varepsilon\left(u_{\eta^*}(t)\right), \kappa_{\eta^*}(t)\right),$$
$$\dot\kappa_{\eta^*}(t) = \dot z^2_{\eta^*}(t) = \eta^{*2}(t) = \varphi\left(\sigma_{\eta^*}(t), \varepsilon\left(u_{\eta^*}(t)\right), \kappa_{\eta^*}(t)\right),$$

a.e. $t \in (0,T)$.

Let now (u, σ, κ) be another solution of Problem P_1 having the regularity

$$u \in W^{1,\infty}(0,T,H_1), \qquad \sigma \in W^{1,\infty}(0,T,\mathcal{H}_1), \qquad \kappa \in W^{1,\infty}(0,T,Y).$$

We denote by $\eta \in L^\infty(0,T,\mathcal{H}\times Y)$ the function defined by

$$\eta(t) = \left(G\left(\sigma(t), \varepsilon\left(u(t)\right), \kappa(t)\right), \varphi\left(\sigma(t), \varepsilon\left(u(t)\right), \kappa(t)\right)\right), \qquad \forall t \in [0,T], \tag{4.15}$$

and let $z_\eta \in W^{1,\infty}(0,T,\mathcal{H}\times Y)$ be the function given by (4.1),(4.2). It results that (u,σ) is a solution of (4.3),(4.4), and using Lemma 4.2 it follows that

$$u = u_\eta, \qquad \sigma = \sigma_\eta. \tag{4.16}$$

Moreover, by (3.16), (3.18), (4.15), and (4.10), we have

$$\kappa = \kappa_\eta. \tag{4.17}$$

Using now (4.11), (4.16), and (4.17), we deduce that $\Lambda\eta = \eta$, and by the uniqueness of the fixed-point of Λ, it results that

$$\eta = \eta^*. \tag{4.18}$$

The uniqueness part in Theorem 4.1 is now a consequence of (4.16)–(4.18).

5. SECOND EXISTENCE AND UNIQUENESS RESULT

The main result of this section is the following.

THEOREM 5.1. *Let (3.1)–(3.5) and (3.10) hold. Then there exists a unique solution of Problem P_2 having the regularity*

$$u \in W^{1,\infty}(0,T,V), \qquad \sigma \in W^{1,\infty}(0,T,\mathcal{H}_1), \qquad \kappa \in W^{1,\infty}(0,T,Y).$$

In order to prove this theorem, we suppose in the sequel that the assumptions of Theorem 5.1 are satisfied. Let us first remark that (3.21) is equivalent to the differential inclusion

$$\varepsilon(u(t)) + \partial\psi_{\Sigma_{ad}(t)}(\sigma(t)) \ni 0, \qquad \forall\, t \in [0,T], \tag{5.1}$$

where $\partial\psi_{\Sigma_{ad}(t)}$ denotes the subdifferential of the indicator function $\psi_{\Sigma_{ad}(t)}$. Since the set $\Sigma_{ad}(t)$ depends on time, we shall replace (5.1) by a differential inclusion associated to a fixed convex set. For this, let us denote

$$\Sigma_0 = \{\tau \in \mathcal{H} \mid \langle \tau, \varepsilon(v)\rangle_{\mathcal{H}} \geq 0,\ \forall\, v \in U_{ad}\}, \tag{5.2}$$

$$\widetilde{\sigma} = \varepsilon(F), \tag{5.3}$$

$$\overline{\sigma} = \sigma - \widetilde{\sigma}, \qquad \overline{\sigma}_0 = \sigma_0 - \widetilde{\sigma}(0). \tag{5.4}$$

From (3.9) and (3.7), we deduce that

$$\Sigma_{ad}(t) = \Sigma_0 + \widetilde{\sigma}(t), \qquad \forall t \in [0,T], \tag{5.5}$$

$$\widetilde{\sigma} \in W^{1,\infty}(0,T,\mathcal{H}_1). \tag{5.6}$$

Moreover, we may easily verify the following result.

LEMMA 5.2. *(u,σ,κ) is a solution of Problem P_2 having the regularity $u \in W^{1,\infty}(0,T,V)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H})$, $\kappa \in W^{1,\infty}(0,T,Y)$, if and only if $(u,\overline{\sigma},\kappa)$ is a solution of the variational problem*

$$\varepsilon(\dot{u}) = \mathcal{E}^{-1}\dot{\overline{\sigma}} - \mathcal{E}^{-1}G(\overline{\sigma}+\widetilde{\sigma},\varepsilon(u),\kappa) + \mathcal{E}^{-1}\dot{\widetilde{\sigma}}, \qquad \text{a.e. on } (0,T), \tag{5.7}$$

$$\dot{\kappa} = \varphi\left(\overline{\sigma}+\widetilde{\sigma},\varepsilon(u),\kappa\right), \qquad \text{a.e. on } (0,T), \tag{5.8}$$

$$\overline{\sigma}(t) \in \Sigma_0, \quad \langle \tau - \overline{\sigma}(t), \varepsilon(u(t))\rangle_{\mathcal{H}} \geq 0, \qquad \forall\,\tau \in \Sigma_0, \quad t \in [0,T], \tag{5.9}$$

$$u(0) = u_0, \qquad \overline{\sigma}(0) = \overline{\sigma}_0, \qquad \kappa(0) = \kappa_0, \tag{5.10}$$

having the regularity $u \in W^{1,\infty}(0,T,V)$, $\overline{\sigma} \in W^{1,\infty}(0,T,\mathcal{H})$, $\kappa \in W^{1,\infty}(0,T,Y)$.

In order to solve problems (5.7)–(5.10), we shall use again a fixed-point method. For this, for each $\eta = (\eta^1,\eta^2) \in L^\infty(0,T,\mathcal{H}\times Y)$, we denote by $z_\eta = (z_\eta^1, z_\eta^2) \in W^{1,\infty}(0,T,\mathcal{H}\times Y)$ the function defined by

$$z_\eta(t) = \int_0^t \eta(s)\,ds + z_0, \qquad \forall t \in [0,T], \tag{5.11}$$

where

$$z_0 = \left(\varepsilon(u_0) - \mathcal{E}^{-1}\sigma_0,\ \kappa_0\right). \tag{5.12}$$

Let us consider the following variational problem.

PROBLEM $P_{2\eta}$. Find $u_\eta : [0,T] \longrightarrow H_1$ and $\sigma_\eta : [0,T] \longrightarrow \mathcal{H}_1$, such that

$$\varepsilon(u_\eta(t)) = \mathcal{E}^{-1}\sigma_\eta(t) + z_\eta^1(t) + \mathcal{E}^{-1}\widetilde{\sigma}(t), \tag{5.13}$$

$$\sigma_\eta(t) \in \Sigma_0, \quad \langle \tau - \sigma_\eta(t), \varepsilon(u_\eta(t))\rangle_{\mathcal{H}} \geq 0, \qquad \forall\,\tau \in \Sigma_0, \tag{5.14}$$

for all $t \in [0,T]$.

LEMMA 5.3. *There exists a unique solution of the variational Problem* $P_{2\eta}$ *having the regularity* $u_\eta \in W^{1,\infty}(0,T,V)$, $\sigma_\eta \in W^{1,\infty}(0,T,\mathcal{H}_1)$. *Moreover,*

$$u_\eta(0) = u_0, \qquad \sigma_\eta(0) = \sigma_0. \tag{5.15}$$

PROOF. Let $t \in [0,T]$. Remarking that Σ_0 is a nonempty closed convex set in $\mathcal{H}$, and using (3.1) and classical results on elliptic variational inequalities, we obtain the existence and uniqueness of a function $\sigma_\eta(t)$ such that

$$\sigma_\eta(t) \in \Sigma_0, \quad \left\langle \tau - \sigma_\eta(t), \mathcal{E}^{-1}\sigma_\eta(t) + z^1_\eta(t) + \mathcal{E}^{-1}\widetilde{\sigma}(t)\right\rangle_{\mathcal{H}} \geq 0, \qquad \forall\, \tau \in \Sigma_0. \tag{5.16}$$

This last inequality is equivalent to

$$\langle \tau - \sigma_\eta(t), \varepsilon_\eta(t)\rangle_{\mathcal{H}} \geq 0, \qquad \forall\, \tau \in \Sigma_0, \tag{5.17}$$

where

$$\varepsilon_\eta(t) = \mathcal{E}^{-1}\sigma_\eta(t) + z^1_\eta(t) + \mathcal{E}^{-1}\widetilde{\sigma}(t). \tag{5.18}$$

Let us introduce the space $\mathcal{V}$ defined by

$$\mathcal{V} = \{\tau \in \mathcal{H} \mid \operatorname{Div}\tau = 0 \text{ in } \Omega,\ \tau\nu = 0 \text{ on } \Gamma_2 \cup \Gamma_3\},$$

where $\tau\nu$ denotes the normal trace of τ. Since $\sigma_\eta(t) \in \Sigma_0$, it results that $\sigma_\eta(t) \pm z \in \Sigma_0$, $\forall\, z \in \mathcal{V}$. Hence, taking $\tau = \sigma_\eta(t) \pm z$ in (5.17), we obtain that $\langle z, \varepsilon_\eta(t)\rangle_{\mathcal{H}} = 0$, $\forall\, z \in \mathcal{V}$. Since the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}$ is the space $\varepsilon(V)$, where V is given by (2.8) (see, for instance, [12, p. 34]), we obtain the existence of the element $u_\eta(t) \in V$ such that

$$\varepsilon_\eta(t) = \varepsilon\,(u_\eta(t))\,. \tag{5.19}$$

Moreover, from (5.16) and (3.1) it results that

$$|\sigma_\eta(t_1) - \sigma_\eta(t_2)|_{\mathcal{H}} \leq C\left(\left|z^1_\eta(t_1) - z^1_\eta(t_2)\right|_{\mathcal{H}} + |\widetilde{\sigma}(t_1) - \widetilde{\sigma}(t_2)|_{\mathcal{H}}\right), \qquad \forall\, t_1, t_2 \in [0,T]. \tag{5.20}$$

Let us remark that since $\sigma_\eta(t) \in \Sigma_0$, we obtain that $\operatorname{Div}\sigma_\eta(t) = 0$. It follows now from (5.6),(5.20), and the time-regularity $z^1_\eta \in W^{1,\infty}(0,T,\mathcal{H})$ that $\sigma_\eta \in W^{1,\infty}(0,T,\mathcal{H}_1)$. Hence, using (5.18), (5.19), and (2.9), we deduce that $u_\eta \in W^{1,\infty}(0,T,V)$.

Finally, using (5.17)–(5.19) it results that (u_η, σ_η) is a solution of Problem $P_{2\eta}$. The uniqueness part in Lemma 5.3, follows from the uniqueness of the solution of (5.16) and Korn's inequality (2.9). The initial conditions (5.15) are a consequence of (3.10), (5.3), (5.4), and the uniqueness of the solution of $P_{2\eta}$ for $t = 0$.

Now let $\kappa_\eta \in W^{1,\infty}(0,T,Y)$ be the function given by

$$\kappa_\eta = z^2_\eta. \tag{5.21}$$

Using the assumptions (3.2), (3.3), (5.11), (5.12), and (5.21), we may consider the operator $\Lambda : L^\infty(0,T,\mathcal{H}\times Y) \longrightarrow L^\infty(0,T,\mathcal{H}\times Y)$ by

$$\Lambda\eta = \left(-\mathcal{E}^{-1}G\,(\sigma_\eta + \widetilde{\sigma}, \varepsilon(u_\eta),\ \kappa_\eta(t))\right), \qquad \forall\, \eta \in L^\infty(0,T,\mathcal{H}\times Y), \tag{5.22}$$

where (u_η, σ_η) is the solution of the variational Problem $P_{2\eta}$.

LEMMA 5.4. *The operator Λ has a unique fixed-point $\eta^* \in L^\infty(0,T,\mathcal{H}\times Y)$.*

PROOF. Let $\eta_1 = (\eta_1^1, \eta_1^2)$ and $\eta_2 = (\eta_2^1, \eta_2^2)$ in $L^\infty(0,T,\mathcal{H}\times Y)$, and let $t \in [0,T]$. Using (5.22), (3.1)–(3.3), and (5.11)–(5.14), we obtain that

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \le C \int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}}\, ds, \qquad \forall\, t \in [0,T].$$

Lemma 5.4 follows now from the same arguments as in Lemma 4.3.

PROOF OF THEOREM 5.1. Let $\eta^* \in L^\infty(0,T,\mathcal{H}\times Y)$ be the fixed-point of Λ, and let $u_{\eta^*} \in W^{1,\infty}(0,T,V)$ and $\sigma_{\eta^*} \in W^{1,\infty}(0,T,\mathcal{H}_1)$ denote the solution of Problem $P_{2\eta^*}$. Let, also, $\kappa_{\eta^*} \in W^{1,\infty}(0,T,Y)$ be the function given by (5.21) for $\eta = \eta^*$. Using the same arguments as in the proof of Theorem 4.1, it follows that $(u_{\eta^*}, \sigma_{\eta^*}, \kappa_{\eta^*})$ is a solution of (5.7)–(5.10), and using Lemma 5.2, we obtain the existence part in Theorem 5.1.

The uniqueness part follows from the uniqueness of the fixed-point of the operator Λ defined by (5.22). It can also be deduced directly from (3.19)–(3.22), (3.1)–(3.3), using a Gronwall-type inequality.

6. BEHAVIOUR OF THE SOLUTION

We start this section by the study of the link between the solutions of the variational Problems P_1 and P_2.

THEOREM 6.1. *Let (3.1)–(3.5) and (3.10) hold, and let be the functions $u \in W^{1,\infty}(0,T,V)$, $\sigma \in W^{1,\infty}(0,T,\mathcal{H}_1)$, and $\kappa \in W^{1,\infty}(0,T,Y)$. Then (u,σ,κ) is a solution of the variational Problem P_1 if and only if (u,σ,κ) is a solution of the variational Problem P_2.*

PROOF. Let $t \in [0,T]$, and let us suppose that (u,σ,κ) is a solution of P_1. Taking $v = 2u(t)$ and $v = 0$ in (3.17), we obtain that

$$\langle \sigma(t), \varepsilon(u(t))\rangle_{\mathcal{H}} = \langle F(t), u(t)\rangle_V, \tag{6.1}$$

and using (3.17) and (3.9), it follows that $\sigma(t) \in \Sigma_{ad}(t)$. The inequality in (3.21) follows from (3.9) and (6.1). Hence, we conclude that (u,σ,κ) is a solution of Problem P_2.

Conversely, let (u,σ,κ) be a solution of P_2. We shall first prove that $u(t) \in U_{ad}$, $\forall\, t \in [0,T]$. Indeed, let $t \in [0,T]$, and let us suppose in the sequel that $u(t) \notin U_{ad}$. We denote by $Pu(t)$ the projection of $u(t)$ on the closed convex set $U_{ad} \subset V$. We have that

$$\langle Pu(t) - u(t), v\rangle_V \ge \langle Pu(t) - u(t), Pu(t)\rangle_V > \langle Pu(t) - u(t), u(t)\rangle_V, \qquad \forall\, v \in U_{ad}.$$

From these inequalities, we obtain that there exists $\alpha \in \mathbf{R}$ such that

$$\langle Pu(t) - u(t), v\rangle_V > \alpha > \langle Pu(t) - u(t), u(t)\rangle_V, \qquad \forall\, v \in U_{ad}. \tag{6.2}$$

Now let $\tilde{\tau}(t)$ be the function defined by

$$\tilde{\tau}(t) = \varepsilon\,(Pu(t) - u(t)) \in \mathcal{H}. \tag{6.3}$$

Using (6.2), (6.3), and (2.10), we deduce that

$$\langle \tilde{\tau}(t), \varepsilon(v)\rangle_{\mathcal{H}} > \alpha > \langle \tilde{\tau}(t), \varepsilon(u(t))\rangle_{\mathcal{H}}, \qquad \forall\, v \in U_{ad}, \tag{6.4}$$

and taking $v = 0$ in (6.4), we obtain that

$$\alpha < 0. \tag{6.5}$$

Let us now suppose that there exists $w \in U_{ad}$ such that

$$\langle \widetilde{\tau}(t), \varepsilon(w)\rangle_{\mathcal{H}} < 0. \tag{6.6}$$

Since $\lambda w \in U_{ad}$ for $\lambda \geq 0$, it follows from (6.4) that

$$\lambda \langle \widetilde{\tau}(t), \varepsilon(w)\rangle_{\mathcal{H}} > \alpha, \qquad \forall \lambda \geq 0,$$

and, passing to the limit when $\lambda \to +\infty$, using (6.6) we obtain that $\alpha \leq -\infty$ which is in contradiction with $\alpha \in \mathbb{R}$. Hence, it results that $\langle \widetilde{\tau}(t), \varepsilon(w)\rangle_{\mathcal{H}} \geq 0$, $\forall w \in U_{ad}$, which implies that $\widetilde{\tau}(t) \in \Sigma_0$ (see (5.2)). Using (5.5), we deduce that $\widetilde{\tau}(t) + \widetilde{\sigma}(t) \in \Sigma_{ad}(t)$, where $\widetilde{\sigma}$ is given by (5.3) and from (6.5), (6.4), (3.21), it follows that

$$0 > \langle \widetilde{\tau}(t), \varepsilon(u(t))\rangle_{\mathcal{H}} \geq \langle \sigma(t) - \widetilde{\sigma}(t), \varepsilon(u(t))\rangle_{\mathcal{H}},$$

which implies that

$$\langle \sigma(t) - \widetilde{\sigma}(t), \varepsilon(u(t))\rangle_{\mathcal{H}} < 0. \tag{6.7}$$

Moreover, since $\sigma(t) - \widetilde{\sigma}(t) \in \Sigma_0$, from (5.4) and (5.9) for $\tau = 2(\sigma(t) - \widetilde{\sigma}(t))$, it results that

$$\langle \sigma(t) - \widetilde{\sigma}(t), \varepsilon(u(t))\rangle_{\mathcal{H}} \geq 0. \tag{6.8}$$

Let us now remark that (6.7) and (6.8) are in contradiction. Therefore, $u(t) \in U_{ad}$. Using (5.3) and (2.10), it follows that $\widetilde{\sigma}(t) \in \Sigma_{ad}(t)$. Moreover, taking $\tau = \widetilde{\sigma}(t)$ in (3.21), we obtain that

$$\langle F(t), u(t)\rangle_V \geq \langle \sigma(t), \varepsilon(u(t))\rangle_{\mathcal{H}}. \tag{6.9}$$

Since $\sigma(t) \in \Sigma_{ad}(t)$ and $u(t) \in U_{ad}$ from (3.9), it results that

$$\langle \sigma(t), \varepsilon(u(t))\rangle_{\mathcal{H}} \geq \langle F(t), u(t)\rangle_V. \tag{6.10}$$

So, from (6.9) and (6.10), we obtain that

$$\langle \sigma(t), \varepsilon(u(t))\rangle_{\mathcal{H}} = \langle F(t), u(t)\rangle_V. \tag{6.11}$$

The inequality in (3.17) follows now from (3.9) and (6.11). Hence, it results that (u, σ, κ) is a solution of Problem P_1.

We shall now introduce a penalized problem of the mechanical Problem P for which we give again, two variational formulations and two existence and uniqueness results. Moreover, if we denote by $(u_h, \sigma_h, \kappa_h)$ the solution of this penalized problem depending on the parameter $h > 0$, we obtain a convergence result of $(u_h, \sigma_h, \kappa_h)$ to the solution (u, σ, κ) of Problem P_1 when $h \to \infty$.

More precisely, let $h > 0$. We consider the following mixed problem.

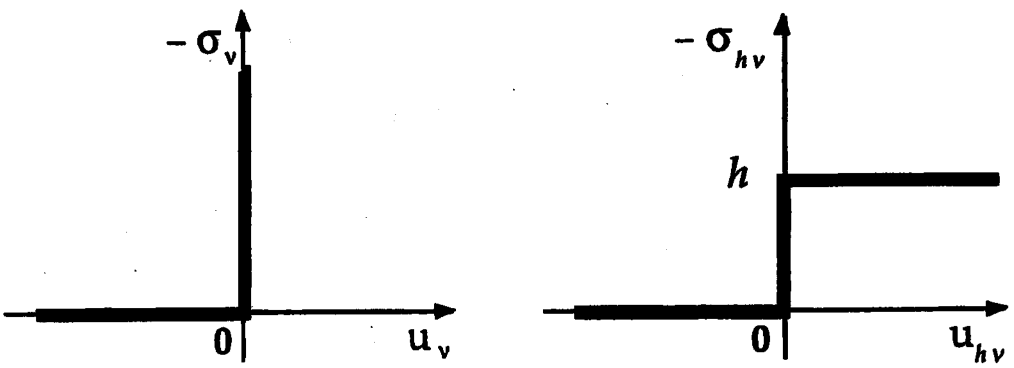

Figure 1. Graphs of the contact boundary conditions of the Problems P and P^h.

PROBLEM P^h. Find the displacement field $u_h : \Omega \times [0,T] \longrightarrow \mathbf{R}^N$, the stress field $\sigma_h : \Omega \times [0,T] \longrightarrow S_N$, and the internal state variable $\kappa_h : \Omega \times [0,T] \longrightarrow \mathbf{R}^M$ such that

$$\dot{\sigma}_h = \mathcal{E}\varepsilon(\dot{u}_h) + G(\sigma_h, \varepsilon(u_h), \kappa_h), \qquad \text{in } \Omega \times (0,T), \tag{6.12}$$

$$\dot{\kappa}_h = \varphi(\sigma_h, \varepsilon(u_h), \kappa_h), \qquad \text{in } \Omega \times (0,T), \tag{6.13}$$

$$\operatorname{Div} \sigma_h + f = 0, \qquad \text{in } \Omega \times (0,T), \tag{6.14}$$

$$u_h = 0, \qquad \text{on } \Gamma_1 \times (0,T), \tag{6.15}$$

$$\sigma_h \nu = g, \qquad \text{on } \Gamma_2 \times (0,T), \tag{6.16}$$

$$\begin{aligned} &u_{h\nu} > 0 \Longrightarrow \sigma_{h\nu} = -h, \\ &u_{h\nu} = 0 \Longrightarrow -h < \sigma_{h\nu} < 0, \qquad \sigma_{h\tau} = 0, \qquad \text{on } \Gamma_3 \times (0,T), \\ &u_{h\nu} < 0 \Longrightarrow \sigma_{h\nu} = 0, \end{aligned} \tag{6.17}$$

$$u_h(0) = u_0, \quad \sigma_h(0) = \sigma_0, \quad \kappa_h = \kappa_0, \quad \text{in } \Omega. \tag{6.18}$$

Let us remark that Problem P^h is similar to Problem P except the fact that the frictionless contact condition (2.6) on Γ_3 was replaced by (6.17). From the mechanical point of view, (6.17) means that the body Ω may leave the foundation and in this case the normal stresses vanish but once there is contact the normal stresses may decrease until a yield which determines the penetrability of Ω in the foundation.

In order to study the penalized Problem P^h, we suppose that (3.1)–(3.5) hold and let us consider the functions $\psi : \mathbf{R} \longrightarrow \mathbf{R}$ and $j : H_1 \longrightarrow \mathbf{R}$ given by

$$\psi(x) = \begin{cases} x \text{ if}, & x > 0, \\ 0 \text{ if}, & x \le 0, \end{cases} \qquad \forall x \in \mathbf{R}, \tag{6.19}$$

$$j(v) = \int_{\Gamma_3} \psi(v_\nu)\, da, \qquad \forall v \in H_1. \tag{6.20}$$

Using the same arguments as in the proof of Lemma 3.1, we can give the following two variational formulations for Problem P^h.

PROBLEM P_1^h. Find the displacement field $u_h : [0,T] \longrightarrow H_1$, the stress field $\sigma_h : [0,T] \longrightarrow \mathcal{H}_1$, and the internal state variable $\kappa_h : [0,T] \longrightarrow Y$ such that

$$\dot{\sigma}_h(t) = \mathcal{E}\varepsilon(\dot{u}_h(t)) + G(\sigma_h(t), \varepsilon(u_h(t)), \kappa_h(t)), \qquad \text{a.e. } t \in (0,T), \tag{6.21}$$

$$\dot{\kappa}_h(t) = \varphi(\sigma_h(t), \varepsilon(u_h(t)), \kappa_h(t)), \qquad \text{a.e. } t \in (0,T), \tag{6.22}$$

$$\begin{aligned} u_h(t) \in V, \quad &\langle \sigma_h(t), \varepsilon(v) - \varepsilon(u_h(t)) \rangle_{\mathcal{H}} + hj(v) - hj(u_h(t)) \ge \langle F(t), v - u_h(t) \rangle_V, \\ &\forall v \in V, \quad t \in [0,T], \end{aligned} \tag{6.23}$$

$$u_h(0) = u_0, \qquad \sigma_h(0) = \sigma_0, \qquad \kappa_h(0) = \kappa_0. \tag{6.24}$$

PROBLEM P_2^h. Find the displacement field $u_h : [0,T] \longrightarrow H_1$, the stress field $\sigma_h : [0,T] \longrightarrow \mathcal{H}_1$, and the internal state variable $\kappa_h : [0,T] \longrightarrow Y$ such that

$$\dot{\sigma}_h(t) = \mathcal{E}\varepsilon(\dot{u}_h(t)) + G(\sigma_h(t), \varepsilon(u_h(t)), \kappa_h(t)), \qquad \text{a.e. } t \in (0,T), \tag{6.25}$$

$$\dot{\kappa}_h(t) = \varphi(\sigma_h(t), \varepsilon(u_h(t)), \kappa_h(t)), \qquad \text{a.e. } t \in (0,T), \tag{6.26}$$

$$\sigma_h(t) \in \Sigma_{\mathrm{ad}}^h(t), \quad \langle \tau - \sigma_h(t), \varepsilon(u_h(t)) \rangle_{\mathcal{H}} \ge 0, \qquad \forall \tau \in \Sigma_{\mathrm{ad}}^h(t), \quad t \in [0,T], \tag{6.27}$$

$$u_h(0) = u_0, \qquad \sigma_h(0) = \sigma_0, \qquad \kappa_h(0) = \kappa_0, \tag{6.28}$$

where

$$\Sigma_{\mathrm{ad}}^h(t) = \{ \tau \in \mathcal{H} \mid \langle \tau, \varepsilon(v) \rangle_{\mathcal{H}} + hj(v) \ge \langle F(t), v \rangle_V \ \forall v \in V \}. \tag{6.29}$$

Let us remark that similarly to (3.14), it results in this case that

$$\langle\sigma_h(t),\varepsilon(u_h(t))\rangle_{\mathcal{H}}+hj(u_h(t))=\langle F(t),u_h(t)\rangle_V,\qquad \forall t\in[0,T]. \tag{6.30}$$

For the study of Problems P_1^h and P_2^h, let us consider the following assumptions:

$$u_0\in V,\quad \sigma_0\in\Sigma_{\mathrm{ad}}^h(0),\quad \kappa_0\in Y,\qquad \langle\sigma_0,\varepsilon(u_0)\rangle_{\mathcal{H}}+hj(u_0)=\langle F(0),u_0\rangle_V. \tag{6.31}$$

As in Section 3, we remark that assumptions (6.31) (which involve regularity conditions on the initial data u_0, σ_0, and κ_0 as well as a compatibility condition between u_0, σ_0, f, and g) are satisfied if $u_0\in H_1$, $\sigma_0\in\mathcal{H}$, $\kappa_0\in Y$, and if u_0 and σ_0 verify (6.14)–(6.17) for $t=0$.

We have the following result.

THEOREM 6.2. *Let (3.1)–(3.5) and (6.31) hold. Then there exists a unique solution of Problem P_1^h having the regularity*

$$u_h\in W^{1,\infty}(0,T,H_1),\qquad \sigma_h\in W^{1,\infty}(0,T,\mathcal{H}_1),\qquad \kappa_h\in W^{1,\infty}(0,T,Y).$$

PROOF. The proof of Theorem 6.2 is similar to the proof of Theorem 4.1 and it can be obtained in three steps, as follows.

(i) For all $\eta\in L^\infty(0,T,\mathcal{H}\times Y)$ there exists a unique $u_{h\eta}\in W^{1,\infty}(0,T,H_1)$, $\sigma_{h\eta}\in W^{1,\infty}(0,T,\mathcal{H}_1)$ solution of the variational Problem $\mathrm{P}_{1\eta}^h$:

$$\sigma_{h\eta}(t)=\mathcal{E}\varepsilon(u_{h\eta}(t))+z_\eta^1(t), \tag{6.32}$$

$$\begin{gathered} u_{h\eta}(t)\in V,\\ \langle\sigma_{h\eta}(t),\varepsilon(v)-\varepsilon(u_{h\eta}(t))\rangle_{\mathcal{H}}+hj(v)-hj(u_{h\eta}(t))\ge\langle F(t),v-u_{h\eta}(t)\rangle_V,\\ \forall v\in V,\end{gathered} \tag{6.33}$$

for all $t\in[0,T]$, where $z_\eta=(z_\eta^1,z_\eta^2)\in W^{1,\infty}(0,T,\mathcal{H}\times Y)$ is given by (4.1),(4.2).

(ii) The operator $\Lambda_h:L^\infty(0,T,\mathcal{H}\times Y)\longrightarrow L^\infty(0,T,\mathcal{H}\times Y)$ defined by

$$\Lambda_h\eta=\left(G\left(\sigma_{h\eta},\varepsilon(u_{h\eta}),\kappa_\eta\right),\varphi\left(\sigma_{h\eta},\varepsilon(u_{h\eta}),\kappa_\eta\right)\right),\qquad \forall\eta\in L^\infty(0,T,\mathcal{H}\times Y), \tag{6.34}$$

where $\kappa_\eta\in W^{1,\infty}(0,T,Y)$ is given by (4.10), has a unique fixed-point $\eta_h^*\in L^\infty(0,T,\mathcal{H}\times Y)$.

(iii) $u_h=u_{h\eta_h^*}$, $\sigma_h=\sigma_{h\eta_h^*}$, $\kappa_h=\kappa_{\eta_h^*}$ is the unique solution of Problem P_1^h.

THEOREM 6.3. *Let (3.1)–(3.5) and (6.31) hold. Then there exists a unique solution of Problem P_2^h having the regularity*

$$u_h\in W^{1,\infty}(0,T,V),\qquad \sigma_h\in W^{1,\infty}(0,T,\mathcal{H}_1),\qquad \kappa_h\in W^{1,\infty}(0,T,Y).$$

PROOF. We use similar arguments as in the proof of Theorem 5.1 replacing Problem $\mathrm{P}_{2\eta}$ by the following problem.

PROBLEM $\mathrm{P}_{2\eta}^h$. Find $u_{h\eta}:[0,T]\longrightarrow H_1$ and $\sigma_{h\eta}:[0,T]\longrightarrow\mathcal{H}_1$, such that

$$\varepsilon\left(u_{h\eta}(t)\right)=\mathcal{E}^{-1}\sigma_{h\eta}(t)+z_\eta^1(t)+\mathcal{E}^{-1}\tilde{\sigma}(t), \tag{6.35}$$

$$\sigma_{h\eta}(t)\in\Sigma_0^h,\quad \langle\tau-\sigma_{h\eta}(t),\varepsilon(u_{h\eta}(t))\rangle_{\mathcal{H}}\ge 0,\qquad \forall\tau\in\Sigma_0^h. \tag{6.36}$$

Here Σ_0^h is given by

$$\Sigma_0^h=\{\tau\in\mathcal{H}\mid\langle\tau,\varepsilon(v)\rangle_{\mathcal{H}}+hj(v)\ge 0\,\forall v\in V\},$$

and $z_\eta=(z_\eta^1,z_\eta^2)\in W^{1,\infty}(0,T,\mathcal{H}\times Y)$ is defined by (5.11),(5.12).

THEOREM 6.4. *Let (3.1)–(3.5) and (6.31) hold, and let be the functions $u_h \in W^{1,\infty}(0,T,V)$, $\sigma_h \in W^{1,\infty}(0,T,\mathcal{H}_1)$, and $\kappa_h \in W^{1,\infty}(0,T,Y)$. Then, $(u_h, \sigma_h, \kappa_h)$ is a solution of the variational Problem P_1^h if and only if $(u_h, \sigma_h, \kappa_h)$ is a solution of the variational Problem P_2^h.*

PROOF. The proof of Theorem 6.4 uses the same arguments as in the proof of Theorem 6.1.

The solution of the variational Problems P_1^h and P_2^h depends on the parameter $h > 0$. The behaviour of this solution when $h \to \infty$ is given by the following result.

THEOREM 6.5. *Let (3.1)–(3.5), (3.10), and (6.31) hold. For all $h > 0$, let $(u_h, \sigma_h, \kappa_h)$ be the solution of Problems P_1^h and P_2^h, and let (u, σ, κ) be the solution of Problems P_1 and P_2. Then, for all $t \in [0,T]$, we have*

$$u_h(t) \longrightarrow u(t) \text{ in } H_1, \quad \sigma_h(t) \longrightarrow \sigma(t) \text{ in } \mathcal{H}_1, \quad \kappa_h(t) \longrightarrow \kappa(t) \text{ in } Y, \qquad \text{when } h \to +\infty. \tag{6.37}$$

In order to prove Theorem 6.5, we need some preliminary results. For this, we suppose in the sequel that the assumptions of Theorem 6.5 are satisfied. We first give an estimation of the difference between the solutions of the variational Problem $P_{1\eta}^h$ constructed with two different functions η.

LEMMA 6.6. *Let $\eta_i = (\eta_i^1, \eta_i^2) \in L^\infty(0,T,\mathcal{H} \times Y)$ $(i = 1,2)$, and let $h > 0$. Then, denoting by $(u_{h\eta_i}, \sigma_{h\eta_i})$ the solution of Problem $P_{1\eta}^h$ for $\eta = \eta_i$, there exists $C > 0$ such that*

$$\left|u_{h\eta_1}(t) - u_{h\eta_2}(t)\right|_{H_1} + \left|\sigma_{h\eta_1}(t) - \sigma_{h\eta_2}(t)\right|_{\mathcal{H}_1} \le C\int_0^t \left|\eta_1^1(s) - \eta_2^1(s)\right|_{\mathcal{H}} ds, \qquad \forall t \in [0,T]. \tag{6.38}$$

PROOF. Let $t \in [0,T]$, and let z_{η_i} be defined by (4.1),(4.2) for $\eta = \eta_i$ $(i = 1,2)$. Using (6.32) and (6.33), we obtain that

$$\begin{gathered} u_{h\eta_i}(t) \in V, \\ \langle \mathcal{E}\varepsilon(u_{h\eta_i}(t)), \varepsilon(v) - \varepsilon(u_{h\eta_i}(t))\rangle_{\mathcal{H}} + \left\langle z_{\eta_i}^1(t), \varepsilon(v) - \varepsilon(u_{h\eta_i}(t))\right\rangle_{\mathcal{H}} \\ + hj(v) - hj(u_{h\eta_i}(t)) \ge \langle F(t), v - u_{h\eta_i}(t)\rangle_V, \\ \forall v \in V, \qquad i = 1,2. \end{gathered} \tag{6.39}$$

It follows now from (6.39), (3.1), and (2.9) that

$$\left|u_{h\eta_1}(t) - u_{h\eta_2}(t)\right|_{H_1} \le C\left|z_{\eta_1}^1(t) - z_{\eta_2}^1(t)\right|_{\mathcal{H}}. \tag{6.40}$$

From (6.32) and (6.39), it results that $\operatorname{Div}\sigma_{h\eta_1}(t) = \operatorname{Div}\sigma_{h\eta_2}(t) = -f(t)$. Hence, from (6.32), (3.1), and (6.40), we deduce that

$$\left|\sigma_{h\eta_1}(t) - \sigma_{h\eta_2}(t)\right|_{H_1} \le C\left|z_{\eta_1}^1(t) - z_{\eta_2}^1(t)\right|_{\mathcal{H}}. \tag{6.41}$$

The inequality (6.38) is now a consequence of (6.40), (6.41), and (4.1).

LEMMA 6.7. *Let $\eta \in L^\infty(0,T,\mathcal{H} \times Y)$, and let $z_\eta \in W^{1,\infty}(0,T,\mathcal{H} \times Y)$, the function defined by (4.1),(4.2). Let, also, (u_η, σ_η) be the solution of Problem $P_{1\eta}$ and for all $h > 0$, let $(u_{h\eta}, \sigma_{h\eta})$ be the solution of Problem $P_{1\eta}^h$. Then, for all $t \in [0,T]$, we have*

$$u_{h\eta}(t) \longrightarrow u_\eta(t) \text{ in } H_1, \quad \sigma_{h\eta}(t) \longrightarrow \sigma_\eta(t) \text{ in } \mathcal{H}_1, \qquad \text{when } h \to +\infty. \tag{6.42}$$

PROOF. Let $t \in [0,T]$. From (6.32) and (6.33), we obtain that

$$\begin{gathered} u_{h\eta}(t) \in V, \\ \langle \mathcal{E}\varepsilon(u_{h\eta}(t)), \varepsilon(v) - \varepsilon(u_{h\eta}(t))\rangle_{\mathcal{H}} + \langle z_\eta^1(t), \varepsilon(v) - \varepsilon(u_{h\eta}(t))\rangle_{\mathcal{H}} \\ + hj(v) - hj(u_{h\eta}(t)) \ge \langle F(t), v - u_{h\eta}(t)\rangle_V, \\ \forall v \in V. \end{gathered} \tag{6.43}$$

Taking $v = 0$ in (6.43) and using (6.19), (6.20), (3.1), and (2.9), it results that

$$C\,|u_{h\eta}(t)|^2_{H_1} + hj(u_{h\eta}(t)) \le \langle F(t), u_{h\eta}(t)\rangle_V + \langle z^1_\eta(t), \varepsilon(u_{h\eta}(t))\rangle_{\mathcal{H}}, \tag{6.44}$$

and after some algebra, we deduce that $(u_{h\eta}(t))_h$ is a bounded sequence in H_1. Therefore, there exists an element $\tilde{u}_\eta(t) \in H_1$ and a subsequence $(u_{h'\eta}(t))_{h'} \subset (u_{h\eta}(t))_h$ such that

$$u_{h'\eta}(t) \rightharpoonup \widetilde{u}_\eta(t) \text{ in } H_1, \qquad \text{when } h' \to 0. \tag{6.45}$$

Moreover, since $(u_{h'\eta}(t))_{h'}$ is a bounded sequence in H_1, it follows from (6.44) that there exists a constant $C > 0$ such that

$$j\,(u_{h'\eta}(t)) \le C\frac{1}{h'}, \qquad \forall\, h' > 0. \tag{6.46}$$

The lower semicontinuity of j, (6.45) and (6.46) imply that $j(\widetilde{u}_\eta(t)) = 0$ and by (6.19), (6.20), and (3.8), we obtain that

$$\widetilde{u}_\eta(t) \in U_{\rm ad}. \tag{6.47}$$

Using (6.43),(6.45) and standard lower semicontinuity arguments it results that

$$\begin{gathered}\langle \mathcal{E}\varepsilon\,(\widetilde{u}_\eta(t))\,, \varepsilon(v) - \varepsilon\,(\widetilde{u}_\eta(t))\rangle_{\mathcal{H}} + \langle z^1_\eta(t), \varepsilon(v) - \varepsilon\,(\widetilde{u}_\eta(t))\rangle_{\mathcal{H}} \ge \langle F(t), v - \widetilde{u}_\eta(t)\rangle_V\,,\\ \forall\, v \in U_{\rm ad}.\end{gathered} \tag{6.48}$$

We remark now from (6.47) and (6.48) that $\widetilde{u}_\eta(t)$ is a solution of (4.6) and from the uniqueness of the solution of this variational inequality we obtain that $\tilde{u}_\eta(t) = u_\eta(t)$. Hence, $u_\eta(t)$ is the unique weak limit of any subsequence of $(u_{h\eta}(t))_h$. We deduce, therefore, that the whole sequence $(u_{h\eta}(t))_h$ is weakly convergent to $u_\eta(t)$ in H_1:

$$u_{h\eta}(t) \rightharpoonup u_\eta(t) \text{ in } H_1, \qquad \text{when } h \to 0. \tag{6.49}$$

In order to obtain the strong convergence, let us remark that from (3.1) and (2.9), it follows that

$$\begin{aligned}C\,|u_{h\eta}(t) - u_\eta(t)|^2_{H_1} \le{}& \langle \mathcal{E}\varepsilon(u_{h\eta}(t)), \varepsilon(u_{h\eta}(t)) - \varepsilon(u_\eta(t))\rangle_{\mathcal{H}}\\ &- \langle \mathcal{E}\varepsilon(u_\eta(t)), \varepsilon(u_{h\eta}(t)) - \varepsilon(u_\eta(t))\rangle_{\mathcal{H}},\end{aligned} \tag{6.50}$$

and putting $v = u_\eta(t)$ in (6.43), we obtain that

$$\begin{aligned}&\langle \mathcal{E}\varepsilon(u_{h\eta}(t)), \varepsilon(u_{h\eta}(t)) - \varepsilon(u_\eta(t))\rangle_{\mathcal{H}}\\ &\qquad \le \langle F(t), u_{h\eta}(t) - u_\eta(t)\rangle_V + \langle z^1_\eta(t), \varepsilon(u_\eta(t)) - \varepsilon(u_{h\eta}(t))\rangle_{\mathcal{H}}.\end{aligned} \tag{6.51}$$

Lemma 6.7 follows now from (6.50), (6.51), (6.49), and (6.32).

Proof of Theorem 6.5. Let $h > 0$ and $t \in [0, T]$. As it results from the proofs of Theorems 4.1 and 6.2, we have that $u = u_{\eta^*}$, $\sigma = \sigma_{\eta^*}$, $\kappa = \kappa_{\eta^*}$, $u_h = u_{h\eta_h^*}$, $\sigma_h = \sigma_{h\eta_h^*}$, and $\kappa_h = \kappa_{\eta_h^*}$, where η^* is the fixed-point of the operator Λ defined by (4.11) and η_h^* is the fixed-point of the operator Λ_h defined by (6.34). Then, denoting by $(u_{h\eta^*}, \sigma_{h\eta^*})$ the solution of Problem $P^h_{1\eta}$ for $\eta = \eta^*$, it results that

$$\begin{aligned}&|u_h(t) - u(t)|_{H_1} + |\sigma_h(t) - \sigma(t)|_{\mathcal{H}_1} + |\kappa_h(t) - \kappa(t)|_Y\\ &\quad \le \left|u_{h\eta_h^*}(t) - u_{h\eta^*}(t)\right|_{H_1} + \left|\sigma_{h\eta_h^*}(t) - \sigma_{h\eta^*}(t)\right|_{\mathcal{H}_1} + \left|\kappa_{\eta_h^*}(t) - \kappa_{\eta^*}(t)\right|_Y\\ &\quad + |u_{h\eta^*}(t) - u_{\eta^*}(t)|_{H_1} + |\sigma_{h\eta^*}(t) - \sigma_{\eta^*}(t)|_{\mathcal{H}_1}.\end{aligned} \tag{6.52}$$

From (4.10), (4.1), and (4.2), it follows that

$$\left|\kappa_{\eta_h^*}(t) - \kappa_{\eta^*}(t)\right|_Y \le \int_0^t \left|\eta_h^{*2}(s) - \eta^{*2}(s)\right|_Y ds. \tag{6.53}$$

Moreover, using Lemma 6.6, we obtain that

$$\left|u_{h\eta_h^*}(t) - u_{h\eta^*}(t)\right|_{H_1} + \left|\sigma_{h\eta_h^*}(t) - \sigma_{h\eta^*}(t)\right|_{\mathcal{H}_1} \leq C\int_0^t \left|\eta_h^{*1}(s) - \eta^{*1}(s)\right|_{\mathcal{H}} ds. \tag{6.54}$$

Therefore, (6.53) and (6.54) imply that

$$\begin{aligned}\left|u_{h\eta_h^*}(t) - u_{h\eta^*}(t)\right|_{H_1} + \left|\sigma_{h\eta_h^*}(t) - \sigma_{h\eta^*}(t)\right|_{\mathcal{H}_1}& \\ + \left|\kappa_{\eta_h^*}(t) - \kappa_{\eta^*}(t)\right|_Y &\leq C\int_0^t |\eta_h^*(s) - \eta^*(s)|_{\mathcal{H}\times Y}\, ds.\end{aligned} \tag{6.55}$$

Using the equalities

$$\eta_h^* = \Lambda_h\eta_h^* = (G(\sigma_h, \varepsilon(u_h), \kappa_h),\ \varphi(\sigma_h, \varepsilon(u_h), \kappa_h)),$$
$$\eta^* = \Lambda\eta^* = (G(\sigma, \varepsilon(u), \kappa),\ \varphi(\sigma, \varepsilon(u), \kappa)),$$

in (6.55), by (3.2) and (3.3) it results that

$$\begin{aligned}&\left|u_{h\eta_h^*}(t) - u_{h\eta^*}(t)\right|_{H_1} + \left|\sigma_{h\eta_h^*}(t) - \sigma_{h\eta^*}(t)\right|_{\mathcal{H}_1} + \left|\kappa_{\eta_h^*}(t) - \kappa_{\eta^*}(t)\right|_Y \\ &\qquad\leq C\int_0^t \left(|u_h(s) - u(s)|_{H_1} + |\sigma_h(s) - \sigma(s)|_{\mathcal{H}_1} + |\kappa_h(s) - \kappa(s)|_Y\right) ds.\end{aligned} \tag{6.56}$$

Let us now introduce the following result: let $m \in W^{1,\infty}(0,T,\mathbf{R})$ such that $m(0) = 0$ and $m(t) \geq 0\ \forall\, t \in [0,T]$, and let $a \geq 0$ and $b > 0$ be two constants. If $\psi \in L^\infty(0,T,\mathbf{R})$ is such that

$$\psi(t) \leq a + m(t) + b\int_0^t \psi(s)\, ds, \qquad \forall\, t \in [0,T],$$

then

$$\psi(t) \leq m(t) + \left(a + b\int_0^t m(s)\, dt\right) e^{bt}, \qquad \forall\, t \in [0,T].$$

Hence, using (6.56) in (6.52) and this last result, we obtain that

$$\begin{aligned}&|u_h(t) - u(t)|_{H_1} + |\sigma_h(t) - \sigma(t)|_{\mathcal{H}_1} + |\kappa_h(t) - \kappa(t)|_Y \\ &\quad\leq |u_{h\eta^*}(t) - u_{\eta^*}(t)|_{H_1} + |\sigma_{h\eta^*}(t) - \sigma_{\eta^*}(t)|_{\mathcal{H}_1} \\ &\quad + C\int_0^t \left(|u_{h\eta^*}(s) - u_{\eta^*}(s)|_{H_1} + |\sigma_{h\eta^*}(s) - \sigma_{\eta^*}(s)|_{\mathcal{H}_1}\right) ds.\end{aligned} \tag{6.57}$$

Moreover, from (6.50), (6.51), (4.3), and (6.32), we deduce that

$$|u_{h\eta^*}(s) - u_{\eta^*}(s)|_{H_1} + |\sigma_{h\eta^*}(s) - \sigma_{\eta^*}(s)|_{\mathcal{H}_1} \leq |F(s)|_V + \left|z_{\eta^*}^1(s)\right|_{\mathcal{H}} + |u_{\eta^*}(s)|_{H_1}, \qquad \forall\, s \in [0,T],$$

and using (6.42), by Lebesgue's Theorem, it follows that

$$C\int_0^t \left(|u_{h\eta^*}(s) - u_{\eta^*}(s)|_{H_1} + |\sigma_{h\eta^*}(s) - \sigma_{\eta^*}(s)|_{\mathcal{H}_1}\right) ds \longrightarrow 0, \qquad \text{when } h \to +\infty. \tag{6.58}$$

Theorem 6.5 is now a consequence of (6.57), (6.42), and (6.58).

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Frictional Contact Problems for Nonlinear Elastic Materials

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Frictional Contact Problems for Nonlinear Elastic Materials

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Descriptif

Il s'agit dans ce papier de l'étude du problème statique de contact bilatéral avec frottement entre un matériau ayant une loi de comportement élastique non linéaire et une fondation rigide. Le contact et le frottement sont modélisés par la loi de Tresca.

On considère un milieu continu occupant un domaine Ω de $\mathbb{R}^N$ $(N = 1, 2, 3)$ et dont la frontière Γ, supposée suffisamment régulière, est divisée en trois parties disjointes Γ_1, Γ_2 et Γ_3. On suppose que le champ des déplacements s'annulle sur Γ_1, que des forces surfaciques f_2 s'appliquent sur Γ_2 et que des forces volumiques f_0 agissent dans Ω. On suppose aussi que le matériau en contact bilatéral avec frottement par la partie Γ_3 de sa frontière avec une fondation rigide. Le problème statique de contact étudié se formule mathématiquement de la façon suivante :

Problème P : Trouver le champ des déplacements $u : \Omega \longrightarrow \mathbb{R}^N$ et le champ des contraintes $\sigma : \Omega \longrightarrow \mathbb{R}_s^{N\times N}$ tels que

$$\begin{aligned}
\sigma = \mathcal{F}(\varepsilon(u)) \quad & \text{dans} \quad \Omega,\\
Div\,\sigma + f = 0 \quad & \text{dans} \quad \Omega,\\
u = 0 \quad & \text{sur} \quad \Gamma_1,\\
\sigma\nu = g \quad & \text{sur} \quad \Gamma_2,\\
u_\nu = 0 \quad & \text{sur} \quad \Gamma_3,\\
|\sigma_\tau| \leq g \quad & \text{sur} \quad \Gamma_3,\\
|\sigma_\tau| < g \quad & \Longrightarrow u_\tau = 0,\\
|\sigma_\tau| = g \quad & \Longrightarrow \sigma_\tau = -\lambda u_\tau,\ \lambda \geq 0.
\end{aligned}$$

On note par $\mathbb{R}_s^{N\times N}$ l'espace des tenseurs symétriques du second ordre sur $\mathbb{R}^N$ et par $\varepsilon(u)$ le tenseur des petites déformations linéarisé. $Div\,\sigma$ désigne la divergence de la fonction tensorielle σ, le vecteur ν est la normale unitaire sortante à Ω, $\sigma\nu$ est le vecteur des contraintes de Cauchy, et u_ν, σ_ν et σ_τ représentent respectivement le déplacement normal, les contraintes normales et tangentielles. Le réel positif g désigne le seuil de frottement.

On prouve l'existence ainsi que l'unicité de deux solutions faibles relatives à deux formulations variationnelles du problème P. On établit aussi un résultat d'équivalence entre ces deux formulations. On introduit ensuite, pour tout paramètre $\mu > 0$, un problème régularisé P^μ du problème P et on démontre un résultat de convergence forte de sa solution vers la solution du problème P quand le paramètre $\mu \to 0$. Une interprétation mécanique de ce résultat de convergence est finalement donnée.

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FRICTIONAL CONTACT PROBLEMS FOR NONLINEAR ELASTIC MATERIALS

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We are interested in the static problem of modelling the frictional contact between an elastic body and a rigid foundation. We assume that the elastic constitutive law is nonlinear, that the contact is bilateral and that the friction is described by Tresca's law. Two equivalent weak formulations of the problem are established and the existence of a unique solution is proved in each case. A regularized problem is also studied and a strong convergence result is proved.

1. Introduction

Only recently progress has been made towards the modelling and analysis of contact processes between deformable bodies. This is due to the considerable difficulties that the process of frictional contact presents in the modelling and analysis due to the complicated surface phenomena involved. Contact problems with or without friction were already studied for instance in (Burguera and Viaño, 1995; Drabla *et al.*, 1998; Duvaut and Lions, 1972; Haslinger and Hlaváček, 1980; 1982; Hlaváček and Nečas, 1981; 1983; Kikuchi and Oden, 1988; Licht, 1985; Shillor and Sofonea, 1998), see also the references therein, in the case of elastic or viscoelastic materials. The case of elasto-visco-plastic materials was considered for instance in (Amassad and Sofonea, 1998; Drabla *et al.*, 1997; Rochdi, 1997; Rochdi and Sofonea, 1997; Sofonea, 1997) and the works cited therein.

In this work, we consider the process of frictional contact between an elastic body which is acted upon by volume forces and surface tractions, and a rigid foundation. We assume that the forces and tractions change slowly in time so that the accelerations in the system are negligible. Neglecting sufficiently the inertial terms in the equations of motion leads to a static approximation of the process. The material's constitutive law is assumed to be nonlinear elastic. The same constitutive law was recently used in (Drabla *et al.*, 1998) for the study of a frictionless contact problem with Signorini's contact conditions. The contact is modelled here with a bilateral condition and the friction with the associated Tresca law. These contact and friction conditions were considered for instance in (Duvaut and Lions, 1972; Licht, 1985) in the case of linear elastic or viscoelastic bodies and in (Amassad and Sofonea, 1998) in the case of elasto-visco-plastic bodies.

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This paper is organized as follows. Section 2 contains the notations and some preliminary material. Section 3 deals with the description of the model for the process and the mathematical statement of the problem. In Section 4, we list the assumptions on the data and set the problem in two variational forms. These are two elliptic variational inequalities: Problems P_1 and P_2. The unknown in the first problem is the displacement field and in the second one it is the stress field. The existence of a unique solution to each problem (Theorem 1, Theorem 2) as well as an equivalence result between the problems P_1 and P_2 (Theorem 3) are established in Section 5. In the last section, we introduce for each nonnegative parameter μ a regularized problem P^μ of Problem P_1 and we prove a strong convergence result of its solution to the solution of Problem P_1 when $\mu \to 0$ (Theorem 4).

The purpose of this work is to extend some known results in linear elasticity to the nonlinear case and to point out the second variational formulation which is important in engineering since it is related to the stress field. Moreover, it deals with a regularization of the problem considered, which is of interest from the numerical point of view.

2. Notation and Preliminaries

In this short section, we present the notation we will use and some preliminary material. For further details we refer the reader to (Duvaut and Lions, 1972; Ionescu and Sofonea, 1993; Kikuchi and Oden, 1988; Panagiotopoulos, 1985). $\mathbf{S}_N$ represents the set of second-order symmetric tensors in $\mathbf{R}^N$. We denote by '$\cdot$' and $|\cdot|$ the inner product and the Euclidean norm on $\mathbf{S}_N$ and $\mathbf{R}^N$. We also use the following notations:

$$H = \Big\{ v = (v_i) \,|\, v_i \in L^2(\Omega) \Big\} = L^2(\Omega)^N$$

$$H_1 = \Big\{ v = (v_i) \,|\, v_i \in H^1(\Omega) \Big\} = H^1(\Omega)^N$$

$$\mathcal{H} = \Big\{ \tau = (\tau_{ij}) \,|\, \tau_{ij} = \tau_{ji} \in L^2(\Omega) \Big\} = L^2(\Omega)_s^{N\times N}$$

$$\mathcal{H}_1 = \Big\{ \tau \in \mathcal{H} \,|\, \operatorname{Div} \tau \in H \Big\}$$

where $i, j = 1, \ldots, N$. H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products given by

$$\langle u, v\rangle_H = \int_\Omega u_i v_i \, dx$$

$$\langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx$$

$$\langle u, v\rangle_{H_1} = \langle u, v\rangle_H + \big\langle \varepsilon(u), \varepsilon(v)\big\rangle_{\mathcal{H}}$$

$$\langle \sigma, \tau\rangle_{\mathcal{H}_1} = \langle \sigma, \tau\rangle_{\mathcal{H}} + \langle \operatorname{Div} \sigma, \operatorname{Div} \tau\rangle_H$$

respectively, where $\varepsilon : H_1 \longrightarrow \mathcal{H}$ and $\text{Div} : \mathcal{H}_1 \longrightarrow H$ are the *deformation* and the *divergence* operators, respectively, defined by

$$\varepsilon(v) = \big(\varepsilon_{ij}(v)\big), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad \text{Div}\,\sigma = (\sigma_{ij,j})$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma)^N$ and let $\gamma : H_1 \longrightarrow H_\Gamma$ be the trace map. Let also ν be the outward unit normal to Γ. For every element $v \in H_1$ we use, when no confusion is likely, the notation v for the trace γv of v on Γ. We denote by v_ν and v_τ the *normal* and the *tangential* components of v on Γ given by $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu \nu$, respectively. Let H'_Γ be the dual of H_Γ and let $\langle \cdot , \cdot \rangle$ denote the duality pairing between H'_Γ and H_Γ. For every $\sigma \in \mathcal{H}_1$ let $\sigma\nu$ be the element of H'_Γ given by

$$\langle \sigma\nu, \gamma v \rangle = \big\langle \sigma, \varepsilon(v) \big\rangle_{\mathcal{H}} + \langle \text{Div}\,\sigma, v \rangle_H \qquad \forall\, v \in H_1 \tag{1}$$

We also denote by σ_ν and σ_τ the *normal* and *tangential* traces of σ (see e.g. Kikuchi and Oden, 1988; Panagiotopoulos, 1985). We recall that if σ is a regular function (say C^1), then

$$\langle \sigma\nu, \gamma v \rangle = \int_\Gamma \sigma\nu \cdot v \, \mathrm{d}a \qquad \forall\, v \in H_1 \tag{2}$$

where $\mathrm{d}a$ is the surface measure element, $\sigma_\nu = (\sigma\nu) \cdot \nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$.

3. Problem Modelling

We model the static process when a nonlinear elastic body is being acted upon by forces and surface tractions and as a result it contacts a rigid foundation. The elastic body occupies a domain Ω of $\mathbb{R}^N$ ($N = 1, 2, 3$) with surface Γ. A volume force of density f_0 is applied on Ω. We assume that Γ is Lipschitz and is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}\,\Gamma_1 > 0$. We assume that the body is clamped on Γ_1 and thus the displacement field vanishes there and that surface tractions f_2 act on Γ_2. The solid is always maintained in frictional contact with a rigid foundation on Γ_3, which means that the body and the foundation have a compliant shape on Γ_3.

We denote by u the displacement vector, σ the stress field and $\varepsilon = \varepsilon(u)$ the small strain tensor. The elastic constitutive law that we consider is $\sigma = \mathcal{F}(\varepsilon(u))$, in which $\mathcal{F}$ is a given nonlinear constitutive function. The condition of bilateral contact between the body and the foundation along Γ_3 is given by $u_\nu = 0$, where u_ν represents the normal displacement. The associated friction law is the static Tresca law:

$$|\sigma_\tau| \le g \quad \text{on} \quad \Gamma_3$$
$$|\sigma_\tau| < g \Longrightarrow u_\tau = 0$$
$$|\sigma_\tau| = g \Longrightarrow \sigma_\tau = -\lambda u_\tau, \quad \lambda \ge 0$$

Here, σ_τ represents the tangential force on the contact boundary Γ_3, g denotes the friction yield limit and u_τ represents the tangential displacement. This friction law, which was already considered by Duvaut and Lions (1972), Licht (1985) and Panagiotopoulos (1985), states that the tangential shear cannot exceed the maximal frictional resistance g. Then, if the inequality holds the surface adheres completely to the foundation and is in the so-called ***stick*** state, and when the equality holds there is relative sliding, the so-called ***slip*** state. Therefore, at each time instant the contact surface Γ_3 is divided into two zones: the stick zone and the slip zone.

The mechanical problem of frictional contact between a nonlinear elastic body and a rigid foundation may be formulated classically as follows:

Problem P: Find a displacement field $u\colon \Omega \longrightarrow \mathbb{R}^N$ and a stress field $\sigma\colon \Omega \longrightarrow \mathbb{S}_N$ such that

$$\sigma = \mathcal{F}\big(\varepsilon(u)\big) \quad \text{in} \quad \Omega \tag{3}$$

$$\operatorname{Div} \sigma + f_0 = 0 \quad \text{in} \quad \Omega \tag{4}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \tag{5}$$

$$\sigma\nu = f_2 \quad \text{on} \quad \Gamma_2 \tag{6}$$

$$\begin{cases} u_\nu = 0 \quad \text{on} \quad \Gamma_3 \\ |\sigma_\tau| \le g \quad \text{on} \quad \Gamma_3 \\ |\sigma_\tau| < g \Longrightarrow u_\tau = 0 \\ |\sigma_\tau| = g \Longrightarrow \sigma_\tau = -\lambda u_\tau, \quad \lambda \ge 0 \end{cases} \tag{7}$$

To study Problem P, we need the following additional notation. Let V denote the closed subspace of H_1 given by

$$V = \Big\{ v \in H_1 \,|\, v = 0 \text{ on } \Gamma_1, \; v_\nu = 0 \text{ on } \Gamma_3 \Big\}$$

Now, Korn's inequality holds, since $\operatorname{meas}\Gamma_1 > 0$. Thus (Duvaut and Lions, 1972; Hlaváček and Nečas, 1981)

$$\big|\varepsilon(v)\big|_{\mathcal{H}} \ge C|v|_{H_1} \qquad \forall\, v \in V \tag{8}$$

Here and below C denotes a positive generic constant which may depend on Ω, Γ_1, Γ_2, Γ_3 and $\mathcal{F}$, but does not depend on the input data f_0, f_2, g, and whose value may vary from place to place.

We consider the inner product $\langle \cdot\,,\cdot \rangle_V$ on V, given by

$$\langle v, w \rangle_V = \big\langle \varepsilon(v), \varepsilon(w) \big\rangle_{\mathcal{H}} \tag{9}$$

It follows from (8) that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V. Therefore $(V, |\cdot|_V)$ is a Hilbert space.

4. Variational Formulations

In this section, we give two variational formulations to Problem P. For that purpose, we assume that the *elasticity operator*

$$\mathcal{F}\colon \Omega \times \mathbb{S}_N \longrightarrow \mathbb{S}_N$$

satisfies the following set of conditions:

$$\left\{\begin{array}{l} \text{(a) there exists } L>0 \text{ such that} \\ \quad \big|\mathcal{F}(\cdot,\varepsilon_1)-\mathcal{F}(\cdot,\varepsilon_2)\big| \le L|\varepsilon_1-\varepsilon_2| \quad \forall\, \varepsilon_1,\varepsilon_2 \in \mathbb{S}_N, \text{ a.e. in } \Omega \\ \\ \text{(b) there exists } M>0 \text{ such that} \\ \quad \big(\mathcal{F}(\cdot,\varepsilon_1)-\mathcal{F}(\cdot,\varepsilon_2)\big)\cdot(\varepsilon_1-\varepsilon_2) \ge M|\varepsilon_1-\varepsilon_2|^2 \quad \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}_N, \text{ a.e. in } \Omega \\ \\ \text{(c) } x \longmapsto \mathcal{F}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \quad \forall \varepsilon \in \mathbb{S}_N \\ \\ \text{(d) } x \longmapsto \mathcal{F}(x,0) \in \mathcal{H}. \end{array}\right. \tag{10}$$

The forces and the tractions satisfy

$$f_0 \in H, \quad f_2 \in L^2(\Gamma_2)^N \tag{11}$$

Moreover, the *friction yield limit* satisfies

$$g \ge 0 \tag{12}$$

Remark 1. Using (10) it is straightforward to show that for all $\tau \in \mathcal{H}$, the function $x \longmapsto \mathcal{F}(x,\tau(x))$ belongs to $\mathcal{H}$. Consequently, it is possible to consider $\mathcal{F}$ as an operator from $\mathcal{H}$ into $\mathcal{H}$. Moreover, $\mathcal{F} : \mathcal{H} \longrightarrow \mathcal{H}$ is a strongly monotone and Lipschitz operator (Sofonea, 1993, p.53). Therefore $\mathcal{F}$ is invertible and its inverse operator $\mathcal{F}^{-1}\colon \mathcal{H} \longrightarrow \mathcal{H}$ is also strongly monotone and Lipschitz.

Next, using (11) and the Riesz representation theorem, we may define the element $f \in V$ by

$$\langle f,v\rangle_V = \langle f_0,v\rangle_H + \langle f_2,\gamma v\rangle_{L^2(\Gamma_2)^N} \qquad \forall v \in V \tag{13}$$

Let also $j\colon V \longrightarrow \mathbb{R}_+$ be the functional

$$j(v) = g\int_{\Gamma_3} |v_\tau|\, da, \quad v \in V \tag{14}$$

Finally, we define the set of "*statically admissible stress fields*" Σ_{ad} by

$$\Sigma_{\text{ad}} = \Big\{z \in \mathcal{H} \,\big|\, \big\langle z,\varepsilon(v)\big\rangle_{\mathcal{H}} + j(v) \ge \langle f,v\rangle_V \quad \forall v \in V\Big\} \tag{15}$$

Lemma 1. *If* (u,σ) *is a regular solution to Problem* P*, then*

$$u \in V, \quad \big\langle \mathcal{F}\big(\varepsilon(u)\big),\ \varepsilon(v)-\varepsilon(u)\big\rangle_{\mathcal{H}} + j(v) - j(u) \ge \langle f, v-u\rangle_V \qquad \forall v \in V \tag{16}$$

$$\sigma \in \Sigma_{\text{ad}}, \quad \big\langle \mathcal{F}^{-1}(\sigma),\ \tau-\sigma\big\rangle_{\mathcal{H}} \ \ge\ 0 \qquad \forall \tau \in \Sigma_{\text{ad}} \tag{17}$$

Proof. First, from (5) and (13) we deduce that $u \in V$. Let $v \in V$. Using (4), (1) and (2), we have

$$\langle \sigma,\ \varepsilon(v) - \varepsilon(u) \rangle_{\mathcal{H}} = \langle f_0, v - u \rangle_H + \int_\Gamma \sigma\nu \cdot (v - u)\,\mathrm{d}a$$

and, from (5), (6) and (11), we obtain

$$\langle \sigma,\ \varepsilon(v) - \varepsilon(u) \rangle_{\mathcal{H}} = \langle f, v - u \rangle_V + \int_{\Gamma_3} \sigma\nu \cdot (v - u)\,\mathrm{d}a$$

Using now (3) and (7), the previous equality leads to

$$\langle \mathcal{F}(\varepsilon(u)),\ \varepsilon(v) - \varepsilon(u) \rangle_{\mathcal{H}} = \langle f, v - u \rangle_V + \int_{\Gamma_3} \sigma_\tau (v_\tau - u_\tau)\,\mathrm{d}a \tag{18}$$

The inequality in (16) follows from (18) and (14) since (7) implies that

$$\sigma_\tau (v_\tau - u_\tau) \ \geq\ g\big(|u_\tau| - |v_\tau|\big) \quad \text{a.e. on } \Gamma_3$$

Putting now $v = 2u$ and $v = 0$ in (16) and taking (14) and (3) into account, we obtain

$$\langle \sigma, \varepsilon(u) \rangle_{\mathcal{H}} + j(u) = \langle f, u \rangle_V \tag{19}$$

Hence, by using (16), (3), (19) and (15) it follows that $\sigma \in \Sigma_{\mathrm{ad}}$. The inequality in (17) is now a consequence of (19) and (15) since $u \in V$ and $\mathcal{F}$ is invertible. ∎

Lemma 1 leads to the following weak formulations for Problem P.

Problem P_1: Find a displacement field $u\colon \Omega \longrightarrow \mathbb{R}^N$ such that

$$u \in V, \quad \langle \mathcal{F}(\varepsilon(u)),\ \varepsilon(v) - \varepsilon(u) \rangle_{\mathcal{H}} + j(v) - j(u) \geq \langle f, v - u \rangle_V \qquad \forall v \in V \tag{20}$$

Problem P_2: Find a stress field $\sigma\colon \Omega \longrightarrow \mathbb{S}_N$ such that

$$\sigma \in \Sigma_{\mathrm{ad}}, \quad \langle \mathcal{F}^{-1}(\sigma),\ \tau - \sigma \rangle_{\mathcal{H}} \geq 0 \qquad \forall \tau \in \Sigma_{\mathrm{ad}} \tag{21}$$

Remark 2. Let us remark that Problems P_1 and P_2 are formally equivalent to Problem P. Indeed, if u represents a regular solution to the variational problem P_1 and σ is defined by $\sigma = \mathcal{F}(\varepsilon(u))$, using arguments in (Duvaut and Lions, 1972) it follows that $\{u, \sigma\}$ is a solution to Problem P. In a similar way, if σ represents a regular solution to the variational problem P_2 and $u \in V$ is given by $\sigma = \mathcal{F}(\varepsilon(u))$ then, using the same arguments, it follows that $\{u, \sigma\}$ is a solution to Problem P. For this reason, we may consider Problems P_1 and P_2 as *variational formulations* to Problem P.

Under the assumptions (10)–(12), in the next section we give the existence and uniqueness results for the variational problems P_1 and P_2 followed by an equivalence result between these two problems.

5. Existence and Uniqueness Results

Theorem 1. *Let (10)–(12) hold. Then there exists a unique solution to Problem P_1.*

Proof. Using the Riesz representation theorem, we may consider the operator A: $V \longrightarrow V$ defined by

$$\langle Av, w\rangle_V = \big\langle \mathcal{F}(\varepsilon(v)), \varepsilon(w)\big\rangle_{\mathcal{H}} \qquad \forall v,\ w \in V$$

Theorem 1 is now a consequence of the theory of elliptic variational inequalities (Brezis, 1968; Kikuchi and Oden, 1980), since (10) and (8) imply that the operator A is strongly monotone and Lipschitz and, since the functional j defined by (14) is proper, convex and lower semicontinuous. ∎

Theorem 2. *Let (10)–(12) hold. Then there exists a unique solution to Problem P_2.*

Proof. Using (9) and the fact that the functional j is nonnegative, we deduce that $\varepsilon(f) \in \Sigma_{\text{ad}}$. Thus, Σ_{ad} given by (15) is a nonempty convex subset of $\mathcal{H}$. Moreover, from Remark 1 we obtain that $\mathcal{F}^{-1}$ is a strongly monotone and Lipschitz operator. Hence, using arguments of the theory of elliptic variational inequalities, it follows that Problem P_2 has a unique solution $\sigma \in \Sigma_{\text{ad}}$. Let us prove now that $\sigma \in \mathcal{H}_1$. Indeed, since $\sigma \in \Sigma_{\text{ad}}$, it results from (15) that $\langle \sigma, \varepsilon(v)\rangle_{\mathcal{H}} + j(v) \geq \langle f, v\rangle_V$ for all $v \in V$. Putting in this inequality $v = \pm\varphi$ where $\varphi \in \mathcal{D}(\Omega)^N$ and using (13), we obtain that $\langle \sigma, \varepsilon(\varphi)\rangle_{\mathcal{D}'(\Omega)^N \times \mathcal{D}(\Omega)^N} = \langle f_0, \varphi\rangle_H$ for all $\varphi \in \mathcal{D}(\Omega)^N$. Thus using (1) yields $\operatorname{Div} \sigma + f_0 = 0$ a.e. in Ω. Finally, the regularity of $\sigma \in \mathcal{H}_1$ is a consequence of the last equality and (11). ∎

The following result deals with the study of the link between the variational problems P_1, P_2 and the constitutive law (3).

Theorem 3. *Let (10)–(12) hold and let (u, σ) be such that $u \in V$ and $\sigma \in \mathcal{H}_1$. Consider the following properties:*

(i) u is the solution to Problem P_1 given in Theorem 1,

(ii) σ is the solution to Problem P_2 given in Theorem 2, and

(iii) u and σ are connected with the elastic constitutive law $\sigma = \mathcal{F}(\varepsilon(u))$.

Then two among these properties imply the third one.

Proof. We start by proving that (i) and (iii) imply (ii). Putting $v = 2u \in V$ and $v = 0 \in V$ in (20) and using (14) and (iii), we deduce that

$$\big\langle \sigma, \varepsilon(u)\big\rangle_{\mathcal{H}} + j(u) = \langle f, u\rangle_V \tag{22}$$

Therfore, (20), (22) and (iii) imply that

$$\sigma \in \Sigma_{\text{ad}} \tag{23}$$

Let now $\tau \in \Sigma_{ad}$. From (iii) it follows that

$$\langle \mathcal{F}^{-1}(\sigma),\ \tau - \sigma\rangle_{\mathcal{H}} = \Big(\langle \tau, \varepsilon(u)\rangle_{\mathcal{H}} + j(u)\Big) - \Big(\langle \sigma, \varepsilon(u)\rangle_{\mathcal{H}} + j(u)\Big) \qquad (24)$$

Property (ii) is now a consequence of (15) and (22)–(24) since $u \in V$.

Let us prove now that (i) and (ii) imply (iii). For this, let $\widetilde{\sigma} \in \mathcal{H}$ be the function $\widetilde{\sigma} = \mathcal{F}(\varepsilon(u))$. Using now the previous step of the proof, it follows that $\widetilde{\sigma}$ is a solution to Problem P_2. The uniqueness of the solution σ to this problem yields Property (iii).

Finally, we will establish that (ii) and (iii) imply (i). For that purpose, we introduce the spaces W and $\mathcal{W}$ defined by

$$W = \Big\{ v \in H_1 \,\big|\, v = 0 \ \text{ on } \ \Gamma_1 \Big\} \supset V$$

$$\mathcal{W} = \Big\{ z \in \mathcal{H} \,\big|\, \text{Div } z = 0 \ \text{ in } \ \Omega, \ \ z\nu = 0 \ \text{ on } \ \Gamma_2 \cup \Gamma_3 \Big\}$$

Using (1), it is straightforward to show that the orthogonal complement of $\mathcal{W}$ in $\mathcal{H}$ is the subspace $\varepsilon(W)$, i.e.

$$\mathcal{W}^{\perp} = \varepsilon(W) \ \text{ in } \ \mathcal{H} \qquad (25)$$

Thus it follows from (15) and (25) that $\sigma \pm z \in \Sigma_{ad}$ for all $z \in \mathcal{W}$. Consequently, taking $\tau = \sigma \pm z$ in (21), it may be concluded that $\langle z, \mathcal{F}^{-1}(\sigma)\rangle_{\mathcal{H}} = 0$ for all $z \in \mathcal{W}$. This implies, by using (25), that there exists $\widetilde{u} \in W$ such that

$$\mathcal{F}^{-1}(\sigma) = \varepsilon(\widetilde{u}) \qquad (26)$$

Let us prove that $\widetilde{u} \in V$. For this, let us suppose that $\widetilde{u} \notin V$. Hence, since V is a closed subspace of W, there exists $\widetilde{\tau} \in \mathcal{H}$ such that

$$\langle \widetilde{\tau}, \varepsilon(v)\rangle_{\mathcal{H}} = 0 \qquad \forall v \in V \qquad (27)$$

and

$$\langle \widetilde{\tau}, \varepsilon(\widetilde{u})\rangle_{\mathcal{H}} < 0 \qquad (28)$$

Since the functional j is nonnegative, it follows from (9) that $\lambda\widetilde{\tau} + \varepsilon(f) \in \Sigma_{ad}$ for all $\lambda \geq 0$. Therefore, if we set $\tau = \lambda\widetilde{\tau} + \varepsilon(f)$ in (21) and use (26), we obtain

$$\langle \sigma - \varepsilon(f),\ \varepsilon(\widetilde{u})\rangle_{\mathcal{H}} \leq \lambda \langle \widetilde{\tau}, \varepsilon(\widetilde{u})\rangle_{\mathcal{H}} \qquad \forall \lambda \geq 0$$

Passing to the limit as $\lambda \to +\infty$, it follows from (28) that $\langle \sigma - \varepsilon(f),\ \varepsilon(\widetilde{u})\rangle_{\mathcal{H}} \leq -\infty$ which is absurd. Consequently, $\widetilde{u} \in V$. Assertion (iii) and (26) yield $\sigma = \mathcal{F}(\varepsilon(u)) = \mathcal{F}(\varepsilon(\widetilde{u}))$. Hence, using (10) and (8), we obtain

$$0 = \big\langle \mathcal{F}(\varepsilon(u)) - \mathcal{F}(\varepsilon(\widetilde{u})),\ \varepsilon(u) - \varepsilon(\widetilde{u})\big\rangle_{\mathcal{H}} \geq C\big|\varepsilon(u) - \varepsilon(\widetilde{u})\big|^2_{\mathcal{H}} \geq C\big|u - \widetilde{u}\big|^2_{H_1}$$

Thus we deduce that $u = \widetilde{u} \in V$.

Let us establish now the inequality in (20). Since the functional j is subdifferentiable, there exists $\overline{\tau} \in \mathcal{H}$ such that

$$\langle \overline{\tau},\ \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} + j(v) - j(u) \geq \langle f, v - u\rangle_V \qquad \forall v \in V \qquad (29)$$

Taking $v = 2u \in V$ and $v = 0 \in V$ in this inequality, we have

$$\langle \overline{\tau}, \varepsilon(u) \rangle_{\mathcal{H}} + j(u) = \langle f, u \rangle_V \tag{30}$$

and, from (29), (30) and (15), we deduce that $\overline{\tau} \in \Sigma_{\mathrm{ad}}$. Taking now $\tau = \overline{\tau}$ in (21) and, using Assertion (iii) and (30), it follows that

$$\langle \mathcal{F}(\varepsilon(u)), \varepsilon(u) \rangle_{\mathcal{H}} + j(u) \leq \langle f, u \rangle_V \tag{31}$$

Moreover, since $\sigma = \mathcal{F}(\varepsilon(u)) \in \Sigma_{\mathrm{ad}}$, we have

$$\langle \mathcal{F}(\varepsilon(u)), \varepsilon(u) \rangle_{\mathcal{H}} + j(u) \geq \langle f, u \rangle_V \tag{32}$$

and

$$\langle \mathcal{F}(\varepsilon(u)), \varepsilon(v) \rangle_{\mathcal{H}} + j(v) \geq \langle f, v \rangle_V \qquad \forall\, v \in V \tag{33}$$

The inequality in (20) is finally a consequence of (31)–(33). This concludes the proof of Theorem 3. ■

Remark 3. A mechanical interpretation of the result obtained in Theorem 3 is the following:

1. If the displacement field u is the solution to Problem P_1, then the stress field σ connected to u by the elastic constitutive law $\sigma = \mathcal{F}(\varepsilon(u))$ is the solution to Problem P_2.
2. If the stress field σ is the solution to Problem P_2, then the displacement field u connected to σ by the elastic constitutive law $\sigma = \mathcal{F}(\varepsilon(u))$ is the solution to Problem P_1.
3. If the displacement field u is the solution to Problem P_1 and the stress field σ is the solution to Problem P_2, then u and σ are connected by the elastic constitutive law $\sigma = \mathcal{F}(\varepsilon(u))$.

6. A Regularized Problem

Due to the nondifferentiability of the functional j given by (14), we introduce a regularized problem P^μ of Problem P_1, depending on a nonnegative parameter μ. We prove the existence of a unique solution u_μ to this problem and we obtain a convergence result of u_μ to the solution of Problem P_1 as $\mu \to 0$.

Indeed, for every parameter $0 \leq \mu < 1$, let $j_\mu : V \longrightarrow \mathbb{R}_+$ be the functional defined by

$$j_\mu(v) = \frac{g}{1+\mu} \int_{\Gamma_3} |v_\tau|^{1+\mu}\, da \qquad \forall\, v \in V \tag{34}$$

Replacing the functional j by j_μ in Problem P_1, we obtain the following regularized problem:

Problem P^μ: Find a displacement field $u_\mu \in H_1$ such that

$$u_\mu \in V, \quad \langle \mathcal{F}(\varepsilon(u_\mu)),\, \varepsilon(v) - \varepsilon(u_\mu) \rangle_{\mathcal{H}} + j_\mu(v) - j_\mu(u_\mu) \geq \langle f, v - u_\mu \rangle_V \qquad \forall\, v \in V \tag{35}$$

Since the functional j_μ is proper, convex and lower semicontinuous, using the same arguments as those used in the proof of Theorem 1, we have

Theorem 4. *Let (10)–(12) hold. Then there exists a unique solution to Problem* P^μ.

Our main interest in this section lies in the behaviour of the solution u_μ of Problem P^μ as $\mu \to 0$. This is the subject of the following result:

Theorem 5. *Let (10)–(12) hold. Then the solution* u_μ *of Problem* P^μ *converges in* V *to the solution* u *of Problem* P_1 *as* $\mu \to 0$, *i.e.*

$$u_\mu \longrightarrow u \quad \text{in } V \quad \text{as } \mu \to 0 \tag{36}$$

Proof. If $v = 0$ in (35), then

$$\langle \mathcal{F}(\varepsilon(u_\mu)),\, \varepsilon(u_\mu)\rangle_{\mathcal{H}} + j_\mu(u_\mu) \;\le\; \langle f, u_\mu\rangle_V \quad \text{for all} \quad 0 \le \mu < 1$$

and, using (10), (9) and the nonnegativity of the functional j_μ, we deduce that the sequence $(u_\mu)_\mu$ is bounded in V. Thus there exist a subsequence denoted again by $(u_\mu)_\mu$ and an element $\tilde{u} \in V$ such that

$$u_\mu \rightharpoonup \tilde{u} \quad \text{weakly in } V \quad \text{as } \mu \to 0 \tag{37}$$

In order to pass to the limit in (35) as $\mu \to 0$, we remark that using (37), (34) and (14) we have

$$\lim_{\mu\to 0} \langle f, v - u_\mu\rangle_V = \langle f, v - \tilde{u}\rangle_V \qquad \forall v \in V \tag{38}$$

and

$$\lim_{\mu\to 0} j_\mu(v) = j(v) \qquad \forall v \in V \tag{39}$$

We will prove now that

$$\liminf_{\mu\to 0} j_\mu(u_\mu) \;\ge\; j(\tilde{u}) \tag{40}$$

Due to the differentiability and the convexity of the functional j_μ given by (34), it follows that

$$j_\mu(u_\mu) - j_\mu(\tilde{u}) \;\ge\; g \int_{\Gamma_3} |\tilde{u}_\tau|^\mu (u_\mu - \tilde{u})\, da \tag{41}$$

Consequently, taking $v = \tilde{u} =$ in (39) and using (41), we deduce that in order to establish (40) it suffices to prove that

$$g \int_{\Gamma_3} |\tilde{u}_\tau|^\mu (u_\mu - \tilde{u})\, da \longrightarrow 0 \quad \text{as } \mu \to 0 \tag{42}$$

Indeed, since the trace map is linear and continuous from H_1 into $L^2(\Gamma)^N$, one can easily deduce from (37) that

$$u_\mu \rightharpoonup \tilde{u} \quad \text{weakly in } L^2(\Gamma_3)^N \quad \text{as } \mu \to 0 \tag{43}$$

Moreover, from the Lebesgue theorem we obtain

$$|\tilde{u}_\tau|^\mu \longrightarrow 1 \quad \text{in} \quad L^2(\Gamma_3)^N \quad \text{as} \quad \mu \to 0 \tag{44}$$

Therefore, using (43) and (44), we establish (42) and consequently (40). In order to pass to the limit in (35) as $\mu \to 0$, we need to prove that

$$\liminf_{\mu \to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \leq \langle \mathcal{F}(\varepsilon(\tilde{u})),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}} \qquad \forall v \in V \tag{45}$$

For this, taking $v = \tilde{u}$ in (35) on the one hand and using the monotonicity of the operator $\mathcal{F}$ (see (10)) on the other hand, we obtain

$$\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \geq j_\mu(u_\mu) - j_\mu(\tilde{u}) + \langle f,\ \tilde{u} - u_\mu\rangle_V$$

and

$$\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \leq \langle \mathcal{F}(\varepsilon(\tilde{u})),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}}$$

Passing to the limit in these inequalities as $\mu \to 0$, from (37)–(40) we see that

$$\liminf_{\mu \to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \geq 0$$

and

$$\limsup_{\mu \to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \leq 0$$

Therefore

$$\lim_{\mu \to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} = 0 \tag{46}$$

Let $v \in V$ and $\theta \in (0,1)$. The monotonicity assumption in (10) applied with u_μ and $w \in V$ given by

$$w = (1-\theta)\tilde{u} + \theta v \tag{47}$$

implies that

$$\begin{aligned}\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} &+ \theta\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}} \\ &\leq \langle \mathcal{F}(\varepsilon(w)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} + \theta\langle \mathcal{F}(\varepsilon(w)),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}}\end{aligned} \tag{48}$$

Using now (46), (37) in (48), we obtain

$$\liminf_{\mu \to 0} \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}} \leq \langle \mathcal{F}(\varepsilon(w)),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}} \tag{49}$$

Moreover, since

$$\begin{aligned}\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} &= \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(\tilde{u}) - \varepsilon(u_\mu)\rangle_{\mathcal{H}} \\ &\quad + \langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v) - \varepsilon(\tilde{u})\rangle_{\mathcal{H}}\end{aligned}$$

from (46) and (49) it follows that

$$\liminf_{\mu\to 0}\big\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(v)-\varepsilon(u_\mu)\big\rangle_{\mathcal{H}} \le \big\langle \mathcal{F}(\varepsilon(w)),\ \varepsilon(v)-\varepsilon(\tilde{u})\big\rangle_{\mathcal{H}} \tag{50}$$

The inequality (45) may be deduced by introducing (47) in (50) and passing to the limit as $\theta \to 0$.

Using now (38)–(40) and (45), we may pass to the limit in (35) as $\mu \to 0$ and obtain that $\tilde{u}$ is a solution to the variational problem (20). Therefore, from the uniqueness of the solution to this problem (see Theorem 1) we deduce that $\tilde{u} = u$. Thus u is the unique weak limit of any subsequence of $(u_\mu)_\mu$. Consequently, the whole sequence $(u_\mu)_\mu$ is weakly convergent in V to u, i.e.

$$u_\mu \rightharpoonup u \quad \text{weakly in } V \quad \text{as } \mu \to 0 \tag{51}$$

In order to obtain (36), let us remark that from (10) and (8) it follows that

$$C|u_\mu - u|_V^2 \le \big\langle \mathcal{F}(\varepsilon(u)),\ \varepsilon(u)-\varepsilon(u_\mu)\big\rangle_{\mathcal{H}} - \big\langle \mathcal{F}(\varepsilon(u_\mu)),\ \varepsilon(u)-\varepsilon(u_\mu)\big\rangle_{\mathcal{H}} \tag{52}$$

where $C > 0$ is a positive constant independent of μ. The strong convergence (36) is finally a consequence of (51) and (46) since $\tilde{u} = u$. ∎

Remark 4. Let u and u_μ be the solutions to the problems P and P^μ given in Theorems 1 and 4, respectively. We define the associated stress fields by

$$\sigma = \mathcal{F}(\varepsilon(u)) \tag{53}$$

and

$$\sigma_\mu = \mathcal{F}(\varepsilon(u_\mu)) \tag{54}$$

Then we have

$$\sigma_\mu \longrightarrow \sigma \quad \text{in } \mathcal{H}_1 \quad \text{as } \mu \to 0 \tag{55}$$

Indeed, it follows from (53), (20) applied with $v = \pm\varphi \in \mathcal{D}(\Omega)^N$ and (1) that

$$\operatorname{Div}\sigma + f_0 = 0 \quad \text{a.e. in } \Omega \tag{56}$$

A similar argument used for (54) and (35) implies that

$$\operatorname{Div}\sigma_\mu + f_0 = 0 \quad \text{a.e. in } \Omega \tag{57}$$

Therefore, by (53)–(54) and (56)–(57) we deduce that

$$|\sigma_\mu - \sigma|_{\mathcal{H}_1} = |\sigma_\mu - \sigma|_{\mathcal{H}} = \big|\mathcal{F}(\varepsilon(u_\mu)) - \mathcal{F}(\varepsilon(u))\big|_{\mathcal{H}} \tag{58}$$

The strong convergence (55) is finally a consequence of (58), (10) and (36).

Remark 5. Let us consider the following contact and friction conditions:

$$u_\nu = 0 \quad \text{on } \Gamma_3, \quad |\sigma_\tau| = -g|u_\tau|^{\mu-1}u_\tau \quad \text{on } \Gamma_3 \tag{59}$$

Using arguments similar to those used in the proof of Lemma 1, one can prove that the solution u_μ to Problem P_μ and the associated stress field σ_μ given by (54) represent a weak solution (in the sense of Lemma 1) to the frictional contact problem (3)–(6), (59).

Remark 6. The strong convergence (36), (55) may be interpreted as follows: the weak solution $\{u, \sigma\}$ to problem (3)–(7) modelling the frictional contact between an elastic body and a rigid foundation may be approximated by the weak solution $\{u_\mu, \sigma_\mu\}$ to problem (3)–(6), (59) which models the frictional contact between the elastic body and the rigid foundation using a more regular friction law. The regularization used here may be of a strong interest in the numerical study of such a type of contact problems.

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Quasistatic Viscoelastic Contact with Normal Compliance and Friction

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Descriptif

Cet article concerne l'étude du problème quasistatique de contact d'un matériau ayant une loi de comportement viscoélastique avec une fondation rigide. Les conditions aux limites de contact considérées obéissent à un modèle général du type conditions de compliance normale.

On considère un milieu continu viscoélastique occupant un domaine Ω de $\mathbb{R}^3$ et dont la frontière Γ, supposée suffisamment régulière, est divisée en trois parties disjointes Γ_1, Γ_2 et Γ_3. On suppose que, pendant l'intervalle de temps $[0,T]$, la partie Γ_1 est encastrée dans une structure fixe, que des forces surfaciques f_2 s'appliquent sur Γ_2 et que des forces volumiques f_0 agissent dans Ω. On suppose aussi qu'un écart g_a sépare la surface de contact potentielle Γ_3 d'une fondation indéformable et que cette distance est mesurée le long de la normale unitaire sortante ν à Ω. Le problème quasistatique de contact qu'on se propose d'étudier se formule de la façon suivante :

Problème P : Trouver le champ des déplacements $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^3$ et le champ des contraintes $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{R}_s^{3\times 3}$ tels que

$$\begin{aligned}
\sigma = \mathcal{A}\varepsilon(\dot{u}) + G\varepsilon(u) \quad & \text{dans} \quad \Omega \times (0,T),\\
Div\,\sigma + f_0 = 0 \quad & \text{dans} \quad \Omega \times (0,T),\\
u = 0 \quad & \text{sur} \quad \Gamma_1 \times (0,T),\\
\sigma\nu = f_2 \quad & \text{sur} \quad \Gamma_2 \times (0,T),\\
-\sigma_\nu = p_\nu(u_\nu - g_a) \quad & \text{sur} \quad \Gamma_3 \times (0,T),\\
|\sigma_\tau| \le p_\tau(u_\nu - g_a) \quad & \text{sur} \quad \Gamma_3 \times (0,T),\\
|\sigma_\tau| < p_\tau(u_\nu - g_a) \; & \Longrightarrow \dot{u}_\tau = 0,\\
|\sigma_\tau| = p_\tau(u_\nu - g_a) \; & \Longrightarrow \sigma_\tau = -\lambda \dot{u}_\tau, \ \lambda \ge 0,\\
u(0) = u_0 \quad & \text{dans} \quad \Omega.
\end{aligned}$$

On note par $\mathbb{R}_s^{3\times 3}$ l'espace des tenseurs symétriques du second ordre sur $\mathbb{R}^3$ et par $\varepsilon(u)$ le tenseur des petites déformations linéarisé. Le point au dessus d'une quantité représente sa dérivée temporelle, $Div\,\sigma$ désigne la divergence de la fonction tensorielle σ et ν la normale unitaire sortante à Ω. $\sigma\nu$ est le vecteur des contraintes de Cauchy, et u_ν, $\dot{u}_\tau$, σ_ν et σ_τ représentent respectivement le déplacement normal, la vitesse tangentielle, les contraintes normales et tangentielles. Les fonctionnelles p_ν et p_τ sont données. La fonctionnelle p_ν représente la pénétration du corps dans la fondation, s'il y a contact, alors que la fonctionnelle p_τ désigne le seuil de frottement.

Il s'agit dans cette publication de l'analyse variationnelle du problème P pour lequel on commence par donner une interprétation mécanique des conditions de compliance normale considérées ici. On poursuit avec une formulation variationnelle du problème P ainsi que de l'existence et l'unicité de la solution. L'étape suivante concerne l'étude de la stabilité du problème par rapport à une perturbation des fonctions de compliance normale p_ν et p_τ. Ceci est très important du point de vue des applications. On s'intéresse aussi à l'étude du problème de contact avec glissement entre un corps viscoélastique et une fondation rigide en mouvement. L'usure du matériau due aux frottements est prise en compte et elle est modélisée par la loi d'Archard. Ce problème se formule de la manière suivante :

Problème PW : Trouver le champ des déplacements $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^3$, le champ des contraintes $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{R}_s^{3\times 3}$ et la fonction usure $w : \Gamma_3 \times [0,T] \longrightarrow \mathbb{R}_+$ tels que

$$\begin{aligned}
\sigma = \mathcal{A}\varepsilon(\dot{u}) + G\varepsilon(u) \quad & \text{dans} \quad \Omega \times (0,T),\\
Div\,\sigma + f_0 = 0 \quad & \text{dans} \quad \Omega \times (0,T),\\
u = 0 \quad & \text{sur} \quad \Gamma_1 \times (0,T),\\
\sigma\nu = f_2 \quad & \text{sur} \quad \Gamma_2 \times (0,T),\\
-\sigma_\nu = p_\nu(u_\nu - w - g_a) \quad & \text{sur} \quad \Gamma_3 \times (0,T),\\
|\sigma_\tau| = p_\tau(u_\nu - w - g_a) \quad & \text{sur} \quad \Gamma_3 \times (0,T),\\
\sigma_\tau = -\lambda\dot{u}_\tau,\ \lambda \geq 0 \quad & \text{sur} \quad \Gamma_3 \times (0,T),\\
\dot{w} = -k_w v^* \sigma_\nu \quad & \text{sur} \quad \Gamma_3 \times (0,T),\\
u(0) = u_0, \quad w(0) = w_0 \quad & \text{dans} \quad \Omega.
\end{aligned}$$

Ici, $k_w > 0$ est le coefficient d'usure supposé constant et $v^* > 0$ est la vitesse de la fondation rigide. A la suite de l'interprétation mécanique des conditions aux limites de frottement avec usure, on établit une formulation variationnelle du problème PW suivie d'un résultat d'existence et d'unicité.

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Quasistatic Viscoelastic Contact with Normal Compliance and Friction

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Abstract. We prove the existence of a unique weak solution for the quasistatic problem of frictional contact between a deformable body and a rigid foundation. The material is assumed to have nonlinear viscoelastic behavior. The contact is modeled with normal compliance and the associated version of Coulomb's law of dry friction. We establish the continuous dependence of the solution on the normal compliance function. Moreover, we prove the existence of a unique solution for the problem of sliding contact with wear.

Key words: viscoelastic material, nonlinear constitutive law, frictional contact, normal compliance, sliding friction, wear.

1. Introduction

Frictional contact between deformable bodies can be frequently found in industry and everyday life such as train wheels with the rails, a shoe with the floor, tectonic plates, the car's braking system, etc. Considerable progress has been made with the modeling and analysis of static contact problems. The mathematical, mechanical and numerical state of the art can be found in the recent proceedings Raous et al. [22]. Only recently, however, have the quasistatic and dynamic problems been considered. The reason lies in the considerable difficulties that the process of frictional contact presents in the modeling and analysis because of the complicated surface phenomena involved. General models for thermoelastic frictional contact, derived from thermodynamical principles, have been obtained in [12], [27] and [28]. Quasistatic contact problems with normal compliance and friction have been considered in [15] and [4], where the existence of weak solutions has been proven. The existence of a weak solution to the, technically very complicated, problem with Signorini's contact condition has been established recently in [7]. The quasistatic frictional contact problem for viscoelastic materials can be found in [25] and the one for elastoviscoplastic materials in [3], [23], [24] and [26]. Dynamic problems with normal compliance were first considered in [19]. The existence of weak solutions to dynamic thermoelastic contact problems with frictional heat generation have been proven in [5] and when wear is taken into account in [6].

In this work we consider the process of frictional contact between a viscoelastic body, which is acted upon by volume forces and surface tractions, and a rigid foundation. We assume that the forces and tractions change slowly in time so that the accelerations in the system are negligible. Neglecting the inertial terms in the equations of motion leads to the quasistatic approximation for the process. The material's constitutive law is assumed to be nonlinear viscoelastic. The contact is modeled with a normal compliance and the friction with the associated Coulomb's law of dry friction. The normal compliance contact condition was proposed in [19] and used in [13], [14], [16] and [5], see also the references therein. This condition allows the interpenetration of the body's surface into the foundation. In [19] and [14] normal compliance was justified by considering the interpenetration and deformation of surface asperities. It was assumed to have the form of a power law. In [2] and [18] it was obtained, via homogenization, from a three-body setting in the limit when the thickness of the thin body situated between the other body and the foundation vanishes. On occasions, it has been employed as a mathematical regularization of Signorini's nonpenetration condition and used as such in numerical solution algorithms. We refer to [11] or [20] for the existence of static problems with Signorini's and Coulomb's conditions. We use a general expression for the normal compliance, similarly to the one in [5] and [6]. In part, the introduction of the normal compliance contact condition, in evolution problems, is motivated by the observation that Signorini's condition, while elegant and easy to explain, leads to discontinuous surface velocities which are associated with infinite tractions on the contact surface. This clearly is physically unrealistic; it leads to severe mathematical and numerical difficulties which do not necessarily represent the physical process. The normal compliance condition predicts large, but finite, contact forces. At any rate, we do not have a completely satisfactory contact condition yet, and maybe it is unrealistic to expect one single condition to model the wide variety of phenomena encountered in frictional contact.

In this paper we establish the existence of a unique solution to the problem, using fixed point arguments. Then we prove the stability of the problem with respect to perturbations of the normal compliance function, which is important from the point of view of applications. We also establish the existence of a weak solution to the problem of sliding contact between a viscoelastic body and a moving rigid body involving the wear of the contacting surface due to friction. The wear is modeled by a version of Archard's law. General models of frictional wear were derived from thermodynamical considerations in [27] and [28] and the dynamic thermoelastic contact problem with surface wear has been analysed in [5].

The paper is organized as follows. Section 2 contains the notations and some preliminary material. In Section 3 we describe the model for the process, set it in a variational form, list the assumptions on the problem data and state our main results. These are the existence of a unique weak solution, Theorem 3.1, and the continuous dependence of the solution on the normal compliance function, Theorem 3.2. The proof of Theorem 3.1 is given in Section 4 and is based on the theory of elliptic variational inequalities and application of fixed point theorems. Theorem 3.2 is established in Section 5, based on the necessary apriori estimates. In Section 6 we describe the contact problem with sliding friction and wear. The existence of a unique weak solution to the problem is stated in Theorem 6.1 and proved using fixed point arguments.

2. Notations and Preliminaries

In this short section we present the notations we shall use and some preliminary material. For further details we refer the reader to [8], [10], [13] or [21]. We denote by S_N the space of second order symmetric tensors on $\mathbb{R}^N$ ($N = 2, 3$), while '$\cdot$' and $|\cdot|$ will represent the inner product and the Euclidean norm on S_N and $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. We shall use the notations

$$H = \{v = (v_i) | v_i \in L^2(\Omega), i = 1, \ldots, N\} = L^2(\Omega)^N,$$

$$H_1 = \{v = (v_i) | v_i \in H^1(\Omega), i = 1, \ldots, N\} = H^1(\Omega)^N,$$

$$\mathcal{H} = \{\tau = (\tau_{ij}) | \tau_{ij} = \tau_{ji} \in L^2(\Omega), i, j = 1, \ldots, N\} = L^2(\Omega)_s^{N\times N},$$

$$\mathcal{H}_1 = \{\tau \in \mathcal{H} | \mathrm{Div}\, \tau \in \mathcal{H}\},$$

where, $i, j = 1, \ldots, N$ and summation over repeated indices is implied. H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with inner products given by

$$\langle u, v\rangle_H = \int_\Omega u_i v_i \,\mathrm{d}x,$$

$$\langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \,\mathrm{d}x,$$

$$\langle u, v\rangle_{H_1} = \langle u, v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}},$$

$$\langle \sigma, \tau\rangle_{\mathcal{H}_1} = \langle \sigma, \tau\rangle_{\mathcal{H}} + \langle \mathrm{Div}\,\sigma, \mathrm{Div}\,\tau\rangle_H,$$

respectively, where $\varepsilon\colon H_1 \to \mathcal{H}$ and $\mathrm{Div}\colon \mathcal{H}_1 \to H$ are the *deformation* and the *divergence* operators, respectively, defined by

$$\varepsilon(v) = (\varepsilon_{ij}(v)), \qquad \varepsilon_{ij}(v) = \tfrac{1}{2}(v_{i,j} + v_{j,i}),$$

$$\mathrm{Div}\,\sigma = (\sigma_{ij,j}).$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively.

Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^N$ and let $\gamma\colon H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$ we use, when no confusion is likely, the notation v for the trace γv of v on Γ. We denote by v_ν and v_τ the *normal* and the *tangential* components of v on Γ given by

$$v_\nu = v \cdot \nu, \qquad v_\tau = v - v_\nu \nu.$$

Let H'_Γ be the dual of H_Γ and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between H'_Γ and H_Γ. For every $\sigma \in \mathcal{H}_1$ let $\sigma\nu$ be the element of H'_Γ given by

$$\langle \sigma\nu, \gamma v \rangle = \langle \sigma, \varepsilon(v) \rangle_{\mathcal{H}} + \langle \operatorname{Div} \sigma, v \rangle_H \quad \forall v \in H_1.$$

We also denote by σ_ν and σ_τ the *normal* and *tangential* traces of σ (see, e.g., [13] or [21]). We recall that if σ is a regular function (say C^1), then

$$\langle \sigma\nu, \gamma v \rangle = \int_\Gamma \sigma\nu \cdot v \, da \quad \forall v \in H_1,$$

where da is the surface measure element, and

$$\sigma_\nu = (\sigma\nu) \cdot \nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

Finally, let $(X, |\cdot|_X)$ be a real normed space, then $C(0, T; X)$ and $C^1(0, T; X)$ denote the spaces of continuous, and continuously differentiable functions from $[0, T]$ to X, with norms

$$|f|_{C(0,T;X)} = \max_{[0,T]} |f(t)|_X,$$

and

$$|f|_{C^1(0,T;X)} = \max_{[0,T]} |f(t)|_X + \max_{[0,T]} |\dot{f}(t)|_X,$$

respectively, where the dot represents the time derivative.

3. The Model and Statement of Results

In this section we describe a model for the process, present its variational formulation, list the assumptions on the problem data and state our main results.

The setting is as follows. A viscoelastic body occupies the domain $\Omega \subset \mathbb{R}^3$ and is acted upon by volume forces and surface tractions. We are interested in the resulting process of evolution of the mechanical state on the time interval $[0, T]$. We assume that a volume force of density f_0 acts in Ω. Γ, the boundary of Ω, is assumed to be Lipschitz, and is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that *means* $\Gamma_1 > 0$. The body is clamped on $\Gamma_1 \times (0, T)$ and so the displacement field vanishes there and surface tractions f_2 act on $\Gamma_2 \times (0, T)$. A gap g_a exists between the potential contact surface Γ_3 and the foundation, and is measured along the outward normal ν.

We denot by u the displacements vector, σ the stress field and $\varepsilon = \varepsilon(u)$ the linearized strain tensor. The viscoelastic constitutive law of the material is chosen as

$$\sigma = \mathcal{A}(\cdot, \dot{\varepsilon}) + \mathcal{G}(\cdot, \varepsilon), \tag{3.1}$$

in which $\mathcal{A}$ and G are given nonlinear constitutive functions. Here and below a dot above a variable represents the time derivative.

We recall that in linear viscoelasticity the stress tensor $\sigma = (\sigma_{ij})$ is given by

$$\sigma_{ij} = a_{ijkl}\varepsilon(\dot{u})_{kl} + d_{ijkl}\varepsilon(u)_{kl},$$

where $\mathcal{A} = (a_{ijkl})$ is the viscosity tensor and $G = (d_{ijkl})$ the elasticity tensor, for $i, j, k, l = 1, \ldots, N$.

Next, we describe the conditions on the potential contact surface Γ_3. We assume the the normal stress satisfies the *normal compliance* condition

$$-\sigma_\nu = p_\nu(\cdot, u_\nu - g_a), \tag{3.2}$$

where u_ν represents the normal displacement, p_ν is a prescribed function, such that $p_\nu(\cdot, r) = 0$ for $r \leqslant 0$, and $u_\nu - g_a$, when positive, represents the penetration of the body's surface asperities into those of the foundation. Such contact condition was proposed in [19] and used in a number of publications, see, e.g., [13], [14], [16], [5] and references there. In this condition the interpenetration is allowed but penalized. In [19] and [14], the following form of the function was employed

$$p_\nu(\cdot, r) = c_\nu (r)_+^{m_\nu},$$

where c_ν is a positive constant, m_ν is a positive exponent and $(r)_+ = \max\{0, r\}$. Formally, Signorini's nonpenetration condition is obtained in the limit $c_\nu \to \infty$. Here we allow for a more general expression, similarly to the one in [5] and [6]. The precise assumption on p_ν will be given below. The associated friction law is chosen as

$$\begin{cases} |\sigma_\tau| \leqslant p_\tau(\cdot, u_\nu - g_a), \\ |\sigma_\tau| < p_\tau(\cdot, u_\nu - g_a) \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau| = p_\tau(\cdot, u_\nu - g_a) \Rightarrow \sigma_\tau = -\lambda \dot{u}_\tau, \ \ \lambda \geqslant 0. \end{cases} \tag{3.3}$$

Here, p_τ is a nonnegative function, the so-called *friction bound*, which satisfies $p_\tau(\cdot, r) = 0$ for $r \leqslant 0$, and additional conditions listed below; $\dot{u}_\tau$ denotes the tangential velocity and σ_τ represents the tangential force on the contact boundary.

This is an appropriate version of Coulomb's law of dry friction. It states that the tangential shear cannot exceed the maximal frictional resistance p_τ. When inequality holds the surface adheres to the foundation and is in the so-called *stick* state, and when equality holds there is a relative sliding, the so-called *slip* state. Therefore, at each time instant the contact surface Γ_3 is divided into three zones: stick, slip and the zone of seperation, in which $u_\nu < g_a$, i.e., there is no contact. The boundaries of these zones are *free boundaries* since they are unknown a priori, and are part of the problem. There is virtually no literature dealing with these free boundaries. In [19] and [14] the following form was used

$$p_\tau(\cdot, r) = c_\tau (r)_+^{m_\tau}.$$

We remark that recently a modified version of Coulomb's law of friction has been derived in [27] and [28], and it is in the form

$$p_\tau = \mu p_\nu (1 - \alpha p_\nu)_+,$$

where α is a small positive material constant related to the wear and penetration hardness of the surface, and μ is the coefficient of friction. The condition (3.3) accomodates such a law.

The mechanical problem of frictional contact of a viscoelastic body may be formulated classically as follows:

Find a displacement field u: $\Omega \times [0, T] \to \mathbb{R}^N$ and a stress field $\sigma : \Omega \times [0, T] \to S_N$ such that

$$\sigma = \mathcal{A}(\cdot, \varepsilon(\dot{u})) + G(\cdot, \varepsilon(u)) \qquad \text{in } \Omega \times (0, T), \tag{3.4}$$

$$\operatorname{Div} \sigma + f_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{3.5}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{3.6}$$

$$\sigma_\nu = f_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{3.7}$$

$$\left\{ \begin{array}{l} \left. \begin{array}{l} -\sigma_\nu = p_\nu(\cdot, u_\nu - g_a) \\ |\sigma_\tau| \leqslant p_\tau(\cdot, u_\nu - g_a), \end{array} \right\} \quad \text{on } \Gamma_3 \times (0, T), \\ |\sigma_\tau| < p_\tau(\cdot, u_\nu - g_a) \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau| = p_\tau(\cdot, u_\nu - g_a) \Rightarrow \sigma_\tau = -\lambda \dot{u}_\tau = 0, \lambda \geqslant 0, \end{array} \right. \tag{3.8}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{3.9}$$

To obtain a variational formulation for problem (3.4)–(3.9) we need the following additional notations. Let V denote the closed subspace of H_1 given by

$$V = \{v \in H_1 | v = 0 \quad \text{on } \Gamma_1\}.$$

Now, Korn's inequality holds, since *meas* $\Gamma_1 > 0$, thus

$$|\varepsilon(u)|_{\mathcal{H}} \geqslant C|u|_{H_1} \quad \forall u \in V, \tag{3.10}$$

see, e.g., [8] or [9]. Here and below C denotes a positive generic constant which may depend on Ω, Γ_1, Γ_2, Γ_3, $\mathcal{A}$, G and T, but does not depend on t nor on the input data f_0, f_2, g_a or u_0, and whose value may vary from place to place. The inner product $\langle \cdot, \cdot \rangle_V$ on V, is chosen as

$$\langle u, v \rangle_V = \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}, \tag{3.11}$$

and it follows form (3.10) that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V. Therefore $(V, |\cdot|_V)$ is a Hilbert space.

In the study of the mechanical problem (3.4)–(3.9) we assume that the *viscosity operator*

$$\mathcal{A}: \Omega \times S_N \to S_N,$$

satisfies

$$\left\{\begin{array}{ll}
\text{(a)} & \text{there exists } L > 0 \text{ such that} \\
& |\mathcal{A}(\cdot, \varepsilon_1) - \mathcal{A}(\cdot, \varepsilon_2)| \leqslant L|\varepsilon_1 - \varepsilon_2| \\
& \forall \varepsilon_1, \varepsilon_2 \in S_N, \ \text{a.e. in } \Omega; \\
\text{(b)} & \text{there exists } m > 0 \text{ such that} \\
& (\mathcal{A}(\cdot, \varepsilon_1) - \mathcal{A}(\cdot, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geqslant m|\varepsilon_1 - \varepsilon_2|^2 \\
& \forall \varepsilon_1, \varepsilon_2 \in S_N, \ \text{a.e. in } \Omega; \\
\text{(c)} & x \mapsto \mathcal{A}(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega \\
& \text{for all } \varepsilon \in S_N; \\
\text{(d)} & x \mapsto \mathcal{A}(x, 0) \in \mathcal{H}.
\end{array}\right. \tag{3.12}$$

The *elasticity operator*

$$G: \Omega \times S_N \to S_N,$$

satisfies

$$\left\{\begin{array}{ll}
\text{(a)} & \text{there exists } L' > 0 \text{ such that} \\
& |G(\cdot, \varepsilon_1) - G(\cdot, \varepsilon_2)| \leqslant L'|\varepsilon_1 - \varepsilon_2| \\
& \forall \varepsilon_1, \varepsilon_2 \in S_N, \ \text{a.e. in } \Omega; \\
\text{(b)} & x \mapsto G(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega \\
& \text{for all } \varepsilon \in S_N; \\
\text{(c)} & x \mapsto G(x, 0) \in \mathcal{H}.
\end{array}\right. \tag{3.13}$$

The *normal compliance* functions

$$p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \quad (r = \nu, \tau),$$

satisfy

$$\begin{cases} \text{(a)} \ \text{there exist } L_r > 0 \text{ such that} \\ \qquad |p_r(\cdot, u_1) - p_r(\cdot, u_2)| \leqslant L_r |u_1 - u_2| \\ \qquad \forall u_1, u_2 \in \mathbb{R}, \ \text{a.e. on } \Gamma_3; \\ \text{(b)} \ x \mapsto p_r(x, u) \text{ is Lebesgue measurable on } \Gamma_3 \\ \qquad \text{for all } u \in \mathbb{R}; \\ \text{(c)} \ x \mapsto p_r(x, u) = 0 \quad u \leqslant 0. \end{cases} \tag{3.14}$$

Examples used in the literature can be found above. The forces and tractions satisfy

$$f_0 \in C(0, T; H), \qquad f_2 \in C(0, T; L^2(\Gamma_2)^N). \tag{3.15}$$

Moreover, the *gap* function satisfies

$$g_a \in L^\infty(\Gamma_3) \quad \text{and} \quad g_a \geqslant 0 \ \text{a.e. on } \Gamma_3, \tag{3.16}$$

and, finally,

$$u_0 \in V. \tag{3.17}$$

Next, we denote by $F(t)$ the element of V' given by

$$\langle F(t), v \rangle_V = \langle f_0(t), v \rangle_H + \langle f_2(t), \gamma v \rangle_{L^2(\Gamma_2)^N}, \tag{3.18}$$

for all $v \in V$ and $t \in [0, T]$, and we note that the conditions (3.15) imply

$$F \in C(0, T; V'). \tag{3.19}$$

Let $j: V \times V \to \mathbb{R}$, $j_\nu: V \times V \to \mathbb{R}$ and $j_\tau: V \times V \to \mathbb{R}$ be the functionals

$$\begin{cases} j_\nu(v, w) = \displaystyle\int_{\Gamma_3} p_\nu(\cdot, v_\nu - g_a) w_\nu \, da, \\ j_\tau(v, w) = \displaystyle\int_{\Gamma_3} p_\tau(\cdot, v_\nu - g_a) |w_\tau| \, da, \\ j(v, w) = j_\nu(v, w) + j_\tau(v, w). \end{cases} \tag{3.20}$$

It is straightforward to show that if $\{u, \sigma\}$ are sufficiently regular functions satisfying (3.4)–(3.8) then

$$\langle \sigma(t), \varepsilon(w) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(u(t), w) - j(u(t), \dot{u}(t)) \geqslant \langle F(t), w - \dot{u}(t)\rangle_V \quad \forall w \in V, \tag{3.21}$$

for all $t \in [0, T]$. Thus, by (3.4), (3.9) and (3.21) we obtain the following variational formulation of problem (3.4)–(3.9)

Problem P: Find a displacement field $u: [0, T] \to V$ and a stress field $\sigma: [0, T] \to \mathcal{H}_1$ such that

$$\sigma(t) = \mathcal{A}(\cdot, \varepsilon(\dot{u}(t))) + G(\cdot, \varepsilon(u(t))), \tag{3.22}$$

$$\langle \sigma(t), \varepsilon(w) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(u(t), w) - j(u(t), \dot{u}(t)) \geqslant \langle F(t), w - \dot{u}(t)\rangle_V \quad \forall w \in V, \tag{3.23}$$

$$u(0) = u_0, \tag{3.24}$$

for all $t \in [0, T]$. A pair of functions $\{u, \sigma\}$ which satisfies (3.22)–(3.24) is called a *weak solution* of problem (3.4)–(3.9). Our main result, which we establish in the next section, is the following:

THEOREM 3.1. *Assume that* (3.12)–(3.17) *hold. Then, there exists a unique solution* $\{u, \sigma\}$ *of problem P. Moreover, the solution satisfies*

$$u \in C^1(0, T; V), \qquad \sigma \in C(0, T; \mathcal{H}_1).$$

We conclude that $\{u, \sigma\}$ is the unique weak solution to problem (3.4)–(3.9).

Next we investigate the behavior of the weak solutions to problem (3.4)–(3.9), with respect to perturbations of the normal compliance functions p_ν and p_τ. To this end, we suppose that (3.12)–(3.17) hold. For every $\alpha \geqslant 0$, let p_r^α be a perturbation of p_r which satisfies (3.14) with the Lipschitz constant $L_r^\alpha (r = \nu, \tau)$. Let us introduce the functionals j^α, j_ν^α and j_τ^α which are obtained from j, j_ν and j_τ by replacing p_ν and p_τ with p_ν^α and p_τ^α respectively. We consider now the following problem:

Problem P^α: For $\alpha \geqslant 0$, find a displacement field $u^\alpha : [0, T] \to V$ and a stress field $\sigma^\alpha: [0, T] \to \mathcal{H}_1$ such that

$$\sigma^\alpha(t) = \mathcal{A}(\cdot, \varepsilon(\dot{u}^\alpha(t))) + G(\cdot, \varepsilon(u^\alpha(t))), \tag{3.25}$$

$$\langle \sigma^\alpha(t), \varepsilon(w) - \varepsilon(\dot{u}^\alpha(t))\rangle_{\mathcal{H}} + j^\alpha(u^\alpha(t), w) - j^\alpha(u^\alpha(t), \dot{u}^\alpha(t)) \geqslant \langle F(t), w - \dot{u}^\alpha(t)\rangle_V \quad \forall w \in V, \tag{3.26}$$

for all $t \in [0, T]$, and

$$u^\alpha(0) = u_0. \tag{3.27}$$

We deduce from Theorem 3.1 that for each $\alpha \geqslant 0$ problem P^α has a unique solution $\{u^\alpha, \sigma^\alpha\}$ satisfying

$$u^\alpha \in C^1(0, T; V), \qquad \sigma^\alpha \in C(0, T; \mathcal{H}_1).$$

Let us suppose now that the normal compliance functions satisfy the following assumptions: there exist $\beta_r \in \mathbb{R}$, $M_r \geqslant 0$, $\varphi_r : \mathbb{R}_+ \to [0, M_r]$ $(r = \nu, \tau)$ and $M > 0$ such that

$$\begin{cases} \text{(a)} \ |p_r^\alpha(\cdot, u) - p_r(\cdot, u)| \leqslant \varphi_r(\alpha)(|u| + \beta_r) \quad \forall u \in \mathbb{R}, \text{ a.e. on } \Gamma_3, \\ \text{(b)} \ \lim_{\alpha \to 0} \varphi_r(\alpha) = 0, \\ \text{(c)} \ L_r^\alpha \leqslant M \quad \forall \alpha \geqslant 0. \end{cases} \tag{3.28}$$

Under these assumptions, we have the following stability result:

THEOREM 3.2. *The solutions $\{u^\alpha, \sigma^\alpha\}$ of Problems P^α converge uniformly to the solution $\{u, \sigma\}$ of Problem P:*

$$u^\alpha \to u \ \text{ in } \ C^1(0, T, V), \qquad \sigma^\alpha \to \sigma \ \text{ in } \ C(0, T, \mathcal{H}_1)$$
$$\text{when } \alpha \to 0. \tag{3.29}$$

The proof will be given in Section 5. In addition to the mathematical interest in this result, it is of importance in applications, since it indicates that small inaccuracies in the condition lead to small inaccuracies in the solutions.

The problem with sliding frictional wear is described and analysed in Section 6 where we establish, in Theorem 6.1, an existence and uniqueness result.

4. Proof of Theorem 3.1

The proof of Theorem 3.1 is based on fixed point arguments, similar to those used in [3], [25] or [26], but in a different setting and different choice of the operators. It will be carried out in several steps. We assume that (3.12)–(3.17) hold. To simplify the notation, we shall not indicate explicitly the dependence on t.

In the first step we assume that the contact displacements are given and so is the elastic part of the stress field. Let $\eta \in C(0, T; \mathcal{H})$ and $g \in C(0, T; V)$, and consider the following variational problem:

Problem $P_{\eta g}$: Find $v_{\eta g}: [0, T] \to \mathcal{H}_1$ such that

$$\sigma_{\eta g}(t) = \mathcal{A}(\cdot, \varepsilon(v_{\eta g}(t))) + \eta(t), \tag{4.1}$$

$$\langle \sigma_{\eta g}(t), \varepsilon(w) - \varepsilon(v_{\eta g}(t)) \rangle_{\mathcal{H}} + j(g(t), w) - j(g(t), v_{\eta g}(t)) \geqslant \langle F(t), w - v_{\eta g}(t) \rangle_V, \tag{4.2}$$

for all $w \in V$ and $t \in [0, T]$. Clearly, we solve the problem for the velocity and stress fields. We note that the use of given g can be found in [11] and [20]. We have the following result:

PROPOSITION 4.1. *There exists a unique solution to problem* $P_{\eta g}$ *such that*

$$v_{\eta g} \in C(0, T; V), \qquad \sigma_{\eta g} \in C(0, T; \mathcal{H}_1).$$

Proof. It follows from classical results for elliptic variational inequalities, see, e.g., [8, 10, 21] that there exists a unique pair $\{v_{\eta g}, \sigma_{\eta g}\}$, $v_{\eta g}(t) \in V$ and $\sigma_{\eta g}(t) \in \mathcal{H}$, which is a solution of (4.1) and (4.2). Choosing $w = v_{\eta g}(t) \pm \varphi$ in (4.2), where $\varphi \in \mathcal{D}(\Omega)^N$, we find

$$\langle \sigma_{\eta g}(t), \varepsilon(\varphi) \rangle_{\mathcal{H}} = \langle F(t), \varphi \rangle_V.$$

Using (3.18) we deduce

$$\text{Div}\, \sigma_{\eta g}(t) + f_0(t) = 0 \quad \text{in } \Omega, \tag{4.3}$$

and then assumption (3.15) and equation (4.3) imply that $\sigma_{\eta g}(t) \in \mathcal{H}_1$.

Now, let $t_1, t_2 \in [0, T]$ and for the sake of simplicity we denote $v_{\eta g}(t_i) = v_i$, $\sigma_{\eta g}(t_i) = \sigma_i$, $g(t_i) = g_i$, $\eta(t_i) = \eta_i$, $F(t_i) = F_i$, for $i = 1, 2$. Using (4.1) and (4.2), and algebraic manipulations we find

$$\begin{aligned} &\langle \mathcal{A}(\cdot, \varepsilon(v_1)) - \mathcal{A}(\cdot, \varepsilon(v_2)), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\ &\quad \leqslant \langle F_1 - F_2, v_1 - v_2 \rangle_V - \langle \eta_1 - \eta_2, \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \\ &\qquad + j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2). \end{aligned} \tag{4.4}$$

Moreover, it follows form (3.10) and (3.12) that

$$\langle \mathcal{A}(\cdot, \varepsilon(v_1)) - \mathcal{A}(\cdot, \varepsilon(v_2)), \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} \geqslant C|v_1 - v_2|_V^2. \tag{4.5}$$

Now,

$$\begin{aligned} &j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2) \\ &\quad = \int_{\Gamma_3} (p_\nu(\cdot, g_{1\nu} - g_a) - p_\nu(\cdot, g_{2\nu} - g_a))(v_{2\nu} - v_{1\nu})\, da \\ &\qquad + \int_{\Gamma_3} (p_\tau(\cdot, g_{1\nu} - g_a) - p_\tau(\cdot, g_{2\nu} - g_a))(|v_{2\tau}| - |v_{1\tau}|)\, da, \end{aligned}$$

and by using (3.14), we see that

$$j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2) \leqslant C|g_1 - g_2|_V |v_1 - v_2|_V. \quad (4.6)$$

Using now (4.4)–(4.6) we find

$$|v_1 - v_2|_V \leqslant C(|F_1 - F_2|_V + |\eta_1 - \eta_2|_{\mathcal{H}} + |g_1 - g_2|_V). \quad (4.7)$$

Moreover, we obtain from (3.12) and (4.1)

$$|\sigma_1 - \sigma_2|_{\mathcal{H}} \leqslant C(|v_1 - v_2|_V + |\eta_1 - \eta_2|_{\mathcal{H}}). \quad (4.8)$$

We obtain from (3.19), (4.7) and (4.8) that $v_{\eta g} \in C(0, T; V)$ and $\sigma_{\eta g} \in C(0, T; \mathcal{H})$, and then it follows from (3.15) and (4.3) that $\sigma_{\eta g} \in C(0, T; \mathcal{H}_1)$. This concludes the proof.

Let us consider now the operator $\Lambda_\eta : C(0, T; V) \to C(0, T; V)$ defined by

$$\Lambda_\eta g = g_\eta \quad g \in C(0, T; V), \quad (4.9)$$

where

$$g_\eta(t) = u_0 + \int_0^t v_{\eta g}(s)\, ds \quad t \in [0, T]. \quad (4.10)$$

We have

PROPOSITION 4.2. *The operator Λ_η has a unique fixed point $g_\eta^* \in C(0, T; V)$.*

Proof. Let $g_1, g_2 \in C(0, T; V)$ and let $\eta \in C(0, T; \mathcal{H})$. For the sake of simplicity we denote by $\{v_i, \sigma_i\}$, $i = 1, 2$, the solutions of problems $P_{\eta g_i}$, i.e., $v_i = v_{\eta g_i}$ and $\sigma_i = \sigma_{\eta g_i}$. Using (4.9) and (4.10) we have

$$|\Lambda_\eta g_1(t) - \Lambda_\eta g_2(t)|_V \leqslant \int_0^t |v_1(s) - v_2(s)|_V\, ds \quad \forall t \in [0, T]. \quad (4.11)$$

Using estimates similar to those in the proof of Proposition 4.1 (see (4.4)–(4.7)) we see that

$$|v_1(s) - v_2(s)|_V \leqslant C|g_1(s) - g_2(s)|_V.$$

Taking into account (4.11) we obtain

$$|\Lambda_\eta g_1(t) - \Lambda_\eta g_2(t)|_V \leqslant C \int_0^t |g_1(s) - g_2(s)|_V\, ds \quad \forall t \in [0, T]. \quad (4.12)$$

Reiterating this inequality n times we are led to

$$|\Lambda_\eta^n g_1(t) - \Lambda_\eta^n g_2(t)|_{C(0,T;V)} \leqslant \frac{C^n}{n!}|g_1 - g_2|_{C(0,T;V)},$$

which implies that for a sufficiently large n the operator Λ_η^n is a contraction on $C(0, T; V)$. Thus, there exists a unique $g_\eta^* \in C(0, T; V)$ such that $\Lambda_\eta^n g_\eta^* = g_\eta^*$, and then, g_η^* is the unique fixed point of Λ_η too.

In the sequel, for $\eta \in C(0, T; \mathcal{H})$, we denote by g_η^* the fixed point given in Proposition 4.2. Let $v_\eta \in C(0, T; V)$ and $\sigma_\eta \in (0, T; \mathcal{H}_1)$ be the functions given by

$$v_\eta = v_{\eta g_\eta^*}, \qquad \sigma_\eta = \sigma_{\eta g_\eta^*}. \tag{4.13}$$

Moreover, using (4.10) and (4.13), we let u_η: $[0, T] \to V$ be the function

$$u_\eta(t) = g_\eta(t)^* = u_0 + \int_0^t v_\eta(s)\,\mathrm{d}s, \tag{4.14}$$

for $t \in [0, T]$, and we define the operator Λ: $C(0, T; \mathcal{H}) \to C(0, T; \mathcal{H})$ by

$$\Lambda\eta(t) = G(\cdot, \varepsilon(u_\eta(t))) \quad \eta \in C(0, T; \mathcal{H}), \quad t \in [0, T]. \tag{4.15}$$

We have

PROPOSITION 4.3. *The operator Λ has a unique fixed point $\eta^* \in C(0, T; \mathcal{H})$.*

Proof. Let $\eta_1, \eta_2 \in C(0, T; \mathcal{H})$ and let $v_i = v_{\eta_i}, \sigma_i = \sigma_{\eta_i}, u_i = u_{\eta_i}, g_i = g_{\eta_i}^*$, for $i = 1, 2$. Using (4.10), (4.13) and (4.14) we have $g_i = u_i$ and from (4.1) and (4.2) we obtain

$$\sigma_i(t) = \mathcal{A}(\cdot, \varepsilon(v_i(t))) + \eta_i(t), \tag{4.16}$$

$$\begin{aligned}&\langle\sigma_i(t), \varepsilon(w) - \varepsilon(v_i(t))\rangle_{\mathcal{H}} + j(u_i(t), w) - j(u_i(t), v_i(t))\\ &\quad \geqslant \langle F(t), w - v_i(t)\rangle_V \quad \forall w \in V,\end{aligned} \tag{4.17}$$

where $i = 1, 2$ and $t \in [0, T]$. It follows from (4.14), (4.16), (4.17) and the estimates in the proof of Proposition 4.1 (see (4.4)–(4.7)) that

$$\begin{aligned}|u_1(t) - u_2(t)|_V &\leqslant \int_0^t |v_1(s) - v_2(s)|_V\,\mathrm{d}s\\ &\leqslant C\int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}}\,\mathrm{d}s + C\int_0^t |u_1(s) - u_2(s)|_V\,\mathrm{d}s,\end{aligned}$$

for $t \in [0, T]$. Using now a Gronwall-type inequality we obtain

$$|u_1(t) - u_2(t)|_V \leqslant C \int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}} \,\mathrm{d}s \quad \forall t \in [0, T],$$

and taking into account (3.13) and (4.14) we are led to

$$|\Lambda\eta_1(t) - \Lambda_{\eta_2}(t)|_{\mathcal{H}} \leqslant C \int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}} \,\mathrm{d}s, \tag{4.18}$$

$\forall t \in [0, T]$, which together with Banach's fixed point theorem implies the Proposition.

We have now all the ingredients needed to prove the theorem.

Proof of Theorem 3.1.
Existence. Let $\eta^* \in C(0, T; \mathcal{H})$ be the fixed point of Λ and let v_{η^*}, and u_{η^*} be the functions given by (4.13) and (4.14) for $\eta = \eta^*$. We show that $\{u_{\eta^*}, \sigma_{\eta^*}\}$ is a solution of probelm P. Indeed, choosing $\eta = \eta^*$, $g = g^*_{\eta^*}$ in (4.1) and (4.2) and using (4.13) we obtain

$$\sigma_{\eta^*} = \mathcal{A}(\cdot, \varepsilon(v_{\eta^*})) + \eta^*, \tag{4.19}$$

$$\langle\sigma_{\eta^*}, \varepsilon(w) - \varepsilon(v_{\eta^*})\rangle_{\mathcal{H}} + j(g^*_{\eta_*}, w) - j(g^*_{\eta^*}, v_{\eta^*}) \\ \geqslant \langle F, w - v_{\eta^*}\rangle_V \quad \forall w \in V, \tag{4.20}$$

for all $t \in [0, T]$. Now, equality (3.22) follows from (4.14), (4.15) and (4.19), since

$$v_{\eta^*} = \dot{u}_{\eta^*}, \qquad \eta^* = \Lambda\eta^* = G(\cdot, \varepsilon(u_{\eta^*})),$$

while the inequality (3.23) follows from (4.10), (4.14) and (4.19), since

$$g^*_{\eta^*} = u_{\eta^*}.$$

The equality (3.24) results from (4.14), and the regularity $u_{\eta^*} \in C^1(0, T; V)$ and $\sigma_{\eta^*} \in C(0, T; \mathcal{H}_1)$ is a consequence of Proposition 4.1, (3.17) and (4.14).

Uniqueness. To prove the uniqueness of the solution let $\{u_{\eta^*}, \sigma_{\eta^*}\}$ be the solution of (3.22)–(3.24) obtained above and let $\{u, \sigma\}$ be another solution such that $u \in C^1(0, T; V)$ and $\sigma \in C(0, T; \mathcal{H}_1)$. We denote by $\eta \in C(0, T; \mathcal{H})$ the function

$$\eta(t) = G(\cdot, \varepsilon(u(t))), \tag{4.21}$$

for $t \in [0, T]$, and let

$$v = \dot{u}. \tag{4.22}$$

Using (3.22) and (3.23) we obtain that $\{v, \sigma\}$ is a solution of the variational problem $P_{\eta u}$ and since this problem has a unique solution $v_{\eta u} \in C(0, T; V), \sigma_{\eta u} \in C(0, T; \mathcal{H}_1)$, we conclude that

$$v = v_{\eta u}, \qquad \sigma = \sigma_{\eta u}. \tag{4.23}$$

Moreover, from (3.24), (4.22) and (4.23) we obtain

$$u(t) = u_0 + \int_0^t v_{\eta u}(s)\, \mathrm{d}s,$$

$t \in [0, T]$, i.e., u is a fixed point of Λ_η, given by (4.9). It follows from Proposition 4.2 that $u = g_\eta^*$ and by (4.23) we have

$$v = v_{\eta g_\eta^*}, \qquad \sigma = \sigma_{\eta g_\eta^*}. \tag{4.24}$$

Then, (4.13) and (4.24) imply

$$v = v_\eta, \qquad \sigma = \sigma_\eta. \tag{4.25}$$

Moreover, it follows from (3.24), (4.14), (4.22) and (4.25) that

$$u = u_\eta. \tag{4.26}$$

Using now (4.15), (4.21) and (4.26) we obtain that $\Lambda\eta = \eta$ and by the uniqueness of the fixed point of Λ we have

$$\eta = \eta^*. \tag{4.27}$$

The uniqueness of the solution is now a consequence of (4.25)–(4.27). The proof of Theorem 3.1 is complete.

5. Continuous Dependence on Contact Conditions

In this section we prove Theorem 3.2. Let $\alpha \geqslant 0$. To simplify the notations, we shall not indicate explicitly the dependence on t. Everywhere below C will represent a positive constant which depends on the data but is independent of α. Using (3.22), (3.23), (3.25) and (3.26), we obtain

$$\begin{aligned} &\langle \mathcal{A}(\cdot, \varepsilon(\dot{u})) - \mathcal{A}(\cdot, \varepsilon(\dot{u}^\alpha)), \varepsilon(\dot{u}) - \varepsilon(\dot{u}^\alpha)\rangle_{\mathcal{H}} \\ &\quad \leqslant -\langle G(\cdot, \varepsilon(u)) - G(\cdot, \varepsilon(u^\alpha)), \varepsilon(\dot{u}) - \varepsilon(\dot{u}^\alpha)\rangle_{\mathcal{H}} \\ &\qquad + j(u, \dot{u}^\alpha) - j(u, \dot{u}) + j^\alpha(u^\alpha, \dot{u}) - j^\alpha(u^\alpha, \dot{u}^\alpha). \end{aligned} \tag{5.1}$$

Moreover, it follows from (3.10), (3.12) and (3.13) that

$$\begin{aligned}&\langle \mathcal{A}(\cdot,\varepsilon(\dot u)) - \mathcal{A}(\cdot,\varepsilon(\dot u^\alpha)), \varepsilon(\dot u) - \varepsilon(\dot u^\alpha)\rangle_{\mathcal{H}} \geqslant C|\dot u - \dot u^\alpha|_V^2,\\ &-\langle G(\cdot.\varepsilon(u)) - G(\cdot,\varepsilon(u^\alpha)), \varepsilon(\dot u) - \varepsilon(\dot u^\alpha)\rangle_{\mathcal{H}} \leqslant C|u - u^\alpha|_V|\dot u - \dot u^\alpha|_V.\end{aligned} \tag{5.2}$$

We note that

$$\begin{aligned}&j(u,\dot u^\alpha) - j(u,\dot u) + j^\alpha(u^\alpha,\dot u) - j^\alpha(u^\alpha,\dot u^\alpha)\\ &\quad= \int_{\Gamma_3}(p_\nu(\cdot,u_\nu - g_a) - p_\nu^\alpha(\cdot,u_\nu^\alpha - g_a))(\dot u_\nu^\alpha - \dot u_\nu)\,\mathrm{d}a\\ &\qquad+\int_{\Gamma_3}(p_\tau(\cdot,u_\nu - g_a) - p_\tau^\alpha(\cdot,u_\nu^\alpha - g_a))(|\dot u_\tau^\alpha| - |\dot u_\tau|)\,\mathrm{d}a,\end{aligned}$$

and, using (3.14) we find

$$\begin{aligned}&j(u,\dot u^\alpha) - j(u,\dot u) + j^\alpha(u^\alpha,\dot u) - j^\alpha(u^\alpha,\dot u^\alpha)\\ &\quad\leqslant C|p_\nu(\cdot,u_\nu - g_a) - p_\nu^\alpha(\cdot,u_\nu^\alpha - g_a)|_{L^2(\Gamma_3)}|\dot u - \dot u^\alpha|_V\\ &\qquad+C|p_\tau(\cdot,u_\nu - g_a) - p_\tau^\alpha(\cdot,u_\nu^\alpha - g_a)|_{L^2(\Gamma_3)}|\dot u - \dot u^\alpha|_V.\end{aligned} \tag{5.3}$$

Now (5.1)–(5.3) imply

$$\begin{aligned}|\dot u - \dot u^\alpha|_V &\leqslant C|u - u^\alpha|_V + C|p_\nu(\cdot,u_\nu - g_a) - p_\nu^\alpha(\cdot,u_\nu^\alpha - g_a)|_{L^2(\Gamma_3)}\\ &\qquad+C|p_\tau(\cdot,u_\nu - g_a) - p_\tau^\alpha(\cdot,u_\nu^\alpha - g_a)|_{L^2(\Gamma_3)}.\end{aligned} \tag{5.4}$$

Let now $r = \nu$ or τ. Then

$$\begin{aligned}&|p_r(\cdot,u_\nu - g_a) - p_r^\alpha(\cdot,u_\nu^\alpha - g_a)|_{L^2(\Gamma_3)}\\ &\quad\leqslant |p_r(\cdot,u_\nu - g_a) - p_r^\alpha(\cdot,u_\nu - g_a)|_{L^2(\Gamma_3)}\\ &\qquad+|p_r^\alpha(\cdot,u_\nu - g_a) - p_r^\alpha(\cdot,u_\nu^\alpha - g_a)|_{L^2(\Gamma_3)},\end{aligned}$$

and, taking into account (3.14) and (3.28), we obtain

$$|p_r(\cdot,u_\nu - g_a) - p_r^\alpha(\cdot,u_\nu^\alpha - g_a)|_{L^2(\Gamma_3)} \leqslant C|u - u^\alpha|_V + C\varphi_r(\alpha). \tag{5.5}$$

Combining now (5.4) and (5.5), we deduce

$$|\dot u - \dot u^\alpha|_V \leqslant C|u - u^\alpha|_V + C(\varphi_\nu(\alpha) + \varphi_\tau(\alpha)). \tag{5.6}$$

Integrating over $(0, t)$ and using (3.24) and (3.27) we find

$$|u - u^\alpha|_V \leqslant \int_0^t |\dot u - \dot u^\alpha|_V\,\mathrm{d}s \leqslant C\int_0^t |u - u^\alpha|_V\,\mathrm{d}s + C(\varphi_\nu(\alpha) + \varphi_\tau(\alpha)),$$

and, using a Gronwall-type inequality, we get

$$|u - u^\alpha|_V \leqslant C(\varphi_\nu(\alpha) + \varphi_\tau(\alpha)). \tag{5.7}$$

Moreover, we obtain from (3.12), (3.13), (3.22) and (3.25) that

$$|\sigma - \sigma^\alpha|_{\mathcal{H}} \leqslant C(|\dot{u} - \dot{u}^\alpha|_V + |u - u^\alpha|_V),$$

and, since by (4.3) $\operatorname{Div}\sigma = \operatorname{Div}\sigma^\alpha = -f_0$, we obtain that

$$|\sigma - \sigma^\alpha|_{\mathcal{H}_1} \leqslant C(|\dot{u} - \dot{u}^\alpha|_V + |u - u^\alpha|_V). \tag{5.8}$$

Using now (5.6)–(5.8) we find

$$|\sigma - \sigma^\alpha|_{\mathcal{H}_1} \leqslant C(\varphi_\nu(\alpha) - \varphi_\tau(\alpha)), \tag{5.9}$$

and finally, it follows from (5.6), (5.7) and (5.9) that

$$|u - u^\alpha|_{C^1(0,T;V)} + |\sigma - \sigma^\alpha|_{C(0,T;\mathcal{H}_1)} \leqslant C(\varphi_\nu(\alpha) + \varphi_\tau(\alpha)).$$

Theorem 3.2 is now a consequence of this inequality and assumption (3.28).

6. The Problem with Wear

In this section we study the problem of wear of the contact surface due to friction. We consider the process as described in Section 3 but, now, the foundation is assumed to move steadily and only sliding contact takes place.

Let Ω be the reference configuration of the body. We use the notations as given above; in particular, Γ_3 represents the contact surface, u denotes the displacements and σ the stress. In addition, we introduce the *wear* function $w: \Gamma_3 \times (0, T) \to \mathbb{R}_+$ which represents the accumulated wear of the surface, the evolution of which is governed by a modified version of Archard's law (see, e.g., [27], [28] or [5] and references therein) which we now describe. The rate form of Archard's law is

$$\dot{w} = -k_w \sigma_\nu |\dot{u}_\tau - v^*|,$$

where k_w is the wear coefficient (very small in practice), v^* is the velocity of the foundation and $|\dot{u}_\tau - v^*|$ is the relative velocity between the contact surface and the foundation. Thus, the rate of the wear depends linearly on the contact stress and the slip. We also suppose that on the time scale of the quasistatic process, the surface rearranges itself in such a way that the velocity $\dot{u}_\tau$ is negligible and so, the slip is just v^*, which for the sake of simplicity is assumed to be a positive constant. Thus, we employ the following version of the wear law

$$\dot{w} = -k_w v^* \sigma_\nu,$$

where k_w = const. > 0 and v^* = const. > 0. The case when k_w is a sufficiently smooth function can be considered, but for the sake of simplicity we keep it a constant. Now, $\sigma_\nu \leqslant 0$ and therefore $\dot{w} \geqslant 0$, which is needed if we are to interpret w as the wear of the surface. We replace the contact conditions (3.8) with

$$\begin{cases} -\sigma_\nu = p_\nu(\cdot, u_\nu - w - g_a), \\ |\sigma_\tau| = p_\tau(\cdot, u_\nu - w - g_a), \qquad \sigma_\tau = -\lambda v^*, \quad \lambda \geqslant 0, \\ \dot{w} = -k_w v^* \sigma_\nu, \end{cases} \tag{6.1}$$

on $\Gamma_3 \times (0, T)$. Note that the wear appears in the normal compliance condition. In this way we take into account the material removal that takes place on the surface, but it makes the problem coupled, i.e., more complicated. We remark that from the mathematical point of view we could use Coulomb's condition (3.3) in (6.1) and obtain the same results, but, then the physical interpretation of the problem would not make much sense.

The classical formulation of the problem of *sliding frictional contact with wear* is given by:
Find a displacement field $u: \Omega \times [0, T] \to \mathbb{R}^N$, a stress field $\sigma: \Omega \times [0, T] \to S_N$ and a wear function $w: \Gamma_3 \times [0, T] \to \mathbb{R}$ such that (3.4)–(3.7), (6.1) and (3.9) hold.

Let F be the function defined by (3.18) and let $h: V \times V \times L^2(\Gamma_3) \to \mathbb{R}$, $h_\nu: V \times V \times L^2(\Gamma_3) \to \mathbb{R}$ and $h_\tau: V \times V \times L^2(\Gamma_3) \to \mathbb{R}$ be the functionals

$$\begin{aligned} h_\nu(u, \xi, w) &= \int_{\Gamma_3} p_\nu(\cdot, u_\nu - w - g_a)\xi_\nu \, da, \\ h_\tau(u, \xi, w) &= \int_{\Gamma_3} p_\tau(\cdot, u_\nu - w - g_a)|\xi_\tau| \, da, \end{aligned} \tag{6.2}$$

and

$$h(u, \xi, w) = h_\nu(u, \xi, w) + h_\tau(u, \xi, w).$$

It is straightforward to show that if $\{u, \sigma, w\}$ are sufficiently regular functions satisfying (3.5)–(3.7) and (6.1) then

$$\begin{aligned} &\langle \sigma(t), \varepsilon(\xi) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + h(u(t), \xi, w(t)) - h(u(t), \dot{u}(t), w(t)) \\ &\quad \geqslant \langle F(t), \xi - \dot{u}(t)\rangle_V, \end{aligned} \tag{6.3}$$

for all $\varepsilon \in V$ and $t \in [0, T]$. Thus, by (3.4), (3.9) and (6.3) the variational formulation of the problem with wear is:

Problem PW: Find a displacement field $u: [0, T] \to V$, a stress field σ : $[0, T] \to \mathcal{H}_1$ and a wear function $w: [0, T] \to L^2(\Gamma_3)$ such that

$$\sigma(t) = \mathcal{A}(\cdot, \varepsilon(\dot{u}(t))) + G(\cdot, \varepsilon(u(t))), \tag{6.4}$$

$$\dot{w}(t) = -k_w v^* \sigma_\nu(t), \tag{6.5}$$

$$\langle \sigma(t), \varepsilon(\xi) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + h(u(t), \xi, w(t)) - h(u(t), \dot{u}(t), w(t))$$

$$\geqslant \langle F(t), \xi - \dot{u}(t)\rangle_V, \tag{6.6}$$

$$u(0) = u_0, \qquad w(0) = 0, \tag{6.7}$$

for all $\xi \in V$ and $t \in [0, T]$. A triplet $\{u, \sigma, w\}$ which satisfies (6.4)–(6.7) is called a weak solution of the problem of sliding frictional contact with wear.

The main result in this section is:

THEOREM 6.1. *Assume that* (3.12)–(3.17) *hold. Then, there exists a unique solution* $\{u, \sigma, w\}$ *of problem* PW. *Moreover, the solution satisfies*

$$u \in C^1(0, T; V), \qquad \sigma \in C(0, T; \mathcal{H}_1), \qquad w \in C^1(0, T; L^2(\Gamma_3)).$$

The proof of Theorem 6.1 will be carried out in several steps. In the first step we assume that the wear is known. Thus, let $\kappa \in C(0, T; L^2(\Gamma_3))$, and consider the following variational problem:

Problem PW_κ: Find $u_\kappa\colon [0, T] \to V$ and $\sigma_\kappa\colon [0, T] \to \mathcal{H}_1$ such that

$$\sigma_\kappa(t) = \mathcal{A}(\cdot, \varepsilon(\dot{u}_\kappa(t))) + G(\cdot, \varepsilon(u_\kappa(t))), \tag{6.8}$$

$$\langle \sigma_\kappa(t), \varepsilon(\xi) - \varepsilon(\dot{u}_\kappa(t))\rangle_{\mathcal{H}} + h(u_\kappa(t), \xi, \kappa(t)) - h(u_\kappa(t), \dot{u}_\kappa(t), \kappa(t))$$

$$\geqslant \langle F(t), \xi - \dot{u}_\kappa(t)\rangle_V, \tag{6.9}$$

$$u_\kappa(0) = u_0, \tag{6.10}$$

for all $\xi \in V$ and $t \in [0, T]$.

As a consequence of Theorem 3.1, we have

PROPOSITION 6.2. *There exists a unique solution* $\{u_\kappa, \sigma_\kappa\}$ *of problem* PW_κ *such that*

$$u_\kappa \in C^1(0, T; V), \qquad \sigma_\kappa \in C(0, T; \mathcal{H}_1).$$

Let us consider now the operator $\mathcal{L}\colon C(0, T; L^2(\Gamma_3)) \to C(0, T; L^2(\Gamma_3))$ defined by

$$\mathcal{L}\kappa(t) = -k_w v^* \int_0^t (\sigma_\kappa)_\nu(s)\, ds \quad \forall t \in [0, T]. \tag{6.11}$$

The second step in the proof of Theorem 6.1 is given in the following result:

PROPOSITION 6.3. *The operator $\mathcal{L}$ has a unique fixed point $\kappa^* \in C(0,T;L^2(\Gamma_3))$.*

Proof. Let $\kappa_1, \kappa_2 \in C(0,T;L^2(\Gamma_3))$. For simplicity we denote by $\{u_i, \sigma_i\}$, $i = 1, 2$, the solutions of problems PW_{κ_i}, i.e., $u_i = u_{\kappa_i}$ and $\sigma = \sigma_{\kappa_i}$. Moreover, we shall not indicate explicitly the dependence on t and the constant C may depend on k_w and v^*. Using (6.11) we have

$$|\mathcal{L}\kappa_1 - \mathcal{L}\kappa_2|_{L^2(\Gamma_3)} \leqslant C \int_0^t |\sigma_1 - \sigma_2|_{\mathcal{H}_1}\,\mathrm{d}s \quad \forall t \in [0,T]. \tag{6.12}$$

Using estimates similar to those in the proof of Proposition 4.1 (see (4.4)–(4.7)) we are led to

$$|\dot{u}_1 - \dot{u}_2|_V \leqslant C(|u_1 - u_2|_V + |\kappa_1 - \kappa_2|_{L^2(\Gamma_3)}). \tag{6.13}$$

Integrating over $(0, t)$ and using (6.10) we find

$$\begin{aligned} |u_1 - u_2|_V &\leqslant \int_0^t |\dot{u}_1 - \dot{u}_2|_V\,\mathrm{d}s \\ &\leqslant C\int_0^t |u_1 - u_2|_V\,\mathrm{d}s + C\int_0^t |\kappa_1 - \kappa_2|_{L^2(\Gamma_3)}\,\mathrm{d}s, \end{aligned}$$

and, using a Gronwell-type inequality, we deduce that

$$|u_1 - u_2|_V \leqslant C\int_0^t |\kappa_1 - \kappa_2|_{L^2(\Gamma_3)}\,\mathrm{d}s, \tag{6.14}$$

for all $t \in [0,T]$. It follows now from (6.13) and (6.14) that

$$|\dot{u}_1 - \dot{u}_2|_V \leqslant C|\kappa_1 - \kappa_2|_{L^2(\Gamma_3)} + C\int_0^t |\kappa_1 - \kappa_2|_{L^2(\Gamma_3)}\,\mathrm{d}s. \tag{6.15}$$

On the other hand, (6.8) and (6.9) imply, taking into account (3.12) and (3.13), that

$$|\sigma_1 - \sigma_2|_{\mathcal{H}_1} \leqslant C(|\dot{u}_1 - \dot{u}_2|_V + |u_1 - u_2|_V),$$

and, using (6.14) and (6.15), we are led to

$$|\sigma_1 - \sigma_2|_{\mathcal{H}_1} \leqslant C|\kappa_1 - \kappa_2|_{L^2(\Gamma_3)} + C\int_0^t |\kappa_1 - \kappa_2|_{L^2(\Gamma_3)}\,\mathrm{d}s. \tag{6.16}$$

Combining now (6.12) with (6.16), we obtain

$$\begin{aligned} &|\mathcal{L}\kappa_1 - \mathcal{L}\kappa_2|_{L^2(\Gamma_3)} \\ &\quad\leqslant C\int_0^t |\kappa_1 - \kappa_2|_{L^2(\Gamma_3)}\,\mathrm{d}s + C\int_0^t\int_0^s |\kappa_1 - \kappa_2|_{L^2(\Gamma_3)}\,\mathrm{d}r\,\mathrm{d}s. \end{aligned} \tag{6.17}$$

Finally, Proposition 6.3 is a consequence of (6.17) and the Banach fixed point theorem.

Proof of Theorem 6.1. Let κ^* be the fixed point of $\mathcal{L}$. Using (6.8)–(6.11), it is easy to verify that $\{u_{\kappa^*}, \sigma_{\kappa^*}, \kappa^*\}$ is the unique solution of Problem PW satisfying the regularity given by Theorem 6.1.

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A Quasistatic Contact Problem with Directional Friction and Damped Response

M. ROCHDI, M. SHILLOR et M. SOFONEA

A Quasistatic Contact Problem with Directional Friction and Damped Response

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Descriptif

On introduit dans ce travail les notions de conditions aux limites de *contact lubrifié* et de *frottements directionnels* et on étudie le problème quasistatique de contact suivant de telles conditions entre un matériau viscoélastique non linéaire et une fondation rigide.

On considère un milieu continu viscoélastique occupant un domaine Ω de $\mathbb{R}^3$ et dont la frontière Γ, supposée suffisamment régulière, est divisée en trois parties disjointes Γ_1, Γ_2 et Γ_3. On suppose que, pendant l'intervalle de temps $[0,T]$, la partie Γ_1 est encastrée dans une structure fixe, que des forces surfaciques f_2 s'appliquent sur Γ_2 et que des forces volumiques f_0 agissent dans Ω. On suppose aussi que la surface de contact potentielle Γ_3 de la fondation est couverte d'un lubrifiant et que les frottements se font suivant deux directions orthogonales du plan tangent à Γ_3. Le problème quasistatique de contact proposé peut se formuler de la façon suivante :

Problème P : Trouver le champ des déplacements $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^3$ et le champ des contraintes $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{R}_s^{3\times 3}$ tels que

$$
\begin{aligned}
\sigma = \mathcal{A}(\varepsilon(u')) + G(\varepsilon(u)) \quad & \text{dans} \quad \Omega \times (0,T),\\
Div\, \sigma + f_0 = 0 \quad & \text{dans} \quad \Omega \times (0,T),\\
u = 0 \quad & \text{sur} \quad \Gamma_1 \times (0,T),\\
\sigma \mathbf{n} = f_2 \quad & \text{sur} \quad \Gamma_2 \times (0,T),\\
-\sigma_n = \beta(u_n')_+ + p_0 \quad & \text{sur} \quad \Gamma_3 \times (0,T),\\
|\sigma_\tau^i| \le \mu_i|\sigma_n|, \quad (i=1,2) \quad & \text{sur} \quad \Gamma_3 \times (0,T),\\
& |\sigma_\tau^i| < \mu_i|\sigma_n| \implies (u_\tau^i)' = 0,\\
& |\sigma_\tau^i| = \mu_i|\sigma_n| \implies \sigma_\tau^i = -\lambda_i (u_\tau^i)', \ \lambda_i \ge 0,\\
u(0) = u_0 \quad & \text{dans} \quad \Omega.
\end{aligned}
$$

On note par $\mathbb{R}_s^{3\times 3}$ l'espace des tenseurs symétriques du second ordre sur $\mathbb{R}^3$ et par $\varepsilon(u)$ le tenseur des petites déformations linéarisé. Le "prime" au dessus d'une quantité représente sa dérivée temporelle, $Div\,\sigma$ désigne la divergence de la fonction tensorielle σ, $\mathbf{n}$ est la normale unitaire sortante à Ω et $\sigma\mathbf{n}$ est le vecteur des contraintes de Cauchy. u_n et σ_n représentent respectivement le déplacement normal et les contraintes normales alors que $\sigma_\tau = \sigma_\tau^1\tau_1 + \sigma_\tau^2\tau_2$ et $u'_\tau = (u_\tau^1)'\tau_1 + (u_\tau^2)'\tau_2$ sont respectivement les contraintes et la vitesse tangentielles où (τ_1, τ_2) est une base orthonormale du plan tangent à Γ_3. Les fonctions β et p_0 désignent respectivement le *coefficient de lubrification* et la *pression du lubrifiant* alors que μ_1 et μ_2 sont les *coefficients de frottement* suivant les directions τ_1 et τ_2.

Pour l'étude du problème P, on commence par donner une interprétation mécanique des conditions aux limites de contact lubrifié et de frottements directionnels. Une fois les hypothèses nécessaires faites, on donne une formulation variationnelle du problème P. On établit ensuite l'existence et l'unicité de la solution quand les coefficients de lubrification et de frottement β, μ_1 et μ_2 sont suffisamment petits. On s'intéresse aussi à l'étude du problème de contact lubrifié avec glissement entre un corps viscoélastique et une fondation rigide en mouvement. L'usure du matériau due aux frottements est prise en compte et elle est modélisée par une variante de la loi d'Archard. Ce problème est formulé de la manière suivante :

Problème PW : Trouver le champ des déplacements $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^3$, le champ des contraintes $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{R}_s^{3\times 3}$ et la fonction usure $w : \Gamma_3 \times [0,T] \longrightarrow \mathbb{R}_+$ tels que

$$\begin{aligned}
\sigma = \mathcal{A}(\varepsilon(u')) + G(\varepsilon(u)) \qquad & \text{dans} \quad \Omega \times (0,T),\\
Div\,\sigma + f_0 = 0 \qquad & \text{dans} \quad \Omega \times (0,T),\\
u = 0 \qquad & \text{sur} \quad \Gamma_1 \times (0,T),\\
\sigma\mathbf{n} = f_2 \qquad & \text{sur} \quad \Gamma_2 \times (0,T),\\
-\sigma_n = \beta u'_n \qquad & \text{sur} \quad \Gamma_3 \times (0,T),\\
|\sigma_\tau^i| = \mu_i|\sigma_n|, \quad (i=1,2) \qquad & \text{sur} \quad \Gamma_3 \times (0,T),\\
\sigma_\tau^i = -\lambda_i(u_\tau^i)',\ \lambda_i \geq 0 \qquad & \text{sur} \quad \Gamma_3 \times (0,T),\\
w = -u_n, \quad w' = -k_w|v^*|\sigma_n \qquad & \text{sur} \quad \Gamma_3 \times (0,T),\\
u(0) = u_0 \qquad & \text{dans} \quad \Omega.
\end{aligned}$$

Ici, $k_w > 0$ est le coefficient d'usure supposé constant et v^* est la vitesse de la fondation rigide. A la suite de l'interprétation mécanique des conditions aux limites de contact lubrifié et de frottements avec usure, on établit une formulation variationnelle du problème PW suivie d'un résultat d'existence et d'unicité.

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A Quasistatic Contact Problem with Directional Friction and Damped Response

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Abstract

The quasistatic contact of a viscoelastic body with a rigid foundation is studied. The material behavior is modeled by a general nonlinear viscoelastic constitutive law. The contact is with directional friction and the foundation's resistance is proportional to the normal velocity. The existence of a unique weak solution to the problem is proved. The sliding frictional contact problem with wear is introduced, too, and the existence of its unique weak solution established. The proofs are based on fixed point theorems and elliptic variational inequalities, and the results hold when the friction and damping coefficients are small.

AMS: 73T05, 35D05, 35M99, 35R45, 73F15

KEY WORDS: Quasistatic contact, directional friction, damped normal response, viscoelastic constitutive law, existence and uniqueness, weak solutions, elliptic variational inequalities, fixed points, sliding contact, wear.

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1. Introduction

This paper deals with a model for the quasistatic frictional contact of a viscoelastic body with a rigid foundation. It is set as a family of variational inequalities parametrized with time. The contact is modeled with damped response and the friction with a directional version of Coulomb's law. Using fixed point arguments and the theory of elliptic variational inequalities we establish the existence of the unique solution to the problem. We also model the process when there is sliding contact with wear and prove the existence of a unique weak solution in the same way.

The quasistatic bilateral contact problem for viscoelastic materials with a nonlocal version of Coulomb's law has been studies recently in [13]. The sliding problem with wear was considered there, too. The results have been established using fixed point theorems, but for different operators, since the contact conditions, and therefore the settings, there and here are different. Such problems are common in industrial settings that involve slowly rotating parts.

There has been considerable interest in the study of quasistatic contact problems recently, see, e.g., [8, 3, 5, 1, 2, 10, 11, 12, 13] and references therein. The quasistatic approximation is obtained when the applied forces in the system vary slowly with time and then the inertial terms in the equations of motion can be assumed negligibly small.

In this paper we invesigate the process of quasistatic frictional contact between a deformable body and a rigid foundation, thus contributing to the theory of quasistatic contact. The body is assumed to be viscoelastic with a nonlinear constitutive law. The contact is modeled with normal damped response, which represents the behavior of a layer of lubricant on the contact surface which supports load via pressure and offers resistance proportionally to surface velocity. In addition, the contact surface may have groves or is corrugated which leads to directional friction, which we model with a version of Coulomb's law with two different friction coefficients in the two principal directions. Thus, the model allows for the possibility of sliding taking place in one direction while the surfaces stick in the other one. Directional friction, in addition, is important in every system where the relative motion of the parts or components is restricted directionally. Then, because of the wear, the surfaces develop directional microstructure which can be described by directional friction.

The model is set as a family of time parametrized elliptic variational inequalities. In a number of steps we establish the existence of a fixed point to an appropriately constructed abstract operator. This holds only when the product of the L^∞ norms of the friction coefficients and the damping coefficient function is small. It is interesting to note that in [11], where the normal compliance contact condition was employed, we did not need such a restriction although a similar method was used.

The rest of the paper is organized as follows. Some preliminary material and notation are given in Section 2. The physical setting and the classical model for the process are presented in Section 3. The assumptions on the problem data are given and so is the weak formulation. Then, our main result, the existence of a unique weak solution, is stated in Theorem 3.1. The proof is given in Section 4. It is based on the theory of elliptic variational inequalities and the Banach fixed point theorem, which requiers the smallness of the product of the coefficients. In Section 5 we model and analyze the process when the body is sliding on the foundation and the wear of the contact surface is taken into account. The structure of this problem is similar to the one in Section 3, and a similar existence and uniqueness result is obtained for it.

2. Notation and preliminaries

In this short section we present the notation we will use and some preliminary material, for further details we refer the reader to [6] or [7]. We denote by S_N the space of second order symmetric tensors on R^N $(N = 2,3)$, while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on S_N and R^N, respectively. Let $\Omega \subset \mathrm{R}^N$ be a bounded domain with a Lipschitz boundary Γ and let $\mathbf{n}$ denote the unit outer normal on Γ.

In the sequel we will use the following real Hilbert spaces $H = L^2(\Omega)^N$, $H_1 = H^1(\Omega)^N$, $\mathcal{H} = \{\xi = (\xi_{ij}) \mid \xi_{ij} = \xi_{ji} \in L^2(\Omega)\}$, $(i,j = 1,...,N)$ and $\mathcal{H}_1 = \{\xi \in \mathcal{H} \mid Div\,\xi \in H\}$, endowed with their canonical inner products and the associated norms denoted by $\langle\cdot,\cdot\rangle_X$ and $|\cdot|_X$, respectively, where X represents any one of these spaces.

Let v_n and v_τ denote the *normal* and the *tangential* components of a vector function v on Γ given by $v_n = v\cdot\mathbf{n}$, $v_\tau = v - v_n\mathbf{n} = v_\tau^1\tau_1 + v_\tau^2\tau_2$, where $\{\tau_1,\tau_2\}$ represents an orthonormal basis on the tangent plane. We also denote by σ_n and σ_τ the *normal* and *tangential* traces of σ (see, e.g., [7]) given by

$$\sigma_n = (\sigma\mathbf{n})\cdot\mathbf{n}, \qquad \sigma_\tau = \sigma\mathbf{n} - \sigma_n\mathbf{n} = \sigma_\tau^1\tau_1 + \sigma_\tau^2\tau_2.$$

We also introduce the *deformation* operator $\varepsilon : H_1 \longrightarrow \mathcal{H}$ and the *divergence* operator $Div : \mathcal{H}_1 \longrightarrow H$ defined by

$$\varepsilon(v) = (\varepsilon_{ij}(v)), \qquad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}),$$

$$Div\,\sigma = (\sigma_{ij,j}).$$

Finally, let $(X, |\cdot|_X)$ be a real normed space. We denote by $C(0,T;X)$ the space of continuous functions from $[0,T]$ to X and by $C^1(0,T;X)$ the space of continuously differentiable functions from $[0,T]$ to X endowed with their canonical norms.

3. Model and statement of results

We model the quasistatic contact process of a viscoelastic body with a rigid foundation as a result of forces and surface tractions which act on it. We set it as a variational inequality, state the assumptions on the data and the existence and uniqueness theorem.

The physical setting is as follows. A viscoelastic body occupies the domain Ω with surface $\Gamma = \partial\Omega$, which is Lipschitz and is divided into three disjoint parts Γ_1, Γ_2 and Γ_3, such that $meas\,\Gamma_1 > 0$. The body is clamped on Γ_1 and so the displacement field vanishes there; surface tractions f_2 act on Γ_2; it is in frictional contact with a rigid foundation on Γ_3. Moreover, volume forces of density f_0 act in Ω.

Let u denote the displacements vector, σ the stress field and $\varepsilon = \varepsilon(u)$ the small strain tensor. The viscoelastic constitutive law that we consider is

$$\sigma = \mathcal{A}(\varepsilon') + G(\varepsilon), \tag{3.1}$$

in which $\mathcal{A}$ and G are given constitutive nonlinear functions, which will be described below. The prime denotes time derivative.

The potential contact surface Γ_3 is assumed to be covered with a lubricant that contains solid particles, such as one of the new smart lubricants, or with worn metallic particles. The resistance of the foundation, actually of the oil layer, is taken proportional to the normal velocity, i.e.,

$$-\sigma_n = \beta(u_n')_+ + p_0, \tag{3.2}$$

where β is the damping resistance function, assumed positive, $(\cdot)_+ = \max\{0, \cdot\}$ is the positive part and p_0 is the oil pressure, which is given and nonnegative. The oil layer presents resistance, or damping, only when the surface moves towards the foundation, but does nothing when it receeds. We remark that all the results below hold unchanged when u_n' is used in (3.2). But then we need to assume that the lubricant layer is thick and its supply is slow, since in such case partial voids would develop.

Assuming that the contacting surfaces have conforming grooves with well defined directions, we use a directional version of Coulomb's law of friction, where the friction coefficients are different in two directions. Let τ_1, τ_2 be two orthonormal tangential vectors on Γ_3, then

$$\begin{aligned} &|\sigma_\tau^i| \le \mu_i|\sigma_n| \quad (i = 1, 2),\\ &|\sigma_\tau^i| < \mu_i|\sigma_n| \implies ({u^i}_\tau)' = 0,\\ &|\sigma_\tau^i| = \mu_i|\sigma_n| \implies \sigma_\tau^i = -\lambda_i({u^i}_\tau)', \ \lambda_i \ge 0. \end{aligned} \tag{3.3}$$

Here, $\sigma_\tau = \sigma_\tau^1\tau_1 + \sigma_\tau^2\tau_2$ represents the tangential traction on the contact boundary, μ_i are the friction coefficients and $({u^i}_\tau)'$ are the tangential velocities in the directions of τ_1 and τ_2. Thus, in each direction τ_1 or τ_2, the tangential shear cannot exceed the maximal frictional resistance, $\mu_1|\sigma_n|$ or $\mu_2|\sigma_n|$, respectively. Then, when the inequality holds the surface adheres to the foundation and is in the so-called *stick* state. When the equality holds there is relative sliding, the so-called *slip* state. The novelty in this directional friction law is that the surface may be sliding in one direction and in a stick state in the other. Therefore, at each time instant the contact surface Γ_3 is divided into three zones: the stick zone, where there is no sliding in either direction; the slip zone where there is slip in both directions; the mixed zone where the surface slides in one direction but not in the other.

Since we are interested in slow evolution we neglect the inertial terms in the equations of motion. The classical formulation of the *quasistatic contact problem with directional friction and normal damped response* is:

Find a displacement field $u : \Omega \times [0,T] \longrightarrow \mathrm{R}^N$ and a stress field $\sigma : \Omega \times [0,T] \longrightarrow S_N$ such that

$$\sigma = \mathcal{A}(\varepsilon(u')) + G(\varepsilon(u)) \quad \text{in} \quad \Omega \times (0,T), \tag{3.4}$$

$$Div\,\sigma + f_0 = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{3.5}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0,T), \tag{3.6}$$

$$\sigma \mathbf{n} = f_2 \quad \text{on} \quad \Gamma_2 \times (0,T), \tag{3.7}$$

$$-\sigma_n = \beta(u_n')_+ + p_0 \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{3.8}$$

$$\begin{aligned} &|\sigma_\tau^i| \le \mu_i |\sigma_n|, \quad (i = 1,2) \quad \text{on} \quad \Gamma_3 \times (0,T), \\ &\quad |\sigma_\tau^i| < \mu_i |\sigma_n| \Longrightarrow ({u^i}_\tau)' = 0, \\ &\quad |\sigma_\tau^i| = \mu_i |\sigma_n| \Longrightarrow \sigma_\tau^i = -\lambda_i ({u^i}_\tau)', \ \lambda_i \ge 0, \end{aligned} \tag{3.9}$$

$$u(0) = u_0 \quad \text{in} \quad \Omega. \tag{3.10}$$

To obtain a variational formulation for problem (3.4)–(3.10) we need additional notation. Let V denote the closed subspace of H_1 given by

$$V = \left\{ v \in H_1 \mid v = 0 \quad \text{on} \quad \Gamma_1 \right\}.$$

Now, Korn's inequality holds, since $meas\,\Gamma_1 > 0$, thus

$$|\varepsilon(u)|_{\mathcal{H}} \ge C|u|_{H_1} \qquad \forall u \in V, \tag{3.11}$$

see, e.g., [6]; here and below C denotes a positive generic constant which may depend on Ω, Γ and $\mathcal{A}$, but does not depend on G, T nor on the input data f_0, f_2, p_0, β, u_0, μ_1 or μ_2, and whose value may vary from place to place.

We consider the inner product $\langle \cdot , \cdot \rangle_V$ on V, given by $\langle u, v \rangle_V = \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}$, and, it follows from (3.11) that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V. Therefore $(V, |\cdot|_V)$ is a Hilbert space.

To study the contact problem (3.4)–(3.10) we assume that the *viscosity operator*

$$\mathcal{A} : \Omega \times S_N \longrightarrow S_N,$$

satisfies

$$\begin{aligned}
&(i). \quad \text{there exists } L > 0 \text{ such that}\\
&\qquad |\mathcal{A}(\cdot,\varepsilon_1) - \mathcal{A}(\cdot,\varepsilon_2)| \le L|\varepsilon_1 - \varepsilon_2| \quad \forall \varepsilon_1, \varepsilon_2 \in S_N, \text{ a.e. in } \Omega,\\
&(ii). \quad \text{there exists } m > 0 \text{ such that}\\
&\qquad (\mathcal{A}(\cdot,\varepsilon_1)) - \mathcal{A}(\cdot,\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \ge m|\varepsilon_1 - \varepsilon_2|^2 \quad \forall \varepsilon_1, \varepsilon_2 \in S_N, \text{ a.e. in } \Omega,\\
&(iii). \quad x \longmapsto \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \quad \forall \varepsilon \in S_N,\\
&(iv). \quad x \longmapsto \mathcal{A}(x,0) \in \mathcal{H}.
\end{aligned} \tag{3.12}$$

The *elasticity operator* $G : \Omega \times S_N \longrightarrow S_N$ satisfies

$$\begin{aligned}
&(i). \quad \text{there exists } \tilde{L} > 0 \text{ such that}\\
&\qquad |G(\cdot,\varepsilon_1) - G(\cdot,\varepsilon_2)| \le \tilde{L}|\varepsilon_1 - \varepsilon_2| \quad \forall \varepsilon_1, \varepsilon_2 \in S_N, \text{ a.e. in } \Omega,\\
&(ii). \quad x \longmapsto G(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \quad \forall \varepsilon \in S_N,\\
&(iii). \quad x \longmapsto G(x,0) \in \mathcal{H}.
\end{aligned} \tag{3.13}$$

The forces and the tractions satisfy

$$f_0 \in C(0,T;H), \qquad f_2 \in C(0,T;L^2(\Gamma_2)^N). \tag{3.14}$$

The *directional coefficients of friction* satisfy $(i = 1, 2)$

$$\mu_i \in L^\infty(\Gamma_3), \qquad \mu_i \ge 0 \quad \text{a.e. on } \Gamma_3; \tag{3.15}$$

the *damping function* β and the oil pressure p_0 satisfy

$$\beta \in L^\infty(\Gamma_3), \qquad \beta(\cdot) \ge \beta^* > 0, \text{ a.e. on } \Gamma_3, \tag{3.16}$$

$$p_0 \in L^\infty(\Gamma_3), \qquad p_0 \ge 0, \text{ a.e. on } \Gamma_3, \tag{3.17}$$

and, finally,

$$u_0 \in V. \tag{3.18}$$

We denote by $f(t)$ the element of V given by

$$\langle f(t), v\rangle_V = \langle f_0(t), v\rangle_H + \langle f_2(t), \gamma v\rangle_{L^2(\Gamma_2)^N}, \tag{3.19}$$

for all $v \in V$ and $t \in [0,T]$, where γv denotes the trace of v on Γ. We note that conditions (3.14) and (3.17) imply

$$f \in C(0,T;V). \tag{3.20}$$

Let $j : V \times V \longrightarrow \mathbb{R}$ be the functional

$$j(v,w) = \int_{\Gamma_3} (\beta(v_n)_+ + p_0)\left(w_n + \mu_1|w_\tau^1| + \mu_2|w_\tau^2|\right)da. \tag{3.21}$$

It is straightforward to show that if $\{u,\sigma\}$ are sufficiently regular functions satisfying (3.5)–(3.9) then

$$\langle\sigma(t),\ \varepsilon(w) - \varepsilon(u'(t))\rangle_{\mathcal{H}} + j(u'(t),w) - j(u'(t),u'(t))$$

$$\geq \langle f(t),\ w - u'(t)\rangle_V, \tag{3.22}$$

for all $w \in V$ and $t \in [0,T]$. Thus, by (3.4), (3.10) and (3.22) we obtain the following weak formulation for Problem (3.4)–(3.10):

Problem P: Find a pair $\{u,\ \sigma\} : [0,T] \longrightarrow V \times \mathcal{H}_1$ such that

$$\sigma(t) = \mathcal{A}(\varepsilon(u'(t))) + G(\varepsilon(u(t))), \tag{3.23}$$

$$\langle\sigma(t),\ \varepsilon(w) - \varepsilon(u'(t))\rangle_{\mathcal{H}} + j(u'(t),w) - j(u'(t),u'(t))$$

$$\geq \langle f(t),\ w - u'(t)\rangle_V, \tag{3.24}$$

$$u(0) = u_0, \tag{3.25}$$

for all $w \in V$ and $t \in [0,T]$.

The main result of this paper, which we establish in the next section, is

Theorem 3.1. *Assume that* (3.12)–(3.18) *hold. There exists* $\alpha_0 > 0$, *which depends only on* Ω, Γ *and* $\mathcal{A}$, *such that if*

$$|\beta|_{L^\infty(\Gamma_3)}\left(|\mu_1|_{L^\infty(\Gamma_3)} + |\mu_2|_{L^\infty(\Gamma_3)} + 1\right) \leq \alpha_0, \tag{3.26}$$

then, there exists a unique solution $\{u,\sigma\}$ *to Problem P. Moreover, the solution satisfies*

$$u \in C^1(0,T;V), \qquad \sigma \in C(0,T;\mathcal{H}_1). \tag{3.27}$$

We conclude that, when $\beta(\mu_1 + \mu_2 + 1)$ is "sufficiently small," Problem (3.4)–(3.10) has a unique weak solution $\{u,\sigma\}$.

4. Proof of Theorem 3.1

The proof of Theorem 3.1 is based on fixed point arguments, similar to those used in [1, 10, 11, 12] but for a different choice of the operators. It will be carried out in several steps. We assume that (3.12)–(3.18) hold. To simplify the notation, we will not indicate explicitly the dependence on t.

In the first step we assume that g, the contacting surface velocity, is given and so is the elastic part of the stress field η. Let $g \in C(0,T;V)$ and $\eta \in C(0,T;\mathcal{H})$. We consider the following variational problem:

Problem $P_{\eta g}$: Find a pair $\{v_{\eta g}, \sigma_{\eta g}\}$, $v_{\eta g} : [0,T] \longrightarrow V$ and $\sigma_{\eta g} : [0,T] \longrightarrow \mathcal{H}_1$, such that

$$\sigma_{\eta g}(t) = \mathcal{A}(\varepsilon(v_{\eta g}(t))) + \eta(t), \tag{4.1}$$

$$\langle \sigma_{\eta g}(t),\ \varepsilon(w) - \varepsilon(v_{\eta g}(t))\rangle_{\mathcal{H}} + j(g(t), w) - j(g(t), v_{\eta g}(t))$$

$$\geq \langle f(t),\ w - v_{\eta g}(t)\rangle_V, \tag{4.2}$$

for all $w \in V$ and $t \in [0,T]$. We have the following result:

Proposition 4.1. *There exists a unique solution to Problem* $P_{\eta g}$ *such that*

$$v_{\eta g} \in C(0,T;V), \qquad \sigma_{\eta g} \in C(0,T;\mathcal{H}_1).$$

Proof. It follows from classical results for elliptic variational inequalities that there exists a unique pair $\{v_{\eta g}, \sigma_{\eta g}\}$, $v_{\eta g}(t) \in V$ and $\sigma_{\eta g}(t) \in \mathcal{H}$, which is a solution to (4.1)–(4.2). Choosing $w = v_{\eta g}(t) \pm \varphi$, where $\varphi \in \mathcal{D}(\Omega)^N$, in (4.2) yields

$$\langle \sigma_{\eta g}(t), \varepsilon(\varphi)\rangle_{\mathcal{H}} = \langle f(t), \varphi\rangle_V.$$

Using (3.19) we deduce

$$Div\, \sigma_{\eta g}(t) + f_0(t) = 0 \quad \text{in} \quad \Omega, \tag{4.3}$$

and then assumption (3.14) and equation (4.3) imply that $\sigma_{\eta g}(t) \in \mathcal{H}_1$.

Now, let $t_1, t_2 \in [0,T]$ and for simplicity we denote $u_{\eta g}(t_i) = u_i$, $\sigma_{\eta g}(t_i) = \sigma_i$, $g(t_i) = g_i$, $\eta(t_i) = \eta_i$, $f(t_i) = f_i$, for $i = 1,2$. Using (4.1) and (4.2) we obtain

$$\langle \mathcal{A}(\varepsilon(v_1)) - \mathcal{A}(\varepsilon(v_2)),\ \varepsilon(v_1) - \varepsilon(v_2)\rangle_{\mathcal{H}} \leq \langle f_1 - f_2,\ v_1 - v_2\rangle_V - \langle \eta_1 - \eta_2,\ \varepsilon(v_1) - \varepsilon(v_2)\rangle_{\mathcal{H}}$$

$$+ j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2). \tag{4.4}$$

Moreover, it follows from (3.11) and (3.12) that

$$\langle \mathcal{A}(\varepsilon(v_1)) - \mathcal{A}(\varepsilon(v_2)),\ \varepsilon(v_1) - \varepsilon(v_2)\rangle_{\mathcal{H}} \geq C|v_1 - v_2|_V^2. \tag{4.5}$$

Now

$$j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2)$$

$$!!! = \int_{\Gamma_3} \beta\Big[(g_{1n})_+ - (g_{2n})_+\Big]\Big[(v_{2n} - v_{1n}) + \mu_1(|v^1_{2\tau}| - |v^1_{1\tau}|) + \mu_2(|v^2_{2\tau}| - |v^2_{1\tau}|)\Big] da, !!!$$

and, therefore

$$j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2)$$

$$\leq C|\beta|_{L^\infty(\Gamma_3)} \Big(|\mu_1|_{L^\infty(\Gamma_3)} + |\mu_2|_{L^\infty(\Gamma_3)} + 1\Big) |g_1 - g_2|_V |v_1 - v_2|_V. \tag{4.6}$$

Using now (4.4)–(4.6) yields

$$|v_1 - v_2|_V \leq C\Big(|f_1 - f_2|_V + |\eta_1 - \eta_2|_{\mathcal{H}} + |\beta|_{L^\infty(\Gamma_3)} \Big(|\mu_1|_{L^\infty(\Gamma_3)} + |\mu_2|_{L^\infty(\Gamma_3)} + 1\Big) |g_1 - g_2|_V\Big). \tag{4.7}$$

Moreover, (3.12) and (4.1) imply

$$|\sigma_1 - \sigma_2|_{\mathcal{H}} \leq C\Big(|v_1 - v_2|_V + |\eta_1 - \eta_2|_{\mathcal{H}}\Big). \tag{4.8}$$

Then, it follows from (4.7) and (4.8) that $v_{\eta g} \in C(0, T; V)$ and $\sigma_{\eta g} \in C(0, T; \mathcal{H})$, and from (3.14) and (4.3) that $\sigma_{\eta g} \in C(0, T; \mathcal{H}_1)$. This concludes the proof.

We define now the operator $\Lambda_\eta : C(0, T; V) \longrightarrow C(0, T; V)$ by

$$\Lambda_\eta g = v_{\eta g} \qquad g \in C(0, T; V). \tag{4.9}$$

Proposition 4.2. *There exists $\alpha_1 > 0$, which depends only on Ω, Γ and $\mathcal{A}$, such that if*

$$|\beta|_{L^\infty(\Gamma_3)} \Big(|\mu_1|_{L^\infty(\Gamma_3)} + |\mu_2|_{L^\infty(\Gamma_3)} + 1\Big) \leq \alpha_1, \tag{4.10}$$

then the operator Λ_η has a unique fixed point $g_\eta \in C(0, T; V)$.

Proof. Let $g_1, g_2 \in C(0, T; V)$ and let $\eta \in C(0, T; \mathcal{H})$. For simplicity we denote by $\{v_i, \sigma_i\}$, $i = 1, 2$, the solutions to problems $P_{\eta g_i}$, i.e., $v_i = v_{\eta g_i}$ and $\sigma_i = \sigma_{\eta g_i}$. Using (4.9) and estimates similar to those in the proof of Proposition 4.1 (see (4.4)–(4.8)) yields

$$|\Lambda_\eta g_1(t) - \Lambda_\eta g_2(t)|_V \leq C|\beta|_{L^\infty(\Gamma_3)} \Big(|\mu_1|_{L^\infty(\Gamma_3)} + |\mu_2|_{L^\infty(\Gamma_3)} + 1\Big) |g_1(t) - g_2(t)|_V.$$

$t \in [0, T]$. Hence, we deduce

$$|\Lambda_\eta g_1 - \Lambda_\eta g_2|_{C(0,T;V)} \leq C|\beta|_{L^\infty(\Gamma_3)} \Big(|\mu_1|_{L^\infty(\Gamma_3)} + |\mu_2|_{L^\infty(\Gamma_3)} + 1\Big) |g_1 - g_2|_{C(0,T;V)}.$$

Proposition 4.2 follows now from this inequality and the Banach fixed point theorem.

In the sequel we assume that (4.10) holds. For every $\eta \in C(0,T;\mathcal{H})$, we denote by g_η the fixed point given in Proposition 4.2. Let $v_\eta \in C(0,T;V)$, $\sigma_\eta \in C(0,T;\mathcal{H}_1)$ and $u_\eta \in C^1(0,T;V)$ be the functions given by

$$v_\eta = v_{\eta g_\eta}, \qquad \sigma_\eta = \sigma_{\eta g_\eta}, \qquad u_\eta(t) = u_0 + \int_0^t v_\eta(s)ds, \tag{4.11}$$

for all $t \in [0,T]$. We define the operator $\Lambda : C(0,T;\mathcal{H}) \longrightarrow C(0,T;\mathcal{H})$ by

$$\Lambda\eta(t) = G(\varepsilon(u_\eta(t))) \qquad \eta \in C(0,T;\mathcal{H}),\ t \in [0,T]. \tag{4.12}$$

Proposition 4.3. *There exists $\alpha_0 > 0$, which depends only on Ω, Γ and $\mathcal{A}$, such that if*

$$|\beta|_{L^\infty(\Gamma_3)}\left(|\mu_1|_{L^\infty(\Gamma_3)} + |\mu_2|_{L^\infty(\Gamma_3)} + 1\right) \le \alpha_0, \tag{4.13}$$

then the operator Λ has a unique fixed point $\eta^* \in C(0,T;\mathcal{H})$.

Proof. Let $\eta_1, \eta_2 \in C(0,T;\mathcal{H})$ and let $v_i = v_{\eta_i}$, $\sigma_i = \sigma_{\eta_i}$, $u_i = u_{\eta_i}$, $g_i = g_{\eta_i}$, for $i = 1,2$. Using Proposition 4.2 we have $g_i = v_i$ and from (4.1) and (4.2) we obtain

$$\sigma_i(t) = \mathcal{A}(\varepsilon(v_i(t))) + \eta_i(t), \tag{4.14}$$

$$\langle\sigma_i(t), \varepsilon(w) - \varepsilon(v_i(t))\rangle_{\mathcal{H}} + j(v_i(t), w) - j(v_i(t), v_i(t))$$

$$\ge \langle f(t), w - v_i(t)\rangle_V \qquad \forall w \in V, \tag{4.15}$$

where $i = 1,2$ and $t \in [0,T]$. It follows from (4.14) and (4.15) and the estimates in the proof of Proposition 4.1 (see (4.7)) that

$$|v_1(t) - v_2(t)|_V \le C|\eta_1(t) - \eta_2(t)|_{\mathcal{H}} + C|\beta|_{L^\infty(\Gamma_3)}\left(|\mu_1|_{L^\infty(\Gamma_3)} + |\mu_2|_{L^\infty(\Gamma_3)} + 1\right)|v_1(t) - v_2(t)|_V.$$

Hence,

$$\left[1 - C|\beta|_{L^\infty(\Gamma_3)}\left(|\mu_1|_{L^\infty(\Gamma_3)} + |\mu_2|_{L^\infty(\Gamma_3)} + 1\right)\right]|v_1(t) - v_2(t)|_V$$

$$\le C|\eta_1(t) - \eta_2(t)|_{\mathcal{H}}. \tag{4.16}$$

Choosing $\alpha_0 < \min(\alpha_1, \frac{1}{C})$ we deduce from (4.10) and (4.16) that

$$|v_1(t) - v_2(t)|_V \le C|\eta_1(t) - \eta_2(t)|_{\mathcal{H}}, \qquad t \in [0,T]. \tag{4.17}$$

Using now (3.13), (4.11) and (4.12) we obtain

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \le \widetilde{L}\int_0^t |v_1(s) - v_2(s)|_V\,ds, \qquad t \in [0,T],$$

and recalling (4.17) it follows that

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \leq C\widetilde{L}\int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}} ds, \qquad t \in [0,T].$$

Reiterating this inequality yields

$$|\Lambda^n\eta_1 - \Lambda^n\eta_2|_{C(0,T;\mathcal{H})} \leq \frac{C^n\widetilde{L}^nT^n}{n!}|\eta_1 - \eta_2|_{C(0,T;\mathcal{H})} \qquad \forall n \in \mathrm{N},$$

which implies that, for n sufficiently large, a power Λ^n of Λ is a contraction on $C(0,T;\mathcal{H})$. Thus, there exists a unique $\eta^* \in C(0,T;\mathcal{H})$ such that $\Lambda^n\eta^* = \eta^*$. Moreover, η^* is also the unique fixed point of Λ.

Proof of Theorem 3.1.

Existence. Assume that (3.26) holds. Let $\eta^* \in C(0,T;\mathcal{H})$ be the fixed point of Λ and let v_{η^*}, σ_{η^*} and u_{η^*} be the functions given by (4.11) for $\eta = \eta^*$. We show that $\{u_{\eta^*}, \sigma_{\eta^*}\}$ is a solution to *Problem P*. Indeed, by choosing $\eta = \eta^*$, $g = g_{\eta^*}$ in (4.1) and (4.2) and using (4.11) we obtain

$$\sigma_{\eta^*} = \mathcal{A}(\varepsilon(v_{\eta^*})) + \eta^*, \tag{4.18}$$

$$\langle\sigma_{\eta^*},\ \varepsilon(w) - \varepsilon(v_{\eta^*})\rangle_{\mathcal{H}} + j(g_{\eta^*}, w) - j(g_{\eta^*}, v_{\eta^*})$$

$$\geq \langle f, w - v_{\eta^*}\rangle_V \qquad \forall w \in V, \tag{4.19}$$

for all $t \in [0,T]$. Now, equality (3.23) follows from (4.11), (4.12) and (4.18), since

$$v_{\eta^*} = u'_{\eta^*}, \qquad \eta^* = \Lambda\eta^* = G(\varepsilon(u_{\eta^*})),$$

while inequality (3.24) follows from (4.9), (4.11) and (4.19), since

$$g_{\eta^*} = \Lambda_{\eta^*}g_{\eta^*} = v_{\eta^* g_{\eta^*}} = v_{\eta^*}.$$

Now, (3.25) results from (4.11), and (3.27) is a consequence of Proposition 4.1, (3.18) and (4.11).

Uniqueness. Let $\{u_{\eta^*}, \sigma_{\eta^*}\}$ be the solution to (3.23)–(3.25) obtained above and let $\{u, \sigma\}$ be another solution such that $u \in C^1(0,T;V)$ and $\sigma \in C(0,T;\mathcal{H}_1)$. We introduce the function $\eta \in C(0,T;\mathcal{H})$ given by

$$\eta(t) = G(\varepsilon(u(t))), \tag{4.20}$$

for $t \in [0,T]$, and let $u' = v = g$. Using (3.23) and (3.24) we obtain that $\{v, \sigma\}$ is a solution to the variational problem $P_{\eta g}$ and since this problem has a unique solution $v_{\eta g} \in C(0,T;V)$,

$\sigma_{\eta g} \in C(0,T;\mathcal{H}_1)$, we conclude that $v = v_{\eta g}$, $\sigma = \sigma_{\eta g}$. Moreover, since $g = v = u'$ we have from (4.9) that $\Lambda_\eta g = v_{\eta g} = v = g$, i.e., g is a fixed point of Λ_η. It follows from Proposition 4.2 that $g = g_\eta$ and then $v = v_{\eta g_\eta}$, $\sigma = \sigma_{\eta g_\eta}$. Then (3.25) and (4.11) imply

$$v = v_\eta, \qquad \sigma = \sigma_\eta, \qquad u = u_\eta. \tag{4.21}$$

Using now (4.12) and (4.20) we obtain that $\Lambda\eta = \eta$ and by the uniqueness of the fixed point $\eta = \eta^*$. The uniqueness of the solution is now a consequence of (4.21). This completes the proof of Theorem 3.1.

5. Sliding frictional contact with wear

In this short section we study the problem of the wear of the contact surface due to sliding friction. We consider the process described in Section 3 when the foundation moves with a steady velocity v^*, and when sliding contact takes place. Recently a similar problem has been analyzed in [13].

We use the setting and notation as above. In addition, we introduce the surface *wear* function $w : \Gamma_3 \times [0,T] \longrightarrow \mathbb{R}_+$ which represents the accumulated wear of the surface, the evolution of which is governed by a modified version of Archard's law (see, e.g., [14] or [4] and references therein) which we now describe. The rate form of Archard's law is

$$w' = -k_w \sigma_n |u'_\tau - v^*|,$$

where k_w is the wear coefficient (very small in practice) and $|u'_\tau - v^*|$ is the relative velocity between the contact surface and the foundation. Thus, the rate of the wear depends linearly on the contact stress and the slip. We also suppose that on the time scale of the quasistatic process, the surface rearranges itself in such a way that the velocity u'_τ is negligible and so the slip is v^*, which for the sake of simplicity is assumed to be a constant vector. Thus, we employ the wear law

$$w' = -k_w |v^*| \sigma_n, \tag{5.1}$$

where k_w =const.> 0. Now, $\sigma_n \leq 0$ and therefore $w' \geq 0$, which allows us to interpret w as the wear of the surface, since the wear of the surface cannot decrease.

Over short periods of time the wear is small, thus, we could assume that $u_n = 0$ on Γ_3, in which case the resulting problem is uncoupled. Here we represent the accumulated effects of wear as the recession of Γ_3 and, since the condition $u_n = 0$ means that the body in its reference configuration is in contact with the foundation, we impose the condition

$$u_n = -w, \tag{5.2}$$

on $\Gamma_3 \times (0,T)$. Consequently, the position of the contact evolves with wear. This in turn leads to the following interesting mathematical problem. Let $\alpha = k_w|v^*|$, which is a positive function, and let $\beta = \frac{1}{\alpha}$. Now, it follows from (5.2) that $u_n' = -w'$ and then (5.1) yields $u_n' = \alpha\sigma_n$. Therefore, instead of (3.9) we consider the following contact conditions

$$\begin{aligned} &\sigma_n = \beta u_n', \quad \text{on} \quad \Gamma_3 \times (0,T), \\ &|\sigma_\tau^i| = -\mu_i\sigma_n, \quad \text{on} \quad \Gamma_3 \times (0,T), \\ &\sigma_\tau^i = -\lambda_i({u^i}_\tau)', \ \lambda_i \geq 0. \end{aligned} \tag{5.3}$$

Here, for the sake of simplicity, we set $p_0 = 0$, which means that the contact does not support any load, say the foundation is a vertical wall that acts as an obstacle to the body's expansion.

The classical formulation of the *sliding frictional contact problem with wear* is:
Find a displacement field $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^N$, a stress field $\sigma : \Omega \times [0,T] \longrightarrow S_N$ and a wear field $w : \Gamma_3 \times [0,T] \longrightarrow \mathbb{R}_+$ such that (3.4)–(3.7), (5.2), (5.3) and (3.10).

Using the same notation and assumptions as above, it is straightforward to verify that this problem has the weak formulation (3.23)–(3.25), when the functional j given by (3.21) is replaced by $j : V \times V \longrightarrow \mathbb{R}_+$ defined by

$$j(v,w) = \int_{\Gamma_3} \beta|v_n| \left(w_n + \mu_1|w_\tau^1| + \mu_2|w_\tau^2|\right) da.$$

This implies that

Theorem 5.1. *Assume that* (3.12)–(3.18) *hold. There exists* $\alpha_0 > 0$, *which depends only on* Ω, Γ *and* $\mathcal{A}$, *such that if*

$$\beta \left(|\mu_1|_{L^\infty(\Gamma_3)} + \mu_2|_{L^\infty(\Gamma_3)} + 1\right) \leq \alpha_0,$$

then, there exists a unique solution $\{u,\sigma,w\}$ *to the sliding problem. Moreover, the solution satisfies*

$$u \in C^1(0,T;V), \qquad \sigma \in C(0,T;\mathcal{H}_1), \qquad w \in C^1(0,T;L^2(\Gamma_3)).$$

If we assume that k_w is a positive function of time and location on the surface we have to repalce β with $|\beta|_{L^\infty(\Gamma_3)}$ in the condition.

We conclude that when $\beta(\mu_1+\mu_2+1)$ is "sufficiently small" the sliding frictional contact problem with wear has a unique weak solution.

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Abstract Evolution Equations for Viscoelastic Frictional Contact Problems

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Descriptif

L'objet ce travail est l'étude d'un problème d'évolution abstrait proposé dans un cadre hilbertien général afin d'unifier l'étude d'une classe de problèmes de contact avec frottement entre un matériau viscoélastique et une fondation rigide.

Soit H un espace de Hilbert, V un sous-espace fermé de H, et A et G deux opérateurs de H dans H. Soit aussi $[0,T]$ un intervalle de temps, $f:[0,T] \longrightarrow V$ et $j : H \times H \longrightarrow]-\infty, +\infty]$. On considère le problème d'évolution abstrait suivant :

Problème P_1: Trouver $x : [0,T] \longrightarrow H$ tel que

$$A\dot{x} + Gx + \partial_2 j(x, \dot{x}) \ni f,$$

$$x(0) = x_0.$$

Le point au dessus d'une quantité représente sa dérivée temporelle et $\partial_2 j$ désigne le sous-différentiel de la fonction j par rapport au second argument. L'originalité et la difficulté de ce problème résident dans la double dépendance de la fonction j par rapport à l'inconnue x et par rapport à sa dérivée $\dot{x}$.

Sous certaines hypothèses faites sur les données, on prouve que le problème P_1 admet une solution unique $x \in C^1(0,T;V)$. La preuve de ce résulat utilise des arguments sur les inéquations variationnelles elliptiques suivis d'une technique de point fixe. On introduit ensuite une formulation "duale" du problème P_1. Il s'agit du problème suivant :

Problème P_2: Trouver $y : [0,T] \longrightarrow H$ tel que

$$y(t) \in \Sigma(t,y), \qquad \frac{d}{dt}(\mathcal{T}y)(t) + \partial\psi_{\Sigma(t,y)}(y(t)) \ni 0.$$

Pour tout $y \in C(0,T;H)$, on a

$$\Sigma(t,y) = \Big\{ z \in H | \ \langle z, v\rangle_H + j(\mathcal{T}y(t), v) \geq \langle f(t), v\rangle_H \quad \forall v \in V \Big\},$$

$\psi_{\Sigma(t,y)}$ est la fonction indicatrice de $\Sigma(t,y)$ et $\mathcal{T}$ est l'opérateur $\mathcal{T} : C(0,T;H) \longrightarrow C^1(0,T;H)$ défini par

$$\mathcal{T}y = x,$$

où $x \in C^1(0,T;H)$ l'unique solution du problème suivant :

$$y = A\dot{x} + Gx,$$

$$x(0) = x_0.$$

On établit alors le résultat d'équivalence entre les problèmes P_1 et P_2 dans le sens suivant :

1) *Si $x \in C^1(0,T;V)$ est solution du problème P_1 alors $y = A\dot{x} + Gx$ est solution du problème P_2.*

2) *Réciproquement, si $y \in C(0,T;H)$ est solution du problème P_2 alors $x = \mathcal{T}y$ est solution du problème P_1.*

La dernière partie de ce travail est consacrée à l'application des résultats obtenus dans le cadre abstrait pour l'étude de quelques problèmes de contact avec frottement entre un matériau viscoélastique et une fondation rigide.

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Abstract evolution equations for viscoelastic frictional contact problems

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Abstract. We analyze a nonlinear abstract evolution problem describing a class of frictional contact processes between a viscoelastic body and a foundation. The problem is set as a time-dependent differential inclusion. The existence of a unique solution is established using the theory of elliptic variational inequalities and Banach's fixed point theorem. A dual formulation of the problem is also introduced and an equivalence result between the two problems is proved. Finally, the abstract results obtained are used to solve some frictional contact problems for viscoelastic materials.

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1. Introduction

Let H be a real Hilbert space, V a closed subspace of H, A and G two operators from H into H. Let also $T > 0$, $f : [0,T] \longrightarrow V$ and $j : H \times H \longrightarrow]-\infty, +\infty]$. We consider the following abstract evolution problem :

Problem P_1: Find $x : [0,T] \longrightarrow H$ such that

$$A\dot{x}(t) + Gx(t) + \partial_2 j(x(t), \dot{x}(t)) \ni f(t), \qquad \forall t \in [0,T], \tag{1.1}$$

$$x(0) = x_0. \tag{1.2}$$

Here the dot represents the time-derivative, $\partial_2 j$ is the subdifferential of the function j with respect to the second argument and x_0 is the initial data.

Evolution problems of the form P_1 arise in the study of quasistatic processes modeling the frictional contact between a viscoelastic body and an obstacle. In this case, the unknown x is the small strain tensor, A and G are operators related to the viscoelastic constitutive law, and (1.1) involves the equilibrium equation as well as the boundary conditions. Here the data f is related to the given body forces and tractions and the function j is determined by the type of the contact boundary conditions. Finally, the initial data x_0 in (1.2) represents the initial strain. Examples of contact problem which may be formulated in the form (1.1)–(1.2) can be found for instance in [15] and [5] (see also the references therein).

Situations which involve frictional contact processes are very common in industry and everyday life such as train wheels with rails, braking pads with the wheel, tectonic plates, etc. The classical formulation of such problems consists of a system of elliptic or evolution equations with a frictional boundary condition on the contact surface. Since, generally, such problems do not have classical solutions, the model is usually restated in a variational formulation. The mathematical analysis of frictional contact problems including existence and uniqueness results has been considered for instance in [6], [13], [8], [7]. New results concerning the modeling and the numerial analysis in contact mechanics can also be found in the recent proceedings [14]. Often, in practice, the main interest lies in the contact stress, since the behavior of the system and especially the surface integrity and wear depend on it. For this reason, in most engineering applications the distribution of the contact stress is of greater importance than the displacements or strains.

The aim of this paper is to study the evolution problem P_1 in order to obtain abstract results which may be applied in the variational analysis of quasistatic viscoelastic problems with friction. As noted above, Problem P_1 represents a variational model for the small strain tensor x and our interest lies also in the stress field. For this reason we propose and analyze a formulation in terms of the stress, the so-called "*dual formulation*".

The paper is structured in the following way. Section 2 deals with an existence and uniqueness result to problem (1.1)–(1.2). It is based on standard arguments for elliptic variational inequalities followed by two applications of Banach's fixed point theorem. In Section 3, the dual formulation of the problem is presented and an equivalence result is established. Finally, in the last section we consider three concrete examples of viscoelastic frictional contact problems. We prove that the variational formulation of these problem are of the form (1.1)–(1.2) and we apply the abstract results of Sections 2 and 3 in order to study these mechanical problems.

2. An existence and uniqueness result

We denote in the sequel by $\langle \cdot, \cdot \rangle_H$ and $|\cdot|_H$ the inner product and the associated norm on H, respectively. Let us remark that V is a real Hilbert space endowed with the inner product of H; thus, we will use sometimes the notation $\langle u, v\rangle_V$, $|u|_V$ instead of $\langle u, v\rangle_H$, $|u|_H$ when $u, v \in V$. We denote, for all $z \in H$, by $D_2 j(z, \cdot)$ and $\partial_2 j(z, \cdot)$ the effective domain and the subdifferential of the function $j(z, \cdot)$, defined by

$$D_2 j(z,\cdot) = \Big\{ u \in H \mid j(z,u) < +\infty \Big\},$$

$$\partial_2 j(z,u) = \Big\{ w \in H \mid j(z,v) - j(z,u) \geq \langle w, v-u\rangle_H \quad \forall v \in H \Big\} \quad \forall u \in H,$$

respectively. In the sequel C will represent a positive generic constant which may depend on A, G, j and T. Finally, $C(0,T;H)$ and $C^1(0,T;H)$ will represent the spaces of continuos and continuously differentiable functions from $[0,T]$ to H with norms

$$|x|_{C(0,T;H)} = \max_{t\in[0,T]} |x(t)|_H, \qquad |x|_{C^1(0,T;H)} = \max_{t\in[0,T]} |x(t)|_H + \max_{t\in[0,T]} |\dot{x}(t)|_H,$$

respectively. The spaces $C(0,T;V)$ and $C^1(0,T;V)$ are defind in a similar way.

In order to study Problem P_1, we assume that the operator $A : H \longrightarrow H$ is Lipschitz continuous and strongly monotone, i.e.,

$$\begin{aligned} &(a) \text{ there exists } L > 0 \text{ such that} \\ &\quad |Au_1 - Au_2| \leq L|u_1 - u_2|_H \quad \forall u_1,\, u_2 \in H, \\ &(b) \text{ there exists } M > 0 \text{ such that} \\ &\quad \langle Au_1 - Au_2, u_1 - u_2\rangle_H \geq M|u_1 - u_2|_H^2 \quad \forall u_1,\, u_2 \in H. \end{aligned} \tag{2.1}$$

The operator $G : H \longrightarrow H$ is Lipschitz continuous, i.e.,

$$\begin{aligned} &\text{there exist } \tilde{L} > 0 \text{ such that} \\ &|Gu_1 - Gu_2|_H \leq \tilde{L}|u_1 - u_2|_H \quad \forall u_1,\, u_2 \in H. \end{aligned} \tag{2.2}$$

The function $j : H \times H \longrightarrow]-\infty, +\infty]$ satisfies

$$\begin{aligned} &(a)\ D_2 j(z,\cdot) = V, \qquad \forall z \in H, \\ &(b)\ j(z,\cdot) \text{ is a continuous seminorm on } V, \qquad \forall z \in H, \\ &(c) \text{ there exists } \tilde{M} > 0 \text{ such that} \\ &\quad j(z_1, v_2) - j(z_1, v_1) + j(z_2, v_1) - j(z_2, v_2) \leq \\ &\quad \tilde{M}|z_1 - z_2|_H |v_1 - v_2|_V \qquad \forall z_1,\, z_2 \in H,\ v_1,\, v_2 \in V. \end{aligned} \tag{2.3}$$

Finally, we assume that

$$f \in C(0,T;V) \tag{2.4}$$

and

$$x_0 \in V. \tag{2.5}$$

The main result of this section is the following.

Theorem 2.1. *Let* (2.1)–(2.5) *hold. Then there exists a unique solution* $x \in C^1(0,T;V)$ *to Problem* P_1.

The proof of Theorem the 2.1 is based on fixed point arguments similar to those used in [2], [15]. It will be established in several steps. We assume in the sequel that (2.1)–(2.5) hold and, to simplify the notation, sometimes we will not indicate explicitly the dependence of various functions on t.

For each $\eta \in C(0,T;H)$ and $g \in C(0,T;V)$, we consider the following problem.

Problem $P_{\eta g}$: Find $x_{\eta g} : [0,T] \longrightarrow V$ such that

$$A\dot{x}_{\eta g}(t) + \eta(t) + \partial_2 j(g(t), \dot{x}_{\eta g}(t)) \ni f(t), \qquad \forall t \in [0,T], \tag{2.6}$$

$$x_{\eta g}(0) = x_0. \tag{2.7}$$

Lemma 2.2. *Problem* $P_{\eta g}$ *has a unique solution* $x_{\eta g} \in C^1(0,T;V)$.

Proof. Using (2.1) and (2.3) it follows from classical results for elliptic variational inequalities that there exists a unique function $z_{\eta g} : [0,T] \to V$ such that

$$\begin{aligned} &(Az_{\eta g}(t), z - z_{\eta g}(t))_H + j(g(t), z) - j(g(t), z_{\eta g}(t)) \geq \\ &\qquad (f(t), z - z_{\eta g}(t))_V - (\eta(t), z - z_{\eta g}(t))_H \qquad \forall z \in V,\ t \in [0,T]. \end{aligned} \tag{2.8}$$

Let t_1, $t_2 \in [0,T]$. For the sake of simplicity, we use the notation $z_{\eta g}(t_i) = z_i$, $g(t_i) = g_i$, $\eta(t_i) = \eta_i$ and $f(t_i) = f_i$, for $i = 1,2$. Using (2.8) after some algebra yields

$$\begin{aligned} (Az_1 - Az_2, z_1 - z_2)_H \leq (f_1 - f_2, z_1 - z_2)_V - (\eta_1 - \eta_2, z_1 - z_2)_H \\ + j(g_1, z_2) - j(g_1, z_1) + j(g_2, z_1) - j(g_2, z_2). \end{aligned} \tag{2.9}$$

From (2.9), (2.1) and (2.3.c), it follows that

$$|z_1 - z_2|_V \leq C(|f_1 - f_2|_V + |\eta_1 - \eta_2|_H + |g_1 - g_2|_V), \tag{2.10}$$

and using (2.4) it follows that $z_{\eta g} \in C(0,T;V)$. Taking (2.8) into account, one can easily verify that the function $x_{\eta g} \in C^1(0,T;V)$ given by

$$x_{\eta g}(t) = x_0 + \int_0^t z_{\eta g}(s)ds, \qquad t \in [0,T], \tag{2.11}$$

is the unique solution to Problem $P_{\eta g}$. This completes the proof of Lemma 2.2.

Let us consider now the operator $\Lambda_\eta : C(0,T;V) \longrightarrow C(0,T;V)$ defined by

$$\Lambda_\eta g = x_{\eta g}, \qquad g \in C(0,T;V). \tag{2.12}$$

We have the following result.

Lemma 2.3. *The operator Λ_η has a unique fixed point $g_\eta \in C(0,T;V)$.*

Proof. Let g_1, $g_2 \in C(0,T;V)$ and $\eta \in C(0,T;H)$. For the sake of simplicity, we set $x_i = x_{\eta g_i}$, $z_i = \dot{x}_{\eta g_i}$ where $x_{\eta g_i}$ is the solution to Problem $P_{\eta g_i}$, $i = 1,2$. We obtain using (2.12) and (2.11) that

$$|\Lambda_\eta g_1(t) - \Lambda_\eta g_2(t)|_V \le \int_0^t |z_1(s) - z_2(s)|_V ds \qquad \forall t \in [0,T]. \tag{2.13}$$

Using estimates similar to those in the proof of Lemma 2.2 (see (2.8)–(2.10)) yields

$$|z_1(t) - z_2(t)|_V \le C|g_1(t) - g_2(t)|_V \qquad \forall t \in [0,T].$$

When applied in (2.13) this implies

$$|\Lambda_\eta g_1(t) - \Lambda_\eta g_2(t)|_V \le C \int_0^t |g_1(s) - g_2(s)|_V ds \qquad \forall t \in [0,T].$$

Reiterating this inequality n times, results in

$$|\Lambda_\eta^n g_1(t) - \Lambda_\eta^n g_2(t)|_{C(0,T;V)} \le \tfrac{C^n}{n!}|g_1 - g_2|_{C(0,T;V)}.$$

The last inequality shows that for a sufficiently large n, the operator Λ_η^n is a contraction on $C(0,T;V)$. Thus, there exists a unique $g_\eta \in C(0,T;V)$ such that $\Lambda_\eta^n g_\eta = g_\eta$. Then it is easy to verify that g_η is the unique fixed point of Λ_η too.

For each $\eta \in C(0,T;H)$, let $x_\eta \in C^1(0,T;V)$ be the function given by

$$x_\eta(t) = x_{\eta g_\eta}(t) = x_0 + \int_0^t z_{\eta g_\eta}(s) ds, \qquad t \in [0,T], \tag{2.14}$$

where g_η is the fixed point of the operator Λ_η given in Lemma 2.3. Let us remark that from (2.12) and (2.14) it results that

$$x_\eta = g_\eta. \tag{2.15}$$

We define now the operator $\Lambda : C(0,T;H) \longrightarrow C(0,T;H)$ by

$$\Lambda\eta = Gx_\eta, \qquad \eta \in C(0,T;H). \tag{2.16}$$

We have the following result.

Lemma 2.4. *The operator Λ has a unique fixed point $\eta^* \in C(0,T;H)$.*

Proof. Let $\eta_1, \eta_2 \in C(0,T;H)$. We use the notation $x_i = x_{\eta_i}$, $z_i = \dot{x}_{\eta_i}$ and $g_i = g_{\eta_i}$ for $i = 1,2$. Using estimates similar to those in the proof of Lemma 2.2 (see (2.8)–(2.10)) and (2.15) it follows that

$$\begin{aligned}|x_1(t) - x_2(t)|_H &\le \int_0^t |z_1(s) - z_2(s)|_H ds \\ &\le C\int_0^t |\eta_1(s) - \eta_2(s)|_H ds + C\int_0^t |x_1(s) - x_2(s)|_H ds,\end{aligned}$$

for all $t \in [0,T]$. Then, using a Gronwall-type inequality yields

$$|x_1(t) - x_2(t)|_H \le C\int_0^t |\eta_1(s) - \eta_2(s)|_H ds \qquad \forall t \in [0,T].$$

Taking (2.2) and (2.16) into account, the last inequality leads to

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_H \le C\int_0^t |\eta_1(s) - \eta_2(s)|_H ds \qquad \forall t \in [0,T]. \tag{2.17}$$

Lemma 2.4 is finally a consequence of (2.17) and Banach's fixed point theorem.

Proof of Theorem 2.1.
Existence part. Let $\eta^* \in C(0,T;H)$ be the fixed point of the operator Λ defined by (2.16). Let also $x_{\eta^*} \in C^1(0,T;V)$ be the function given by (2.14) for $\eta = \eta^*$ and let $z_{\eta^*} = \dot{x}_{\eta^*}$. We show that x_{η^*} is a solution of Problem P_1. Indeed, choosing $\eta = \eta^*$, $g = g_{\eta^*}$ in (2.8), we obtain that

$$\langle Az_{\eta^*}(t), z - z_{\eta^*}(t)\rangle_H + \langle \eta^*(t), z - z_{\eta^*}(t)\rangle_H +$$

$$j(g_{\eta^*}(t), z) - j(g_{\eta^*}(t), z_{\eta^*}(t)) \ge \langle f(t), z - z_{\eta^*}(t)\rangle_V,$$

for all $z \in V$ and $t \in [0,T]$. Since $z_{\eta^*} = \dot{x}_{\eta^*}$, $\eta^* = Gx_{\eta^*}$ and $g_{\eta^*} = x_{\eta^*}$ (see (2.15), (2.16)), the last inequality implies that

$$\begin{aligned}&\langle A\dot{x}_{\eta^*}(t), z - \dot{x}_{\eta^*}(t)\rangle_H + \langle Gx_{\eta^*}(t), z - \dot{x}_{\eta^*}(t)\rangle_H + \\ &j(x_{\eta^*}(t), z) - j(x_{\eta^*}(t), \dot{x}_{\eta^*}(t)) \ge \langle f(t), z - \dot{x}_{\eta^*}(t)\rangle_V,\end{aligned} \tag{2.18}$$

for all $z \in V$ and $t \in [0,T]$. Finally, it results from (2.18) and (2.15) that x_{η^*} is a solution to Problem P_1.

Uniqueness part. Let $x \in C^1(0,T;V)$ be another solution to Problem P_1. Introducing the function $\eta \in C(0,T;H)$ given by

$$\eta = Gx \tag{2.19}$$

and using (2.1), (2.6), it results that x is a solution to Problem $P_{\eta x}$. Since by Lemma 2.2 this problem has a unique solution denoted $x_{\eta x}$, we conclude that

$$x_{\eta x} = x\,. \tag{2.20}$$

Consequently, (2.20) and (2.12) imply that x is the fixed point of the operator Λ_η. It follows then from Lemma 2.3 that $x = g_\eta$. Moreover, by (2.15) and the last equality, we deduce that

$$x = x_\eta\,. \tag{2.21}$$

Using (2.16), (2.19) and (2.21) we obtain $\Lambda\eta = \eta$ and by the uniqueness of the fixed point of Λ we have

$$\eta = \eta^*\,. \tag{2.22}$$

The uniqueness of the solution to Problem P_1 is finally a consequence of (2.21)–(2.22).

3. A dual formulation of the problem

In this section we introduce and analyze a new formulation of the Problem P_1, the so-called "*dual formulation*". To that end, we need the following preliminary result.

Lemma 3.1. *Let* (2.1), (2.2) *and* (2.5) *hold. Then, for all* $y \in C(0,T;H)$, *there exists a unique function* $x \in C^1(0,T;H)$ *such that*

$$y(t) = A\dot{x}(t) + Gx(t) \qquad \forall t \in [0,T], \tag{3.1}$$

$$x(0) = x_0. \tag{3.2}$$

Proof. Let $y \in C(0,T;H)$. It follows from (2.1) that the operator A is invertible and that its inverse A^{-1} is also a Lipschitz continuous operator. Thus, we may consider the operator $\phi:\ C(0,T;H) \longrightarrow C(0,T;H)$ defined by

$$\phi z(t) = A^{-1}y(t) - A^{-1}G\Big(x_0 + \int_0^t z(s)ds\Big) \qquad \forall t \in [0,T]. \tag{3.3}$$

Let now z_1, $z_2 \in C(0,T;H)$. Using (2.1), (2.2) and (3.3) we obtain

$$|\phi z_1(t) - \phi z_2(t)|_H \leq C \int_0^t |z_1(s) - z_2(s)|_H ds \qquad \forall t \in [0,T]. \tag{3.4}$$

Reiterating this inequality n times yields

$$|\phi^n z_1(t) - \phi^n z_2(t)|_{C(0,T;H)} \leq \tfrac{C^n}{n!} |z_1 - z_2|_{C(0,T;H)}.$$

The last inequality implies that for n sufficiently large a power ϕ^n of ϕ is a contraction on $C(0,T;H)$. Thus, ϕ^n has a unique fixed point $z^* \in C(0,T;H)$ which is also the fixed point of the operator ϕ. Finally, it is straightforward to see that the function $x \in C^1(0,T;H)$ given by

$$x(t) = x_0 + \int_0^t z^*(s)ds, \qquad t \in [0,T],$$

is the unique solution to problem (3.1)–(3.2).

The previous lemma allows us to define the operator $\mathcal{T} : C(0,T;H) \longrightarrow C^1(0,T;H)$ by

$$\mathcal{T}y = x, \tag{3.5}$$

where x is the unique solution to problem (3.1)–(3.2).

In order to establish an equivalent formulation of Problem P_1, we introduce the subset $\Sigma(t,y)$ of H given, for all $y \in C(0,T;H)$ and $t \in [0,T]$, by

$$\Sigma(t,y) = \Big\{\sigma \in H \mid (\sigma, z)_H + j(\mathcal{T}y(t), z) \geq \langle f(t), z\rangle_V \quad \forall z \in V\Big\}.$$

We also denote by $\psi_{\Sigma(t,y)}$ the indicator function of $\Sigma(t,y)$ defined by

$$\psi_{\Sigma(t,y)}(\sigma) = \begin{cases} 0 & \text{if } \sigma \in \Sigma(t,y) \\ +\infty & \text{if } \sigma \notin \Sigma(t,y) \end{cases}$$

for all $\sigma \in H$, and let $\partial\psi_{\Sigma(t,y)}$ represent in the sequel the subdifferential of this function.

Let us consider now the following problem.

<u>*Problem P_2*</u>: Find $y : [0,T] \longrightarrow H$ such that

$$y(t) \in \Sigma(t,y), \qquad \frac{d}{dt}(\mathcal{T}y)(t) + \partial\psi_{\Sigma(t,y)}(y(t)) \ni 0 \qquad \forall t \in [0,T]. \tag{3.6}$$

The connection between Problems P_1 and P_2 is given by the following result.

Theorem 3.2. *Let* (2.1)–(2.5) *hold.*
1) *Let* $x \in C^1(0,T;V)$ *be the solution to Problem* P_1. *Then the element* $y \in C(0,T;H)$ *given by* $y = A\dot{x} + Gx$ *is a solution to Problem* P_2.
2) *Conversely, let* $y \in C(0,T;H)$ *be a solution to Problem* P_2. *Then the element* $x = \mathcal{T}y$ *is the solution to Problem* P_1.

Proof.
1) Let $x \in C^1(0,T;V)$ be the solution to Problem P_1 given in Theorem 2.1. Let also $y \in C(0,T;H)$ be the function defined by

$$y(t) = A\dot{x}(t) + Gx(t) \qquad \forall t \in [0,T]. \tag{3.7}$$

It results from (1.1), (3.7) and (2.3.a) that

$$\langle y(t), z - \dot{x}(t)\rangle_H + j(x(t), z) - j(x(t), \dot{x}(t)) \geq \langle f(t), z - \dot{x}(t)\rangle_V, \tag{3.8}$$

for all $z \in V$ and $t \in [0,T]$. Taking $z = 2\dot{x}(t)$ and $z = 0$ in (3.8) and using (2.3.b) we deduce that

$$\langle y(t), \dot{x}(t)\rangle_H + j(x(t), \dot{x}(t)) = \langle f(t), \dot{x}(t)\rangle_V, \tag{3.9}$$

for all $t \in [0,T]$. By (3.7) and (1.2) we deduce that $x = \mathcal{T}y$ and using (3.8) and (3.9) we obtain that $y(t) \in \Sigma(t,y)$ for all $t \in [0,T]$. Moreover, since $\dot{x}(t) \in V$ for all $t \in [0,T]$, we have

$$\langle z, \dot{x}(t)\rangle_H + j(\mathcal{T}y(t), \dot{x}(t)) \geq \langle f(t), \dot{x}(t)\rangle_V \qquad \forall z \in \Sigma(t,y). \tag{3.10}$$

Then by (3.9) and (3.10) it results that

$$\langle z - y(t), \frac{d}{dt}(\mathcal{T}y)(t)\rangle_H \geq 0 \qquad \forall z \in \Sigma(t,y),\ t \in [0,T],$$

which concludes the first part in Theorem 3.2.

2) Conversely, let $y \in C(0,T;H)$ be a solution to Problem P_2. Since $x = \mathcal{T}y$, it results from (3.6) that

$$\langle z - y(t), \dot{x}(t)\rangle_H \geq 0 \qquad \forall z \in \Sigma(t,y),\ t \in [0,T]. \tag{3.11}$$

Let $t \in [0,T]$. Since $y(t) \pm z \in \Sigma(t,y)$ for all $z \in V^\perp$, it follows from (3.11) that $\langle z, \dot{x}(t)\rangle = 0$ for all $z \in V^\perp$. This implies that $\dot{x}(t) \in V^{\perp\perp} = V$, and using (2.5) yields $x(t) \in V$. Thus, since $x \in C^1(0,T;H)$, we obtain that $x \in C^1(0,T;V)$. Let us prove now that x is the solution to Problem P_1. Due to the subdifferentiability of the function $j(x(t),\cdot)$ in $\dot{x}(t) \in V$, it results that there exists a function $\tilde{z}$: $[0,T] \longrightarrow H$ such that

$$j(x(t),z) - j(x(t),\dot{x}(t)) \geq \langle \tilde{z}(t), z - \dot{x}(t)\rangle_H,$$

for all $z \in V$. Therefore, the previous inequality leads to

$$\langle f(t) - \tilde{z}(t), z - \dot{x}(t)\rangle_H + j(x(t),z) - j(x(t),\dot{x}(t)) \geq \langle f(t), z - \dot{x}(t)\rangle_V, \quad (3.12)$$

for all $z \in V$. Taking $z = 2\dot{x}(t)$ and $z = 0$ in (3.12), using (2.3.b) we obtain

$$\langle f(t) - \tilde{z}(t), \dot{x}(t)\rangle_H + j(x(t),\dot{x}(t)) = \langle f(t), \dot{x}(t)\rangle_V. \quad (3.13)$$

From (3.12) and (3.13), it results that $f(t) - \tilde{z}(t) \in \Sigma(t,y)$. Therefore, using $f(t) - \tilde{z}(t)$ as a test function in (3.6) and keeping in mind (3.5), it follows that

$$\langle f(t) - \tilde{z}(t), \dot{x}(t)\rangle_H \geq \langle y(t), \dot{x}(t)\rangle.$$

Adding $j(x(t),\dot{x}(t))$ to this inequality and using (3.13) we find

$$\langle f(t), \dot{x}(t)\rangle_V \geq \langle y(t), \dot{x}(t)\rangle_H + j(x(t),\dot{x}(t)). \quad (3.14)$$

On the other hand, since $y(t) \in \Sigma(t,y)$, $\dot{x}(t) \in V$ and $\mathcal{T}y = x$, we obtain that

$$\langle y(t), \dot{x}(t)\rangle_H + j(x(t),\dot{x}(t)) \geq \langle f(t), \dot{x}(t)\rangle_V, \quad (3.15)$$

and

$$\langle y(t), z\rangle_H + j(x(t),z) \geq \langle f(t), z\rangle_V \qquad \forall z \in V. \quad (3.16)$$

It follows now from (3.14)–(3.16) that

$$\langle y(t), z - \dot{x}(t)\rangle_H + j(x(t),z) - j(x(t),\dot{x}(t)) \geq \langle f(t), z - \dot{x}(t)\rangle_V \qquad \forall z \in V. \quad (3.17)$$

Finally, (1.1) is a consequence of (3.17), (3.1) and (2.3.a) and since the equality $x = \mathcal{T}y$ implies (1.2), we conclude that x is the solution to problem P_1.

Remark 3.3. Having in mind the mechanical examples (see Section 4) as well as Theorem 3.2, it is natural to qualify Problem P_2 as a "*dual formulation*" of the evolution problem P_1. Moreover, Theorems 2.1 and 3.2 imply that under the assumptions (2.1)–(2.5) Problem P_2 has a unique solution $y \in C(0,T;H)$.

4. Applications in contact mechanics

In order to apply the previous results, we will present in this section a number of quasistatic viscoelastic contact problems with friction which may be formulated in the form P_1 or P_2 and such that (2.1)–(2.5) hold.

The physical setting is as follows. A viscoelastic body occupying the domain Ω in $\mathbb{R}^N$ ($N = 2,3$) is acted upon by volume forces and surface tractions. We are interested in the resulting quasistatic evolution process of the mechanical state of the body on the time interval $[0,T]$. We assume that a volume force of density f_0 acts in $\Omega \times (0,T)$. The boundary Γ of Ω is assumed to be Lipschitz, and is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas\,\Gamma_1 > 0$. The body is clamped on $\Gamma_1 \times (0,T)$ and so the displacement field vanishes there, and surface tractions of density f_2 act on $\Gamma_2 \times (0,T)$.

This mechanical setting may be formulated mathematically as follows :

$$\sigma = A\varepsilon(\dot{u}) + G\varepsilon(u) \quad \text{in } \Omega \times (0,T), \tag{4.1}$$

$$Div\,\sigma + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{4.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{4.3}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{4.4}$$

$$u(0) = u_0 \quad \text{in } \Omega, \tag{4.5}$$

where the unknowns are the displacement field $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^N$ and the stress field $\sigma : \Omega \times [0,T] \longrightarrow S_N$. Here $\varepsilon(u)$ denotes the small strain tensor, ν represents the outward unit normal to Ω, u_0 is the initial displacement and S_N denotes the set of second order symmetric tensors on $\mathbb{R}^N$. The viscoelastic constitutive law of the material is given by (4.1) where A and G are nonlinear constitutive functions.

Next, we present some additional notation and list the assumptions on the data. Let " $\cdot$ " be the inner product on $\mathbb{R}^N$ and S_N and let $|\cdot|$ represent the Euclidean norms on $\mathbb{R}^N$ and S_N. We denote by H the real Hilbert space

$$H = \left\{ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega) \quad \forall i,j = 1,\dots,N \right\}$$

endowed with its canonical inner product given by

$$\langle \sigma, \tau \rangle_H = \int_\Omega \sigma_{ij}\tau_{ij}\, dx.$$

We assume in the sequel that:

$$\begin{cases} A:\Omega\times S_N\to S_N \quad \text{and} \\ \text{(a) there exists } L>0 \text{ such that} \\ \qquad |A(x,\varepsilon_1)-A(x,\varepsilon_2)|\le L|\varepsilon_1-\varepsilon_2| \quad \forall \varepsilon_1,\varepsilon_1\in S_N, \text{ a.e. } x\in\Omega, \\ \text{(b) there exists } M>0 \text{ such that} \\ \qquad (A(x,\varepsilon_1)-A(x,\varepsilon_2))\cdot(\varepsilon_1-\varepsilon_2)\ge M\,|\varepsilon_1-\varepsilon_2|^2 \quad \forall \varepsilon_1,\varepsilon_1\in S_N, \\ \qquad \text{a.e. } x\in\Omega, \\ \text{(c) for any } \varepsilon\in S_N,\ x\mapsto A(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \text{(d) the mapping } x\mapsto A(x,0)\in H; \end{cases} \tag{4.6}$$

$$\begin{cases} G:\Omega\times S_N\to S_N \quad \text{and} \\ \text{(a) there exists an } \tilde{L}>0 \text{ such that} \\ \qquad |G(x,\varepsilon_1)-G(x,\varepsilon_2)|\le \tilde{L}\,|\varepsilon_1-\varepsilon_2| \quad \forall \varepsilon_1,\varepsilon_2\in S_N, \text{ a.e. in } \Omega, \\ \text{(b) for any } \varepsilon\in S_N,\ x\mapsto G(x,\varepsilon) \text{ is measurable,} \\ \text{(c) the mapping } x\mapsto G(x,0)\in H. \end{cases} \tag{4.7}$$

Then, we may consider $A: H \longrightarrow H$ and $G: H \longrightarrow H$ such that assumptions (2.1) and (2.2) are satisfied. We also assume that the body forces and tractions satisfy

$$f_0\in C(0,T;L^2(\Omega)^N), \qquad f_2\in C(0,T;L^2(\Gamma_2)^N) \tag{4.8}$$

and, finally, we consider a function $g:\Gamma_3\to\mathbb{R}$ such that

$$g\in L^\infty(\Gamma_3) \qquad \text{and} \qquad g\ge 0 \ \text{ a.e. on } \Gamma_3. \tag{4.9}$$

In order to complete the mechanical setting of the problem, we assume that the viscoelastic body may come into frictional contact with an obstacle along the part Γ_3 of its boundary. Everywhere in the sequel u_ν represents the normal displacement, $\dot{u}_\tau$ denotes the tangential velocity, σ_τ represents the tangential force on the contact boundary and σ_ν is the normal stress. Moreover, for every vector field $v\in H^1(\Omega)^N$, we denote by v_ν and v_τ the normal and the tangential components of v on the boundary given by

$$v_\nu = v\cdot\nu, \qquad v_\tau = v - v_\nu\nu,$$

respectively. Finally, $\varepsilon: H^1(\Omega)^N\to H$ will represent in the sequel the linearized deformation operator given by

$$\varepsilon(v)=(\varepsilon_{ij}(v)), \qquad \varepsilon_{ij}(v)=\frac{1}{2}(v_{i,j}+v_{j,i})$$

for all $v \in H^1(\Omega)^N$, where a subscript that follows a comma indicates a partial derivative, i.e. $v_{i,j} = \dfrac{\partial v_i}{\partial x_j}$.

The examples of frictional boundary conditions are the followings.

Example 4.1. *Bilateral contact with Tresca's friction law.*

This contact condition can be found in [6], [13] and more recently in [1], [2]. It is in the form :

$$\begin{aligned} u_\nu = 0 \quad &\text{on} \quad \Gamma_3 \times (0,T), \\ |\sigma_\tau| \leq g \quad &\text{on} \quad \Gamma_3 \times (0,T), \\ |\sigma_\tau| < g &\Longrightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau| = g &\Longrightarrow \sigma_\tau = -\lambda \dot{u}_\tau, \quad \lambda \geq 0 \end{aligned} \tag{4.10}$$

where g is the friction bound, i.e. the magnitude of the limiting friction traction at which slip begins. The contact is assumed to be bilateral, i.e. there is no loss of contact during the process.

It is straightforward to see that if $\{u, \sigma\}$ are regular functions satisfying (4.2)–(4.4) and (4.10) then $u(t) \in U$ and

$$\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_H + \varphi(v) - \varphi(\dot{u}(t)) \geq L(t, v - \dot{u}(t)) \quad \forall v \in U, t \in [0,T] \tag{4.11}$$

where

$$U = \Big\{ v \in H^1(\Omega)^N \mid v = 0 \quad \text{on} \quad \Gamma_1, \quad v_\nu = 0 \quad \text{on} \quad \Gamma_3 \Big\},$$

$$\varphi : U \longrightarrow \mathbb{R}_+, \qquad \varphi(v) = \int_{\Gamma_3} g|v_\tau| \, da,$$

$$L : [0,T] \times U \longrightarrow \mathbb{R}, \qquad L(t,v) = \int_\Omega f_0(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot v \, da.$$

Using (4.1), (4.5) and (4.11) we obtain the following weak formulation of the mechanical problem (4.1)–(4.5), (4.10) : find a displacement field $u : [0,T] \longrightarrow U$ such that

$$\begin{aligned} &\langle \mathcal{A}\varepsilon(\dot{u}(t)), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_H + \langle \mathcal{G}\varepsilon(u(t)), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_H + \\ &\qquad \varphi(v) - \varphi(\dot{u}(t)) \geq L(t, v - \dot{u}(t)) \qquad \forall v \in U, \quad t \in [0,T], \end{aligned} \tag{4.12}$$

$$u(0) = u_0. \tag{4.13}$$

Let V denote the subspace of H given by

$$V = \varepsilon(U) = \Big\{ \varepsilon(v) \mid v \in U \Big\}. \tag{4.14}$$

We recall that U is a real Hilbert space endowed with the inner product of $H^1(\Omega)^N$. Moreover, using Korn's inequality (see for instance [12] p.79) it follows that V is a closed subspace of H and the deformation operator $\varepsilon : U \longrightarrow V$ is a linear and continuous operator. Denoting by ε^{-1} the inverse of ε, we also obtain that $\varepsilon^{-1} : V \longrightarrow U$ is a linear and continuous operator. This property allows us to restate (4.12)–(4.13) in an equivalent form, by considering as unknown the strain tensor $\varepsilon(u)$. Thus, with the previous notation, it results that the variational problem (4.12)–(4.13) is of the form (1.1)–(1.2) where

$$x = \varepsilon(u), \qquad x_0 = \varepsilon(u_0) \tag{4.15}$$

$$\langle f(t), z\rangle_V = L(t, \varepsilon^{-1}(z)) \qquad \forall z \in V,\ t \in [0,T] \tag{4.16}$$

$$j(w,z) = \begin{cases} \varphi(\varepsilon^{-1}(z)) & \text{if } z \in V \\ +\infty & \text{if } z \notin V \end{cases} \qquad \forall w \in H. \tag{4.17}$$

We remark that in this example the function j depends only on the second argument, and we denote in the sequel $j(z) = j(w,z)$ for all $w, z \in H$. Moreover, using (4.8) and (4.9) it is straightforward to see that the assumptions (2.3)–(2.4) are satisfied, and, if the initial displacement satisfies

$$u_0 \in U, \tag{4.18}$$

using (4.14) and (4.15) we deduce that x_0 satisfies (2.5). Therefore, by Theorem 2.1 we obtain the existence and the uniqueness of the solution of the problem (4.12)–(4.13) such that $u \in C^1(0,T;U)$.

We denote in the sequel by $\Sigma(t)$ the set

$$\Sigma(t) = \Big\{\tau \in H \mid \langle \tau, z\rangle_H + j(z) \geq \langle f(t), z\rangle_V \quad \forall z \in V\Big\} \tag{4.19}$$

for all $t \in [0,T]$. The problem (4.12)–(4.13) is formulated as an evolution variational inequality for the displacement field u. It represents the primal formulation of the mechanical problem (4.1)–(4.5), (4.10). It follows from Section 3 that the dual formulation of this problem, in terms of the stress, is given by: find a stress field $\sigma : [0,T] \longrightarrow H$ such that

$$\sigma(t) \in \Sigma(t), \qquad \langle \frac{d}{dt}(\mathcal{T}\sigma)(t), \tau - \sigma(t)\rangle_H \geq 0 \qquad \forall \tau \in \Sigma(t),\ t \in [0,T]. \tag{4.20}$$

Here, for every $\sigma \in C(0,T;H)$, $\mathcal{T}\sigma$ denotes the unique function $\tau \in C^1(0,T;H)$ which satisfies

$$\sigma(t) = A\dot{\tau}(t) + G\tau(t) \qquad \forall t \in [0,T], \tag{4.21}$$

$$\tau(0) = \varepsilon(u_0). \tag{4.22}$$

Using now Remark 3.3 we obtain that the dual formulation (4.20) has a unique solution $\sigma \in C(0,T;H)$.

We conclude that under the assumption (4.6)–(4.9) and (4.18), the mechanical problem (4.1)–(4.5), (4.10) has a unique weak solution $\{u,\sigma\}$ which satisfies $u \in C^1(0,T;H)$, $\sigma \in C(0,T;H)$. More results concering the study of this problem using the primal variational formulation (4.12)–(4.13) and the dual variational formulation (4.20) were obtained recently in [5].

Example 4.2. *A simplified version of Coulomb's friction law.*

We consider now the problem for a viscoelastic body with the following friction boundary conditions:

$$\begin{aligned} &\sigma_\nu = f_3 \quad \text{on} \quad \Gamma_3 \times (0,T),\\ &|\sigma_\tau| \le \mu|\sigma_\nu| \quad \text{on} \quad \Gamma_3 \times (0,T),\\ &|\sigma_\tau| < \mu|\sigma_\nu| \Longrightarrow \dot{u}_\tau = 0,\\ &|\sigma_\tau| = \mu|\sigma_\nu| \Longrightarrow \sigma_\tau = -\lambda\dot{u}_\tau, \quad \lambda \ge 0. \end{aligned} \tag{4.23}$$

Here f_3 is a given function which satisfies $f_3 \in L^2(\Gamma_3)$ and μ is the coefficient of friction which satisfies (4.9). The friction law (4.23) represents a simplified version of Coulomb's friction law and is was already used in [6], [13], [7] in order to model friction problems for elastic, viscoelastic or viscoplastic materials.

The primal variational formulation of the mechanical problem (4.1)–(4.5), (4.23) is given by (4.12)–(4.13) where

$$U = \left\{ v \in H^1(\Omega)^N \mid v = 0 \quad \text{on} \quad \Gamma_1 \right\},$$

$$\varphi : U \longrightarrow \mathbb{R}_+, \qquad \varphi(v) = \int_{\Gamma_3} \mu|f_3||v_\tau|\,da$$

$$L : [0,T] \times U \longrightarrow \mathbb{R} \qquad L(t,v) = \int_\Omega f_0(t)\cdot v\,dx + \int_{\Gamma_2} f_2(t)\cdot v\,da + \int_{\Gamma_3} f_3 v_\nu\,da.$$

Using again Korn's inequality and notation (4.14)–(4.17) we deduce that this variational formulation is of the form (1.1)–(1.2), in which j depends only on the second argument. Therefore, under the assumptions (4.6)–(4.9), (4.18), Theorems 2.1 and 3.2 may be applied. It results that the dual formulation of the problem is given by (4.20) where for every $\sigma \in C(0,T;H)$, $T\sigma$ denotes the unique function $\tau \in C^1(0,T;H)$ such that (4.21) and (4.22) hold and $\Sigma(t)$ is defined by (4.19), with the corresponding j and f. We conclude that the mechanical problem (4.1)–(4.5), (4.23) has a unique weak solution $\{u,\sigma\}$ which satisfies $u \in C^1(0,T;U)$, $\sigma \in C(0,T;H)$.

Example 4.3. *Normal Compliance contact conditions with friction.*

We assume that the normal stress on Γ_3 satisfies a general form of the so-called normal compliance condition used for instance in [8]–[11], [3], [4] and [15]. The associate friction law is an appropriate version of Coulomb's law of dry friction. More precisely, we assume the following boundary conditions:

$$\begin{aligned} &-\sigma_\nu = p_\nu(u_\nu - g) \qquad \text{on} \quad \Gamma_3 \times (0,T), \\ &|\sigma_\tau| \le p_\tau(u_\nu - g) \qquad \text{on} \quad \Gamma_3 \times (0,T), \\ &\quad |\sigma_\tau| < p_\tau(u_\nu - g) \Longrightarrow \dot{u}_\tau = 0, \\ &\quad |\sigma_\tau| = p_\tau(u_\nu - g) \Longrightarrow \sigma_\tau = -\lambda \dot{u}_\tau, \qquad \lambda \ge 0. \end{aligned} \tag{4.24}$$

Here g represents the gap function between Γ_3 and the obstacle and $p_r : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ $(r = \nu, \tau)$ are the normal compliance functions which satisfy

$$\left\{ \begin{array}{ll} \text{(a)} & \text{there exists } L_r > 0 \text{ such that} \\ & |p_r(\cdot, u_1) - p_r(\cdot, u_2)| \le L_r |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}, \quad \text{a.e. on } \Gamma_3, \\ \text{(b)} & x \mapsto p_r(x, u) \text{ is Lebesgue measurable on } \Gamma_3 \quad \text{for all} \quad u \in \mathbb{R}, \\ \text{(c)} & x \mapsto p_r(x, u) = 0 \quad \text{for} \quad u \le 0. \end{array} \right. \tag{4.25}$$

It was proved in [15] that problem (4.1)–(4.5), (4.24) has a variational formulation of the form: find a displacement field $u : [0,T] \longrightarrow U$ which satisfies (4.13) and

$$\begin{aligned} &\langle A\varepsilon(\dot{u}(t)), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_H + \langle G\varepsilon(u(t)), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_H + \\ &\varphi(u(t), v) - \varphi(u(t), \dot{u}(t)) \ge L(t, v - \dot{u}(t)) \qquad \forall v \in U, \quad t \in [0,T] \end{aligned} \tag{4.26}$$

where

$$U = \left\{ v \in H^1(\Omega)^N \mid v = 0 \quad \text{on} \quad \Gamma_1 \right\},$$

$$\varphi : U \times U \longrightarrow \mathbb{R}_+, \qquad \varphi(u, v) = \int_{\Gamma_3} \left[p_r(u_\nu - g)|v_\nu| + p_\tau(u_\nu - g)|w_\tau| \right] da$$

$$L : [0,T] \times U \longrightarrow \mathbb{R}, \qquad L(t, v) = \int_\Omega f_0(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot v \, da.$$

As usual, we now reformulate the problem (4.26), (4.13) in terms of strains. Thus, using Korn's inequality and notation (4.14)–(4.16) we deduce that the variational formulation (4.26), (4.13) is of the form (1.1)–(1.2) where the function j is given by

$$j(w, z) = \left\{ \begin{array}{ll} \varphi(\varepsilon^{-1}(w), \varepsilon^{-1}(z)) & \text{if } z \in V \\ +\infty & \text{if } z \notin V \end{array} \right. \qquad \forall w \in H.$$

We remark that in this example the function j depends on both arguments. Moreover, the assumptions (4.25) for $r = \nu, \tau$ and (4.9) imply (2.3). Therefore, under the assumptions (4.6)–(4.9), (4.18) and (4.25), Theorems 2.1 and 3.2 may be applied. Thus the dual formulation of the mechanical problem (4.1)–(4.5), (4.24) is given by: find a stress field $\sigma : [0,T] \longrightarrow H$ such that

$$\sigma(t) \in \Sigma(t,\sigma), \qquad \langle \frac{d}{dt}(\mathcal{T}\sigma)(t), \tau - \sigma(t)\rangle_H \geq 0 \qquad \forall \tau \in \Sigma(t,\sigma),\ t \in [0,T] \quad (4.27)$$

which represents a variational formulation in terms of the stress. We recall that in (4.27), for every $\sigma \in C(0,T;H)$, $\mathcal{T}\sigma$ denotes the unique function $\tau \in C^1(0,T;H)$ which satisfies (4.21) and (4.22) and the set $\Sigma(t,\sigma)$ is given by

$$\Sigma(t,\sigma) = \Big\{ \tau \in H \mid \langle \tau, z\rangle_H + j(\mathcal{T}\sigma(t), z) \geq \langle f(t), z\rangle_V \quad \forall z \in V \Big\},$$

for all $t \in [0,T]$. We conclude that problem (4.1)–(4.5), (4.24) has a unique weak solution $\{u,\sigma\}$ which satisfies $u \in C^1(0,T;U)$, $\sigma \in C(0,T;H)$. More results concerning the study of this problem using the primal formulation (4.26), (4.13) were obtained recently in [15].

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Reduction of Second Order Unilateral Singular Systems. Applications in Mechanics

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Reduction of Second Order Unilateral Singular Systems. Applications in Mechanics

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Descriptif

Cet article concerne la présentation de méthodes qui permettent d'étudier des systèmes unilatéraux du second ordre mettant en jeu des masses singulières, des amortisseurs et des matrices de raideur. Des méthodes de réduction sont utilisées pour transformer des inclusions différentielles du second ordre en inclusions différentielles du premier ordre puis des résultats d'existence classiques sont utilisés afin d'obtenir des solutions. Pour illuster cette approche théorique, des applications en mécanique unilatérale sont présentées en faisant intervenir des problèmes avec frottement et impact.

Une inclusion différentielle du second ordre peut être formulée de la manière suivante : Trouver une fonction $q : [0, T] \longrightarrow \mathbb{R}^N$, $t \longmapsto q(t)$ telle que

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \in f(t) + F(t, q(t), \dot{q}(t)), \quad \text{p.p. } t \in]0, T[, \tag{1}$$

$$q(0) = q_0, \quad \dot{q}(0) = q_1 \tag{2}$$

où M est une matrice de masse, C est une matrice d'amortissement, K est une matrice de raideur, $f : [0, T] \longrightarrow \mathbb{R}^N$ est une fonction vectorielle associée aux forces agissant sur le système et $F : [0, T] \times \mathbb{R}^N \longrightarrow \mathcal{P}(\mathbb{R}^N)$ est une fonction multivoque, c'est-à-dire une fonction de $[0, T] \times \mathbb{R}^N$ dans l'ensemble $\mathcal{P}(\mathbb{R}^N)$ de toutes les parties de $\mathbb{R}^N$, qui définit, pour tout $t \in [0, T]$, un graphe dans $\mathbb{R}^N$ utilisé pour exprimer les forces de réaction unilatérale. La formulation du système (1)–(2) peut être complétée par des lois d'impact s'il y a présence de phénomènes de collision entre corps rigides.

Le but de ce travail est discuter des stratégies d'étude de systèmes du type (1)–(2) pour des matrices M, C et K singulières. En particulier, il s'agit de développer des techniques de réduction de telle sorte que les systèmes réduits obtenus puissent être étudiés (résolus) grâce aux méthodes standard dont on dispose. Plus précisément, l'objectif principal est de développer des modèles de réduction de façon à ce que des systèmes du type (1)–(2) puissent être écrits sous la forme du premier ordre suivante :

$$\dot{y}(t) \in \Phi(t, y(t)), \quad \text{p.p. } t \in]0, T[, \tag{3}$$

$$y(0) = y_0, \tag{4}$$

où $y_0 \in \mathbb{R}^n$ et Φ est une application de $[0, T] \times \mathbb{R}^n$ dans l'ensemble de toutes les parties de $\mathbb{R}^n$.

Les inclusions différentielles de ce type ont fait l'objet de plusieurs études (cf. références de l'article).

La première partie de ce travail concerne la méthode de réduction de Jordan pour les inclusions différentielles du premier ordre. Cette méthode est appliquée à la fin de cet article pour étudier un problème en biomécanique. Dans une deuxième partie, on étudie une condition nécessaire pour utiliser le rèsultats précédents pour les inclusions différentielles du second ordre. Des modèles de réduction sont ensuite étudiés et des résultats qu'on trouve dans la littérature sont établis ici dans un contexte plus général sous des conditions affaiblies. Ces résultats sont d'une importance particulière pour la réduction de problèmes en Economie puisque les matrices rencontrées dans ce domaine n'ont souvent pas les mêmes bonnes propriétés que celles rencontées en Mécanique. Dans la dernière partie de cet article, trois problèmes intéressants en ingénierie sont revisités par le biais des méthodes de réduction établies. Il s'agit de systèmes mécaniques mettant en jeu des corps rigides sujets à des contraintes unilatérales.

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Reduction of Second Order Unilateral Singular Systems. Applications in Mechanics

The aim of this paper is to discuss the mathematical strategies permitting the treatment of second order unilateral systems involving singular mass, damping, and stiffness matrices. Reduction methods are used here to transform second order differential inclusions in first order ones and classical results on differential inclusions are considered in order to obtain solutions. Friction and impact problems arising in Unilateral Mechanics are studied so as to illustrate the theoretical approach.

Key words: unilateral problems in mechanics, friction Coulomb's law, unilateral constraints, impact laws, differential inclusions, variational inequalities, reduction methods, matrix analysis, difference methods for differential inclusions, nonsmooth damped spring-mass systems

MSC (1991): 70H30, 70F20, 70Q05, 49J40, 49S05

1. Introduction

Many mechanisms consist of parts that can be considered as perfectly rigid bodies. Some of these parts may come into contact or separate from each other, however they do not penetrate each other. That means that forces of constraints or reaction need to be included in the formulation of mathematical models for such processes. The use of forces of constraints to specify the contact phenomena introduces serious mathematical difficulties into these models. This is the reason why the place of unilateral constraints received in publications in classical mechanics is very modest in comparison with the abundance of unilateral constraints in engineering systems.

Using modern tools of convex analysis, MOREAU [16], [17] has recently proposed rigorous mathematical expressions of normal contact laws, Coulomb's friction law, shock laws, etc., which lead to differential inclusions.

A second order differential inclusion model can be formulated as follows:

Find $q: [0,T] \longrightarrow \mathbb{R}^N$, $t \longmapsto q(t)$ such that

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \in f(t) + F(t, q(t), \dot{q}(t)), \quad \text{a.e.} \quad t \in (0,T), \tag{1.1}$$

where M is the mass matrix, C is the damping matrix, K is the stiffness matrix, $f: [0,T] \longrightarrow \mathbb{R}^N$ is avector-valued function related to the given forces acting on the system, and $F: [0,T] \times \mathbb{R}^N \longrightarrow \mathcal{P}(\mathbb{R}^N)$ is a set-valued function, i.e. a function from $[0,T] \times \mathbb{R}^N$ onto the set $\mathcal{P}(\mathbb{R}^N)$ of all subsets of $\mathbb{R}^N$, that defines, for each $t \in [0,T]$, a graph in $\mathbb{R}^N$ used to express the unilateral reaction forces. Usual initial conditions such as

$$q(0) = q_0, \qquad \dot{q}(0) = q_1,$$

and impact laws (provided that the system under consideration involves rigid body collisions) are generally introduced to complete the formulation of the model.

Until now only some special cases of second order differential inclusions have been studied. See for example the works of CHOLET [5], FRÉMOND [10], MARQUES [15], and MOREAU [16], [17]. However, various problems formulated as in (1.1) cannot be studied using the current theoretical results in the mathematical literature. In particular, if the matrices M, K, and C involved in the model (1.1) are singular, then most of the known results do not apply.

However, singularities occur frequently in models of the dynamics of multi-body systems. Indeed, most problems in Mechanics are formulated in terms of parameters $q_1, \dots, q_N$ making the element q appearing in (1.1). The formulation of usual damped spring-mass systems lead to the so-called mass matrices, damping matrices, and stiffness matrices that are inparts determined by the physical characteristics of the rigid bodies, springs, and dampers involved in the system. This part of the matrix formulation of the whole model is rarely the cause of singularities. However, a whole model may encompass the matrix formulation of bilateral constraints whose geometric or kinematical effects are expressed by equalities of the form

$$A_1\dot{q} + A_2 q = b,$$

with A_1, $A_2 \in \mathbb{R}^{M\times N}$ $(M < N)$, $b \in \mathbb{R}^M$. This last first order system introduces a zero matrix block in the whole mass matrix that can be the cause of the singularities of this last one (see e.g. [3]).

The use of zero-mass points for example to denote a connection between two springs or two dampers may also be the cause of singularities in the resulting models (see e.g. [1]). The treatment of some forces like reaction forces or friction forces as unknows of the problem may also introduce singularities in the whole model. Problems of the form (1.1) with singular matrices M, C, and K, and $F \equiv 0$ occur also informulating the dynamic of various problems in Economics.

The aim of this paper is to discuss the mathematical strategies permitting the treatment of the model (1.1) for matrices M, C, and K allowed to be singular. In particular, we develop reduction techniques in such a way that the resulting reduced model can be studied by means of standard arguments. More precisely, the main goal of this paper is to develop reduction model strategies so as to rewrite second order singular differential unilateral systems in the following first order form:

$$\begin{cases} \dot{y}(t) \in \Phi(t, y(t)), & \text{a.e.} \quad t \in (0, T), \\ y(0) = y_0, \end{cases} \tag{1.2}$$

where $y_0 \in \mathbb{R}^n$ and Φ is a map from $[0, T] \times \mathbb{R}^n$ into the set of all subsets of $\mathbb{R}^n$.

Differential inclusions of this type have been the subject of many papers and, for more details, we refer the reader to the book of FILIPPOV [9], the survey of DONTCHEV and LEMPIO [7] and the paper of LEMPIO and VELIOV [14]. The following theorem (see [7], [19], and [9]) gives sufficient conditions for existence.

Theorem 1.1: *Suppose that Φ satisfies the conditions*

(i) *Φ is nonempty, compact, and convex-valued on $[0, T] \times \mathbb{R}^n$,*
(ii) *$\Phi(t, \cdot)$ is upper semicontinuous, for all $t \in [0, T]$,*
(iii) *$\Phi(\cdot, x)$ is measurable, for all $x \in \mathbb{R}^n$,.*
(iv) *there exist constants k_1 and k_2 such that*

$$\|z\| \le k_1 \|x\| + k_2, \qquad \forall z \in \Phi(t, x), \quad x \in \mathbb{R}^n, \qquad t \in [0, T].$$

Then Problem (1.2) has a least one solution, i.e. an absolutely continuous function y that satisfies (1.2).

If, in addition, the map Φ possesses the decomposition

$$\Phi(t, x) = \theta(t, x) - \beta(x),$$

where $\theta: [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a single-valued function satisfying the one-sided Lipschitz condition

$$(\theta(t, x_1) - \theta(t, x_2))^T (x_1 - x_2) \le L \|x_1 - x_2\|^2,$$

uniformly for all $t \in [0, T]$ and $\beta: \mathbb{R}^n \longrightarrow \mathcal{P}(\mathbb{R}^n)$ is a monotone set-valued mapping, then the solution of Problem (1.2) is necessarily unique.

Models as the one formulated in (1.2) can be solved by means of appropriate set-valued version of classical difference methods like Euler's method and 4-stage Runge-Kutta method.

The Euler Method is here outlined. For $q \in N \backslash \{0\}$, a grid $0 = t_0 < t_1 < \ldots < t_q = T$ is chosen with stepsize $h = (T - t_0)/q = t_j - t_{j-1}$ $(j = 1, \ldots, q)$. Let

$$\eta_0 = y_0,$$

and, for $j = 0, \ldots, q-1$, the vector η_{j+1} is computed by the formula

$$\eta_{j+1} \in \eta_j + h\Phi(t_j, \eta_j). \tag{1.3}$$

Then one sets

$$\eta^q(t) = \eta_j + \frac{1}{h} (t - t_j) (\eta_{j+1} - \eta_j), \tag{1.4}$$

for $t_j \le t \le t_{j+1}$, $j = 0, \ldots, q-1$. The piecewise linear function η^q yields an approximation of the solution of Problem (1.2).

A 4-Stage Runge-Kutta scheme can be outlined in a similar way:

$$\begin{cases} \eta_0 = y_0, \\ \eta_{j+1} = \eta_j + \dfrac{h}{6} (k_{j1} + 2k_{j2} + 2k_{j3} + k_{j4}), \end{cases} \tag{1.5}$$

with

$$\begin{gathered} k_{j1} \in \Phi(t_j, \eta_j), \qquad k_{j2} \in \Phi\left(t_j + \frac{h}{2}, \eta_j + \frac{k_{j1}}{2}\right), \qquad k_{j3} \in \Phi\left(t_j + \frac{h}{2}, \eta_j + \frac{k_{j2}}{2}\right), \\ k_{j4} \in \Phi(t_j + h, \eta_j + k_{j3}). \end{gathered} \tag{1.6}$$

The convergence properties of these schemes are overviewed in [7]. Various problems in Unilateral Mechanics have been recently treated in [19].

Theorem 1.1 can be applied to study a great variety of models of unilateral phenomena like dry friction, debonding effects and delamination effects. However, the assumptions required on Φ are too strong to encompass frictionless

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normal contact laws expressing non-penetration constraints and reactions. Indeed, simple dynamic models or reduced dynamic models involving such unilateral constraints are generally governed by a system of differential inclusions of the type

$$\ddot{q} \in f(t, q, \dot{q}) + \partial\psi_K(q)\,, \tag{1.6}$$

where $f: [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a single-valued function and $\partial\psi_K$ denotes the convex subdifferential of the indicator function of some nonempty closed convex set $K \subset \mathbb{R}^N$ defined by the geometric constraints imposed on q. Impact laws are also usually considered so as to complete the formulation of the problem in consideration. In practical situations, the set K can also be nonconvex, but the theory we use here applies only in the convex case.

The differential inclusion (1.6) reduces to (1.2) by setting $y = (q\ \ \dot{q})^T$, $n = 2N$ and

$$\Phi(t, y) = (y_2 f(t, y_1, y_2) + \partial\psi_K(y_1))^T.$$

It is clear that Φ does not satisfy the sublinear growth condition (iv) in Theorem 1.1. We have indeed

$$\partial\psi_K(x) = N_K(x)\,,$$

where $N_K(x)$ denotes the normal cone of K at x, that is

$$N_K(x) = \{w \in \mathbb{R}^N : w^T z \le 0,\ \forall z \in T_K(x)\}\,,$$

where

$$T_K(x) = \overline{\bigcup_{\lambda > 0} \lambda(K - x)}\,.$$

In the following, we remind a result proved by PAOLI and SCHATZMAN [18]. It gives a weak solution to Problem (1.6) coupled with the impact law

$$\dot{q}(t_+) = -e\dot{q}_N(t_-) + \dot{q}_T(t_-)\,, \qquad \forall t \in [0,T]: q(t) \in \partial K\,,$$

where $\dot{q}_N$ and $\dot{q}_T$ denote the projection of $\dot{q}$ onto $\mathbb{R}\nu(q(t))$ and $(\mathbb{R}\nu(q(t)))^\perp$, respectively. The parameter e iscalled the recovery coefficient.

Theorem 1.2: *Let K be a closed convex subset of $\mathbb{R}^N$ with nonempty interior and a regular boundary ∂K of class C^2 in the sense that there exists a unique mapping $\nu: \partial K \longrightarrow \mathbb{R}^N$ of class C^1 such that*

$$N_K(x) = \mathbb{R}_+ \nu(x)\,, \qquad \forall x \in \partial K\,.$$

Let

(i) $f: [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *continuous,*
(ii) $f(t, \cdot, \cdot)$ *Lipschitz continuous for all* $t \in [0,T]$.

Let also $q_0 \in K$, $q_1 \in T_K(q_0)$, and $e \in (0,1]$ be given. Then there exists $q: [0,T] \longrightarrow \mathbb{R}^N$ Lipschitz continuous such that

(a) $\dot{q}$ *has bounded variations,*
(b) $q(0) = q_0$,
(c) $\dot{q}(0_+) = q_1$,
(d) $q(t) \in K$, $\forall t \in [0,T]$,
(e) $\langle \ddot{q} - f(t, q, \dot{q}),\ \varphi - q\rangle \ge 0$, $\forall \varphi \in C^0([0,T]; K)$,
(f) $\dot{q}(t_+) = -e\dot{q}_N(t_-) + \dot{q}_T(t_-)$, $\forall t \in [0,T]$ *such that* $q(t) \in \partial K$.

Note that the expression $\langle \ddot{q} - f(t, q, \dot{q}),\ \varphi - q\rangle \ge 0$ (considered in the sense of distributions) constitutes a weak formulation for (1.6).

The solution outlined in Theorem 1.2 is obtained as the limit (in $W^{1,p}(0,T;\mathbb{R}^N)$), for all $p \in [1, +\infty[$) when $\lambda \to 0$ of a subsequence of the solutions of the system

$$\begin{cases} \ddot{q}_\lambda + \dfrac{2}{\sqrt{\lambda}} \left(\dfrac{\ln(1/e)}{\sqrt{\pi^2 + (\ln(e))^2}} \right) G(q_\lambda - P_K(q_\lambda), \dot{q}_\lambda) + \dfrac{q_\lambda - P_K(q_\lambda)}{\lambda} = f(t, q_\lambda, \dot{q}_\lambda)\,, \\ q_\lambda(0) = q_0\,, \\ \dot{q}_\lambda(0) = q_1\,, \end{cases} \tag{1.7}$$

where

$$G(u, v) = \begin{cases} (u^T v)\, u/|u|^2 & \text{if} \quad u \neq 0\,, \\ 0 & \text{if} \quad u = 0\,, \end{cases}$$

and P_K is the projection map on K. Note also here that the operator $(u - P_K u)/\lambda$ is the Yosida approximant of the operator $\partial\psi_K$.

In the case of elastic impact, i.e. $e = 1$, the first equation in (1.7) reduces to

$$\ddot{q}_\lambda + \frac{q_\lambda - P_K(q_\lambda)}{\lambda} = f(t, q_\lambda, \dot{q}_\lambda). \tag{1.8}$$

This case will be considered later in this paper.

In this paper, three interesting engineering problems are revisited by means of advanced mathematical tools in unilateral analysis. We show how matrix-reduction methods can be used together with Theorem 1.1, Theorem 1.2 and the difference methods outlined in this introduction to provide a complete analysis of these problems.

Jordan's reduction method for first order differential inclusions is outlined in Section 2 and used in Section 5 to discuss a problem in biomechanics. In Section 2, we discuss a mathematical condition that is required to specialize the result of Section 2 to the second order differential inclusions. Model reduction techniques are studied in Section 4. The results proved in [1] are established here in a more general framework. Symmetry and positive semi-definite properties assumed in [1] on the involved matrices have here been relaxed. The generalized results developed in Section 4 are of particular interest to reduce problems in Economics because they often do not present the "nice" properties encountered in most problems of Mechanics (see [8]). Limitated in space, problems in Economics are not discussed in this paper. The theoretical results discussed here are however illustrated in this sense in [8]. Note also that not-necessarily symmetric matrices appear in the mathematical formulation of electric power systems [2]. In Section 5, we present some examples of mechanical systems involving rigid bodies subject to unilateral constraints. These examples illustrate the methodology developed in this paper to study unilateral problems in Mechanics.

2. Jordan's reduction of first order differential inclusions

Let us first discuss the first order model:

Find

$$x : [0, T] \longrightarrow \mathbb{R}^n, \qquad t \longmapsto x(t),$$

such that

$$E\dot{x}(t) \in Ax(t) + H(t) + G(t, x(t)), \quad \text{a.e.} \quad t \in (0, T), \tag{2.1}$$

where $E, A \in \mathbb{R}^{n\times n}$ are singular matrices, $H : [0, T] \longrightarrow \mathbb{R}^n$ is a given vector-valued function and $G : [0, T] \times \mathbb{R}^n \longrightarrow \mathcal{P}(\mathbb{R}^n)$ denotes a set-valued function. Together with (2.1), we may consider some initial condition like

$$x(0) = c. \tag{2.2}$$

Definition 2.1: One says that the pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ forms a regular matrix pencil provided that there exists $\lambda \in \mathbb{R}$ such that

$$\operatorname{rank}(\lambda E - A) = n.$$

Remark 2.2: Assume that $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is a regular matrix pencil. If

$$G(t, y) = -\partial_y \Psi(t, y), \qquad \forall t \in [0, T], \qquad y \in \mathbb{R}^n,$$

where $\Psi : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous functional in the second variable for any $t \in [0, T]$ and ∂_y denotes the convex subdifferential operator with respect to the second variable, then (2.1) is equivalent to the variational inequality

$$(E\dot{x}(t) - Ax(t) - H(t))^T (v - x(t)) + \Psi(t, v) - \Psi(t, x(t)) \geq 0, \qquad \forall v \in \mathbb{R}^n.$$

Setting $x = e^{\lambda t} u$, we see that (2.1)–(2.2) reduces to

$$\begin{cases} E\dot{u}(t) \in (A - \lambda E)\, u(t) + e^{-\lambda t} H(t) + e^{-\lambda t} G(t, e^{\lambda t} u(t)), \\ u(0) = c. \end{cases} \tag{2.3}$$

The matrix $\lambda E - A$ is regular and the matrix $\hat{E}_\lambda = (\lambda E - A)^{-1} E$ is well defined. The relation (2.3) may be formulated as

$$\begin{cases} -\hat{E}_\lambda \dot{u}(t) = u(t) + \bar{h}(t) + \bar{g}(t), \quad \text{a.e.} \quad t \in (0, T), \\ u(0) = c, \end{cases} \tag{2.4}$$

where

$$\bar{h}(t) = (A - \lambda E)^{-1} e^{-\lambda t} H(t),$$

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and

$$\bar{g}(t) \in (A - \lambda E)^{-1} \, e^{-\lambda t} \, G(t, e^{\lambda t} \, u(t)) .$$

Set $p := \dim(\ker(\hat{E}_\lambda))$. The Jordan form J of the singular matrix $\hat{E}_\lambda = TJT^{-1}$ has the structure

$$J = \begin{pmatrix} W & 0 \\ 0 & N \end{pmatrix},$$

where $W \in \mathbb{R}^{(n-p)\times(n-p)}$ contains all the jordan blocks corresponding to the nonzero eigenvalues of $\hat{E}_\lambda$ and $N \in \mathbb{R}^{p\times p}$ is nilpotent of order $k \le p$ (see e.g. [4]). From (2.4) we deduce that

$$\begin{cases} -JT^{-1}\dot{u} = T^{-1}u + T^{-1}\bar{h} + T^{-1}\bar{g} , \\ u(0) = c . \end{cases}$$

Setting $v = T^{-1}u$, we get

$$\begin{cases} -J\dot{v} = v + h + g , \\ v(0) = T^{-1}c , \end{cases}$$

with

$$g(t) \in T^{-1}(A - \lambda E)^{-1} \, e^{-\lambda t} \, G(t, e^{\lambda t} \, Tv(t)) ,$$

and

$$h(t) = T^{-1}(A - \lambda E)^{-1} \, e^{-\lambda t} \, H(t) .$$

Let us now define the rectangular block matrices

$$\Lambda_1 = (I_{(n-p)\times(n-p)} \mid 0_{(n-p)\times p}) ,$$
$$\Lambda_2 = (0_{p\times(n-p)} \mid I_{p\times p}) ,$$

and set

$$v_i = \Lambda_i v , \qquad h_i = \Lambda_i h , \qquad g_i = \Lambda_i g \quad \text{and} \quad c_i = \Lambda_i T^{-1} c , \qquad i = 1, 2 .$$

We see that

$$\begin{cases} -W\dot{v}_1 = v_1 + h_1 + g_1 , \qquad -N\dot{v}_2 = v_2 + h_2 + g_2 , \\ v_1(0) = c_1 , \qquad v_2(0) = c_2 . \end{cases} \tag{2.5}$$

System (2.5) is coupled through the relation

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in T^{-1}(A - \lambda E)^{-1} \, e^{-\lambda t} \, G\left(t, e^{\lambda t} \, T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) .$$

If $G \equiv 0$ then System (2.5) is decoupled since in this case, it reduces to

$$\begin{cases} \dot{v}_1 = -W^{-1}v_1 - W^{-1}h_1 , \\ v_1(0) = c_1 , \end{cases} \tag{2.6}$$

$$\begin{cases} N\dot{v}_2 = -v_2 - h_2 , \\ v_2(0) = c_2 . \end{cases} \tag{2.7}$$

System (2.6) can be solved by means of standard methods. Remind that

$$v_1(t) = e^{-W^{-1}t} \, c_1 - \int_0^t e^{-W^{-1}(t-\tau)} \, W^{-1} h_1(\tau) \, d\tau ,$$

provided that h_1 is continuous on $[0, T]$. Under the additional smoothness assumption, that h_2 is of class C^{k-1} on $[0, T]$, the system (2.7) is solved by

$$v_2(t) = -h_2(t) + N\dot{h}_2(t) - N^2\ddot{h}_2(t) + \ldots + (-1)^k \, N^{k-1} h_2^{(k-1)}(t) .$$

That means that the initial condition of the initial value c cannot be chosen arbitrarily since $c_2 = \Lambda_2 c$ is determined by $h_2 = \Lambda_2 h$ and its derivatives $\dot{h}_2, \ldots, h_2^{(k-1)}$. The initial condition needs indeed to satisfy the consistency condition

$$\Lambda_2 T^{-1} c = -h_2(0) + N\dot{h}_2(0) + \ldots + (-1)^k \, N^{k-1} h_2^{(k-1)}(0) . \tag{2.8}$$

Remark 2.3: If $h_2 = 0$, then it is clear that if $c \in \mathrm{R}(\hat{E}_\lambda^k)$ then the consistency condition (2.8) is satisfied. Indeed

$$\hat{E}_\lambda^k = T \begin{pmatrix} W^k & 0 \\ 0 & 0 \end{pmatrix} T^{-1},$$

and if $c \in \mathrm{R}(\hat{E}_\lambda^k)$, then $c = \hat{E}_\lambda^k x$ for some $x \in \mathbf{R}^N$ and clearly

$$\Lambda_2 T^{-1} c = \Lambda_2 \begin{pmatrix} W^k & 0 \\ 0 & 0 \end{pmatrix} T^{-1} x = 0 .$$

Assume now that $T^{-1}(A - \lambda E)^{-1} \,\mathrm{e}^{-\lambda t}\, G\left(t, \mathrm{e}^{\lambda t}\, T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$ is decoupled as

$$T^{-1}(A - \lambda E)^{-1} \,\mathrm{e}^{-\lambda t}\, G\left(t, \mathrm{e}^{\lambda t}\, T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} \Phi(t, v_1) \\ 0_{k\times k} \end{pmatrix},$$

where $\Phi\colon [0,T] \times \mathbf{R}^{n-k} \longrightarrow \mathcal{P}(\mathbf{R}^{n-k})$ is some set-valued function. Such reduction may be expected for various problems in mechanics since the unilateral forces are introduced in the formulation of the usual equations of motions that are in fact not the cause of the singularities occurring in the whole model. Then the systems (2.6) and (2.7) reduce to

$$\begin{cases} \dot{v}_1 \in -W^{-1} v_1 - W^{-1} h_1 - W^{-1} \Phi(t, v_1), \\ v_1(0) = c_1 , \end{cases} \tag{2.9}$$

$$\begin{cases} N \dot{v}_2 = -v_2 - h_2 , \\ v_2(0) = c_2 . \end{cases} \tag{2.10}$$

The differential inclusion (2.9) reduces now to a standard one that has been studied in the mathematical literature while (2.10) can be solved as described here-above.

3. Second order differential inclusions with singular matrices

We consider in this section a second order system:

Find $q\colon [0,T] \longrightarrow \mathbf{R}^N$ such that

$$\begin{cases} M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \in f(t) + F(t, q(t), \dot{q}(t)), \quad \text{a.e} \quad t \in (0,T), \\ q(0) = q_0 , \\ \dot{q}(0) = q_1 , \end{cases} \tag{3.1}$$

where M, C, $K \in \mathbf{R}^{N\times N}$ denote mass, damping, and stiffness matrices, respectively, and $f\colon [0,T] \longrightarrow \mathbf{R}^N$ denotes a vector-valued function while $F\colon [0,T] \times \mathbf{R}^N \times \mathbf{R}^N \longrightarrow \mathcal{P}(\mathbf{R}^N)$ is a set-valued function.

This system can be rewritten in the first order form (2.1) by defining

$$x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}, \qquad E = \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ -K & -C \end{pmatrix}, \tag{3.2}$$

$$H(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \qquad G(t, x) = \begin{pmatrix} 0 \\ F(t, x_1(t), x_2(t)) \end{pmatrix}, \qquad c = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}.$$

Note that if the matrix M is singular then E is singular, too. On the other hand, if K is singular then A is also singular.

Let us define by

$$P(\lambda) = \det\left(\lambda^2 M + \lambda C + K\right)$$

the characteristic matrix polynomial associated to the second order system in (3.1).

It is clear that the approach stated in the previous section can be specialized to the second order differential inclusion (3.1) as soon as the matrices E and A in (3.2) forms a regular matrix pencil. The following theorem provides a general result in this sense.

Theorem 3.1: *There exists $\lambda \in \mathbf{R}$ such that $P(\lambda) \neq 0$ if and only if* $\operatorname{rank}(\lambda E - A) = 2N$, *i.e., the matrices E and A form a regular matrix pencil.*

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Proof: Suppose that there exists $\lambda \in \mathbb{R}$ such that $P(\lambda) \neq 0$. If $\lambda \neq 0$ then

$$\det(\lambda E - A) = \det\begin{pmatrix} \lambda I & -I \\ K & \lambda M + C \end{pmatrix} = \det\begin{pmatrix} \lambda I & -I \\ \lambda \frac{K}{\lambda} & \lambda M + C \end{pmatrix}$$

$$= \lambda^N \det\begin{pmatrix} I & -I \\ \frac{K}{\lambda} & \lambda M + C \end{pmatrix} = \lambda^N \det\left(\lambda M + C + \frac{K}{\lambda}\right) = \det(\lambda^2 M + \lambda C + K).$$

It results that there exists $\lambda \in \mathbb{R}\backslash\{0\}$ such that $P(\lambda) \neq 0$ if and only if $\det(\lambda E - A) \neq 0$.

If $\lambda = 0$ we have

$$\det(-A) = \det\begin{pmatrix} 0 & -I \\ K & C \end{pmatrix} = P(0).$$

Thus $P(0) \neq 0$ if and only if K is regular. □

Remark 3.2: Condition $P(\lambda) \neq 0$ is more general than the one (discussed in [1]) requiring the invertibility of $M + C + K$. For example, if

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we see that

$$M + C + K = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

is singular, while

$$4M + 2C + K = \begin{pmatrix} 5 & -6 \\ -6 & 6 \end{pmatrix}$$

is regular, that is $P(2) \neq 0$.

We define $\hat{M}_\lambda = \lambda^2 M + \lambda C + K$ and $\hat{E}_\lambda = (\lambda E - A)^{-1} E$. Note that

$$\hat{E}_\lambda = \begin{pmatrix} \hat{M}_\lambda^{-1}(\lambda M + C) & \hat{M}_\lambda^{-1} M \\ -\hat{M}_\lambda^{-1} K & \lambda \hat{M}_\lambda^{-1} M \end{pmatrix}.$$

Indeed, assume that $\lambda \neq 0$. Then we check that

$$\begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} \lambda I & -I \\ K & \lambda M + C \end{pmatrix} \begin{pmatrix} \hat{M}_\lambda^{-1}(\lambda M + C) & \hat{M}_\lambda^{-1} M \\ -\hat{M}_\lambda^{-1} K & \lambda \hat{M}_\lambda^{-1} M \end{pmatrix}.$$

Only the relation $0 = K\hat{M}_\lambda^{-1}(\lambda M + C) - (\lambda M + C)\, \hat{M}_\lambda^{-1} K$ needs some attention. We have indeed

$$K\hat{M}_\lambda^{-1}(\lambda M + C) - (\lambda M + C)\, \hat{M}_\lambda^{-1} K = K\hat{M}_\lambda^{-1} \left(\lambda M + C + \frac{K}{\lambda}\right) - K\hat{M}_\lambda^{-1} \frac{K}{\lambda}$$

$$- \left(\lambda M + C + \frac{K}{\lambda}\right) \hat{M}_\lambda^{-1} K + K\hat{M}_\lambda^{-1} \frac{K}{\lambda} = K\hat{M}_\lambda^{-1} \frac{\hat{M}_\lambda}{\lambda} - \frac{\hat{M}_\lambda}{\lambda} \hat{M}_\lambda^{-1} K = 0.$$

If $\lambda = 0$ then K is regular and we check that

$$\begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} 0 & -I \\ K & C \end{pmatrix} \begin{pmatrix} K^{-1}C & K^{-1}M \\ -I & 0 \end{pmatrix}.$$

The following two propositions give few properties of the matrix $\hat{E}_\lambda$ in a general framework. The approach used here is similar to the one developed in [1].

Proposition 3.3: *Let $\lambda \in \mathbb{R}$ such that $P(\lambda) \neq 0$. Then the matrix $\hat{E}_\lambda$ satisfies the following properties:*

(i) $\operatorname{rank}(\hat{E}_\lambda^2) = \operatorname{rank}(M) + \operatorname{rank}\begin{pmatrix} C \\ M \end{pmatrix}$,

(ii) $\operatorname{ind}(\hat{E}_\lambda) \leq 1$ *if and only if* $\operatorname{rank}\begin{pmatrix} C \\ M \end{pmatrix} = N$,

(iii) $\operatorname{ind}(\hat{E}_\lambda) = 0$ *if and only if M is invertible.*

Proof: (i) Let

$$y_k = \begin{pmatrix} y_{k1} \\ y_{k2} \end{pmatrix},$$

and $y_{k+1} = \hat{E}_\lambda y_k$ where $y_{k1}, y_{k2} \in \mathbb{R}^N$ for $k = 1, 2$. We suppose that

$$\hat{E}_\lambda^2 y_1 = 0.$$

Using the notation above, we may write

$$y_3 = \hat{E}_\lambda y_2 = \hat{E}_\lambda^2 y_1 = 0.$$

Let us first check that $\ker(\hat{E}_\lambda^2) \subset \ker\begin{pmatrix} C & M \\ M & 0 \end{pmatrix}$. Reminding the definition of $\hat{E}_\lambda$, we obtain $Ey_2 = (\lambda E - A)\, y_3 = 0$ and $E\hat{E}_\lambda y_1 = Ey_2$. It results that

$$y_{21} = \hat{M}_\lambda^{-1}[(\lambda M + C)\, y_{11} + My_{12}] = 0, \tag{3.3}$$

and

$$My_{22} = M\hat{M}_\lambda^{-1}[-Ky_{11} + \lambda My_{12}] = 0. \tag{3.4}$$

Then, using (3.3) and (3.4), we see that

$$\lambda My_{21} - My_{22} = My_{11},$$

and, since $\lambda My_{21} - My_{22} = 0$, we deduce that $My_{11} = 0$. Then from (3.3), we obtain

$$Cy_{11} + My_{12} = 0.$$

Consequently, we have

$$\begin{pmatrix} C & M \\ M & 0 \end{pmatrix} y_1 = \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} = 0.$$

Conversely, assume that $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \ker\begin{pmatrix} C & M \\ M & 0 \end{pmatrix}$, we have

$$\hat{E}_\lambda y = \begin{pmatrix} \hat{M}_\lambda^{-1}[(\lambda M + C)\, y_1 + My_2] \\ \hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) \end{pmatrix}.$$

Using the fact that $Cy_1 + My_2 = 0$ and $My_1 = 0$, we deduce

$$\hat{E}_\lambda y = \begin{pmatrix} 0 \\ \hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) \end{pmatrix}.$$

Then we obtain

$$\hat{E}_\lambda^2 y = \hat{E}_\lambda \begin{pmatrix} 0 \\ \hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) \end{pmatrix} = \begin{pmatrix} \hat{M}_\lambda^{-1} M\hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) \\ \lambda\hat{M}_\lambda^{-1} M\hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) \end{pmatrix}.$$

Using the fact that $-K = \lambda^2 M + \lambda C - \hat{M}_\lambda$, we obtain

$$\begin{aligned} \hat{M}_\lambda^{-1} M\hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) &= \hat{M}_\lambda^{-1} M\hat{M}_\lambda^{-1}((\lambda^2 M + \lambda C)\, y_1 + \lambda My_2 - \hat{M}_\lambda y_1), \\ &= \hat{M}_\lambda^{-1} M(\hat{M}_\lambda^{-1}(\lambda^2 My_1 + \lambda(Cy_1 + My_2))) - \hat{M}_\lambda^{-1} My_1. \end{aligned}$$

Reminding that $Cy_1 + My_2 = 0$ and $My_1 = 0$, we see that

$$\hat{M}_\lambda^{-1} M\hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) = 0,$$

which implies that

$$\hat{E}_\lambda^2 y = 0.$$

Finally, we deduce that

$$\ker(\hat{E}_\lambda^2) = \ker\begin{pmatrix} C & M \\ M & 0 \end{pmatrix},$$

and

$$\operatorname{rank}(\hat{E}_\lambda^2) = \operatorname{rank}(M) + \operatorname{rank}\begin{pmatrix} C \\ M \end{pmatrix}.$$

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(ii) Recall that $\operatorname{ind}(\hat{E}_\lambda) \le 1$ if and only if $\operatorname{rank}(\hat{E}_\lambda) = \operatorname{rank}(\hat{E}_\lambda^2)$. Since

$$\operatorname{rank}(\hat{E}_\lambda) = \operatorname{rank}(E) = N + \operatorname{rank}(M),$$

and then $\operatorname{rank}(\hat{E}_\lambda) = \operatorname{rank}(\hat{E}_\lambda^2)$ if and only if

$$\operatorname{rank}(M) + \operatorname{rank}\begin{pmatrix} C \\ M \end{pmatrix} = N + \operatorname{rank}(M),$$

that is

$$\operatorname{rank}\begin{pmatrix} C \\ M \end{pmatrix} = N,$$

which implies the desired result.

(iii) Finally, $\operatorname{ind}(\hat{E}_\lambda) = 0$ if and only if $\operatorname{rank}(\hat{E}_\lambda) = 2N$. Since

$$\operatorname{rank}(\hat{E}_\lambda) = \operatorname{rank}(E) = N + \operatorname{rank}(M),$$

it follows that $\operatorname{rank}(\hat{E}_\lambda) = 2N$ if and only if M is invertible. □

Definition 3.4: We say that a matrix $A \in \mathbb{R}^{N\times N}$ has Property (K) if

$$\{x \in \mathbb{R}^N : x^T A x = 0\} \subset \ker(A).$$

The next proposition gives further information on the matrix $\hat{E}_\lambda$.

Proposition 3.5: *Let $\lambda \in \mathbb{R}$ such that $P(\lambda) \neq 0$. Then the matrix $\hat{E}_\lambda$ satisfies the following properties:*

(i) $\ker(\lambda M + C) \subset \ker(M)$, *then*

$$\operatorname{rank}(\hat{E}_\lambda^2) = \operatorname{rank}(M) + \operatorname{rank}(\lambda M + C).$$

(ii) *If*
- (a) $\hat{M}_\lambda$ *has Property* (K),
- (b) C *has Property* (K),
- (c) $\ker(C) = \ker(C^T)$,
- (d) $\ker(M) = \ker(M^T)$,

then

$$\operatorname{ind}(\hat{E}_\lambda) \le 2.$$

Proof: (i) It is clear that

$$\ker\begin{pmatrix} C \\ M \end{pmatrix} = \ker(M) \cap \ker(C) \subset \ker(\lambda M + C).$$

Then, if we suppose $x \in \ker(\lambda M + C) \subset \ker(M)$, it follows immediately that $x \in \ker(C)$, which implies $\ker(\lambda M + C) \subset \ker(M) \cap \ker(C)$. Hence

$$\ker\begin{pmatrix} C \\ M \end{pmatrix} = \ker(\lambda M + C).$$

We also deduce

$$\operatorname{rank}\begin{pmatrix} C \\ M \end{pmatrix} = \operatorname{rank}(\lambda M + C),$$

and thus by (i) in Proposition 3.3, we get

$$\operatorname{rank}(\hat{E}_\lambda^2) = \operatorname{rank}(M) + \operatorname{rank}(\lambda M + C).$$

(ii) We know that if $\ker(\hat{E}_\lambda^3) \subset \ker(\hat{E}_\lambda^2)$ then $\operatorname{ind}(\hat{E}_\lambda^2) \le 2$. Let

$$y_k = \begin{pmatrix} y_{k1} \\ y_{k2} \end{pmatrix},$$

and $y_{k+1} = \hat{E}_\lambda y_k$, where $y_{k1}, y_{k2} \in \mathbb{R}^N$ for $k = 0, 1$. We suppose that

$$\hat{E}_\lambda^3 y_0 = 0.$$

Since $y_1 = \hat{E}_\lambda y_0$, we have

$$\begin{aligned} y_{11} &= \hat{M}_\lambda^{-1}((\lambda M + C)\, y_{01} + M y_{02})\,, \\ y_{12} &= \hat{M}_\lambda^{-1}(-K y_{01} + \lambda M y_{02})\,. \end{aligned} \tag{3.5}$$

Moreover, y_1 belongs to $\ker(\hat{E}_\lambda^2)$ and thus

$$\begin{aligned} C y_{11} + M y_{12} &= 0\,, \\ M y_{11} &= 0\,. \end{aligned} \tag{3.6}$$

From $(3.6)_1$, we get

$$y_{11}^T C y_{11} + y_{11}^T M y_{12} = 0\,.$$

Since $\ker(M) = \ker(M^T)$ and C has Property (K) we obtain $y_{11} \in \ker(C) = \ker(C^T)$ and then by (3.6) it results that $M y_{12} = 0$. Using (3.5), we get

$$y_{11}^T \hat{M}_\lambda y_{11} = y_{11}^T (\lambda M + C)\, y_{01} + y_{11}^T M y_{02}\,,$$

and we obtain

$$y_{11}^T \hat{M}_\lambda y_{11} = 0\,.$$

Assumption (ii)(a) together with the regularity of $\hat{M}_\lambda$ yields

$$y_{11} = 0\,.$$

Then, using the fact that $M y_{12} = 0$, we deduce

$$\lambda M y_{11} - M y_{12} = M \hat{M}_\lambda^{-1} (\lambda^2 M + \lambda C + K)\, y_{01} = M y_{01} = 0\,,$$

and, from $(3.5)_1$, we obtain

$$C y_{01} + M y_{02} = 0\,.$$

Finally, we have

$$\begin{aligned} C y_{01} + M y_{02} &= 0\,, \\ M y_{01} &= 0\,, \end{aligned}$$

which yields $y_0 \in \ker(\hat{E}_\lambda^2)$. □

Remark 3.6: i) If C is positive semi-definite and M is symmetric and positive semi-definite then

$$\ker(\lambda M + C) \subset \ker(M)\,, \qquad \forall \lambda > 0\,.$$

Indeed, if $x \in \ker(\lambda M + C)$ then $x^T(\lambda M + C)\, x = 0$. However $x^T M x \geq 0$, $x^T C x \geq 0$ and thus $x^T M x = x^T C x = 0$. The matrix M being symmetric, it results that $x \in \ker(M)$.

ii) More generally, if C is positive semi-definite and M is cocoercive (see e.g. [12]), i.e., there exists $\alpha > 0$ such that $x^T M x \geq \alpha \|Mx\|^2$, then

$$\ker(\lambda M + C) \subset \ker(M)\,, \qquad \forall \lambda > 0\,.$$

As above, we see that if $x \in \ker(\lambda M + C)$, then in particular $x^T M x = 0$. It results that $\|Mx\| = 0$ and then $x \in \ker(M)$.

4. Model reduction

The aim of this section is to present a mathematical approach allowing the reduction of the second order system (3.1).

Define $\Gamma = \{\lambda \in \mathbb{R} : P(\lambda) \neq 0\}$, $N_1(\lambda) := \mathrm{def}(\lambda M + C)$, $N_2(\lambda) := \mathrm{def}(M) - N_1(\lambda)$ for all $\lambda \in \Gamma$, and $N_3 = \mathrm{rank}(M)$. Here, for a matrix A, $\mathrm{def}(A) := \dim(\ker(A))$. It is clear that $N_1(\lambda) + N_2(\lambda) + N_3 = N$. We assume that M is singular but non-zero. In the rest of this section, we note for simplicity $N_1 = N_1(\lambda)$ and $N_2 = N_2(\lambda)$.

Theorem 4.1: *Let the following assumptions hold:*

$$\ker(M) = \ker(M^T)\,, \tag{4.1}$$

There exists $\lambda \in \Gamma$ such that (i) $\ker(\lambda M + C) \subset \ker(M)$,

(ii) $\ker(\lambda M + C) = \ker(\lambda M + C^T)$, (4.2)

(iii) $N_1 > 0\,, \qquad N_2 > 0\,.$

Let $U = \{x_1, \ldots, x_N\}$ be an orthonormal basis for $\mathbb{R}^N$ such that $\{x_1, \ldots, x_{N_2+N_1}\}$ is an orthonormal basis for $\ker(M)$ *and $\{x_1, \ldots, x_{N_1}\}$ is an orthonormal basis for* $\ker(\lambda M + C)$. *Then we have*

$$U^T M U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_{33} \end{pmatrix}, \qquad U^T C U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{pmatrix},$$

where $M_{33} \in \mathbb{R}^{N_3 \times N_3}$ is regular and $C_{ij} \in \mathbb{R}^{N_i \times N_j}$, $i, j = 2, 3$.

Proof: Under the previous notation, we get the scheme

$$\overbrace{\underbrace{\underbrace{x_1, x_2, \ldots, x_{N_1}}_{\ker(\lambda M + C)}, x_{N_1+1}, \ldots, x_{N_1+N_2}}_{\ker(M)}, x_{N_1+N_2+1}, \ldots, x_N}^{\mathbb{R}^N}.$$

We have

$$x_i^T M x_j = 0, \qquad 1 \le i \le N, \qquad 1 \le j \le N_1 + N_2,$$

and

$$x_i^T C x_j = x_i^T (\lambda M + C)\, x_j = 0, \qquad 1 \le i \le N, \qquad 1 \le j \le N_1.$$

Thus, we may write

$$U^T M U = \begin{pmatrix} 0 & 0 & M_{13} \\ 0 & 0 & M_{23} \\ 0 & 0 & M_{33} \end{pmatrix}, \qquad U^T C U = \begin{pmatrix} 0 & C_{12} & C_{13} \\ 0 & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{pmatrix}, \qquad U^T K U = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}.$$

Moreover, using assumption (4.1), we obtain

$$x_i^T M x_j = (M^T x_i)^T\, x_j = 0, \qquad 1 \le i \le N_1 + N_2, \qquad N_1 + N_2 + 1 \le j \le N,$$

so that

$$M_{13} = M_{23} = 0.$$

Consequently, we deduce that the matrix M_{33} is regular since $\operatorname{rank}(M_{33}) = \operatorname{rank}(M) = N_3$. On the other hand, using assumption (4.2), we see that

$$x_i^T C x_j = (C^T x_i)^T\, x_j = ((\lambda M + C^T)\, x_i)^T\, x_j = 0, \qquad 1 \le i \le N_1, \qquad N_1 + 1 \le j \le N,$$

and thus

$$C_{12} = C_{13} = 0. \qquad \square$$

Remark 4.2: i) Condition $\ker(M) = \ker(M^T)$ is for instance satisfied in the following cases: a) M symmetric, b) M skew-symmetric, c) M positive semi-definite.

ii) Condition $\ker(\lambda M + C) = \ker(\lambda M + C^T)$ is for instance satisfied in the following cases: a) C symmetric, b) M symmetric and $\lambda M + C$ positive semi-definite.

To go further, we suppose now that the matrix $U^T \hat{M}_\lambda U$ has all leading principal minors non-zero. We will refer to this assumption as Condition (H_U). Note that this property is for example ensured as soon as we suppose that $\hat{M}_\lambda$ is positive definite.

Theorem 4.3: *Let assumptions* (4.1) *and* (4.2) *hold together with*

$$\begin{aligned} &\ker(\lambda^2 M + K) \perp \ker(\lambda M + C), \\ &\ker(\lambda^2 M + K) \oplus \ker(\lambda M + C) = \ker(M), \end{aligned} \tag{4.3}$$

$$\ker(\lambda^2 M + K) = \ker(\lambda^2 M + K^T). \tag{4.4}$$

Then

$$U^T K U = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & 0 & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix}.$$

Moreover, if $\hat{M}_\lambda$ satisfies Condition (H_U), then $K_{11} \in \mathbb{R}^{N_1 \times N_1}$ and $C_{22} \in \mathbb{R}^{N_2 \times N_2}$ are regular.

Proof: Assumption (4.3) yields $\ker(\lambda^2 M + K) \subset \ker(M)$ and thus

$$Kx_j = (\lambda^2 M + K)\, x_j = 0\,, \qquad N_1 + 1 \le j \le N_1 + N_2\,.$$

Then it results that

$$K_{12} = K_{22} = K_{32} = 0\,.$$

On the other hand, we have

$$x_i^T K x_j = (K^T x_i)^T x_j = ((\lambda^2 M + K^T)\, x_i)^T x_j\,, \qquad N_1 + 1 \le i \le N_1 + N_2\,, \qquad 1 \le j \le N\,,$$

and thus, by assumption (4.4), we get

$$x_i^T K x_j = 0\,, \qquad N_1 + 1 \le i \le N_1 + N_2\,, \qquad 1 \le j \le N\,.$$

Then it results that

$$K_{21} = K_{23} = 0\,.$$

Moreover, $\begin{pmatrix} K_{11} & 0 \\ 0 & \lambda C_{22} \end{pmatrix}$ is a principal submatrix of $U^T \hat{M}_\lambda U$ so that K_{11} and C_{22} are regular. □

The special case $N_2 = 0$, i.e. $\ker(M) = \ker(\lambda M + C)$, is now treated.

Theorem 4.4: *Let the following assumptions hold:*

$$\ker(M) = \ker(M^T)\,, \tag{4.5}$$

There exists $\lambda \in \Gamma$ *such that* (i) $\ker(\lambda M + C) \subset \ker(M)\,,$

$$\text{(ii)}\ \ker(\lambda M + C) = \ker(\lambda M + C^T)\,, \tag{4.6}$$

$$\text{(iii)}\ N_1 = N_1 + N_2 > 0\,.$$

Let $U = \{x_1, \ldots, x_N\}$ *be an orthonormal basis for* $\mathbb{R}^N$ *such that* $\{x_1, \ldots, x_{N_1}\}$ *is an orthonormal basis for* $\ker(M)$. *Then we have*

$$U^T M U = \begin{pmatrix} 0 & 0 \\ 0 & M_{22} \end{pmatrix}, \qquad U^T C U = \begin{pmatrix} 0 & 0 \\ 0 & C_{22} \end{pmatrix},$$

where $M_{22} \in \mathbb{R}^{N_3 \times N_3}$ *is regular,* $C_{22} \in \mathbb{R}^{N_3 \times N_3}$. *If, in addition,*

$$\hat{M}_\lambda \text{ satisfies Condition } (H_U)\,, \tag{4.7}$$

then $K_{11} \in \mathbb{R}^{N_1 \times N_1}$ *is regular.*

Proof: Under the previous notation, we get the scheme

$$\overbrace{\underbrace{x_1, x_2, \ldots, x_{N_1}}_{\substack{\ker(M) \\ = \\ \ker(\lambda M + C)}}, x_{N_1+1}, \ldots, x_N}^{\mathbb{R}^N}\,.$$

As in Theorem 4.1, we check that

$$U^T M U = \begin{pmatrix} 0 & M_{12} \\ 0 & M_{22} \end{pmatrix}, \qquad U^T C U = \begin{pmatrix} 0 & C_{12} \\ 0 & C_{22} \end{pmatrix}, \qquad U^T K U = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

Moreover, using assumption (4.5), we obtain

$$x_i^T M x_j = (M^T x_i)^T x_j = 0\,, \qquad 1 \le i \le N_1\,, \qquad N_1 + 1 \le j \le N\,.$$

Then it results that $M_{12} = 0$. Consequently, we deduce that the matrix M_{22} is regular since $\operatorname{rank}(M_{22}) = \operatorname{rank}(M) = N_3$.

On the other hand, using assumption (4.6), we get

$$x_i^T C x_j = (C^T x_i)^T x_j = ((\lambda M + C^T)\, x_i)^T x_j = 0\,, \qquad 1 \le i \le N_1\,, \qquad N_1 + 1 \le j \le N\,,$$

and thus $C_{12} = 0$.

If condition (4.7) holds then K_{11} is regular as a principal submatrix of $U^T \hat{M}_\lambda U$. □

The following corollary shows that under assumptions of Theorem 4.3 System (3.1) can be reduced to a regular second order system of differential inclusions.

Corollary 4.5: *Assume that assumptions* (4.1)–(4.5) *hold and let* $S \in \mathbb{R}^{N \times N_3}$ *be the matrix defined by*

$$S = U \begin{pmatrix} -K_{11}^{-1} K_{13} \\ -C_{22}^{-1} C_{23} \\ I \end{pmatrix}.$$

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Then the matrix $S^T MS$ is regular.

Proof: Using the results of Theorem 4.3, it easy to check that

$$S^T MS = M_{33}\,, \qquad S^T CS = C_{33} - C_{32}C_{22}^{-1}C_{23}\,, \qquad S^T KS = K_{33} - K_{31}K_{11}^{-1}K_{13}\,.$$

Consequently, using the transformation $q = S\bar{q}$, System (3.1) reduces to

$$M_{33}\ddot{\bar{q}} + (C_{33} - C_{32}C_{22}^{-1}C_{23})\,\dot{\bar{q}} + (K_{33} - K_{31}K_{11}^{-1}K_{13})\,\bar{q} \in S^T f + S^T F(t, S\bar{q}, S\dot{\bar{q}})\,. \qquad \square$$

The following corollary which follows from Theorem 4.4 gives another case in which a reduction in the number of degrees of freedom can be achieved.

Corollary 4.6: *Assume that assumptions* (4.5)–(4.6) *hold and let $S \in \mathbb{R}^{N \times N_3}$ be the matrix defined by*

$$S = U\begin{pmatrix} -K_{11}^{-1}K_{12} \\ I \end{pmatrix}.$$

Then the matrix $S^T MS$ is regular.

Proof: Using the results of Theorem 4.4, it easy to check that

$$S^T MS = M_{22}\,, \qquad S^T CS = C_{22}\,, \qquad S^T KS = K_{22} - K_{21}K_{11}^{-1}K_{12}\,.$$

Consequently, using the transformation $q = S\bar{q}$, System (3.1) reduces to

$$M_{22}\ddot{\bar{q}} + C_{22}\dot{\bar{q}} + (K_{22} - K_{21}K_{11}^{-1}K_{12})\,\bar{q} \in S^T f + S^T F(t, S\bar{q}, S\dot{\bar{q}})\,. \qquad \square$$

5. Applications

We present in this section some mechanical applications of the theoretical results given previously. Examples 5.1 and 5.3 are dedicated to illustrate the results of Section 4 while Example 5.4 illustrates the Jordan reduction technique presented in Section 2.

Example 5.1: Consider the spring mass system of Fig. 5.1. The mass m is constrained to move only in the vertical direction. The design of the system invokes two linear springs with positive spring constants k_1 and k_2 and two

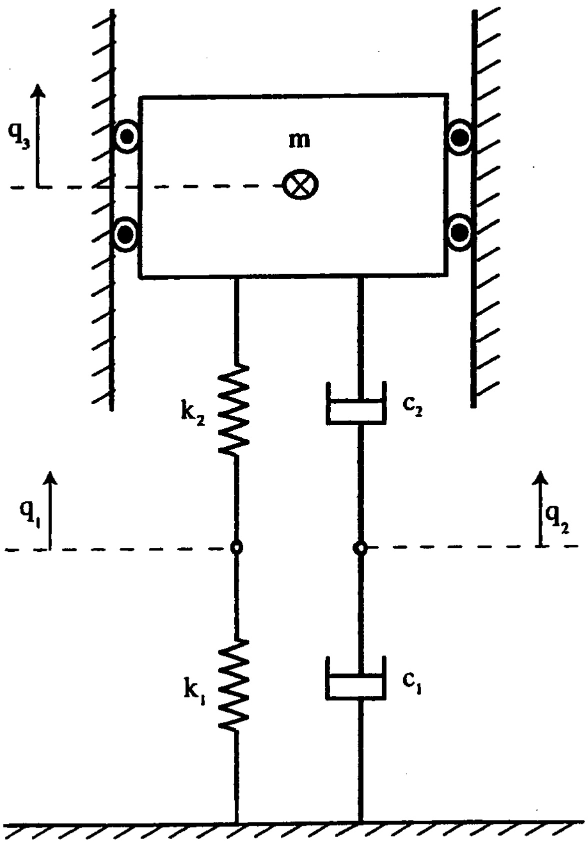


Fig. 5.1

linear viscous dampers with positive damping coefficients c_1 and c_2. The mass m has displacement q_3, the massless joint between the springs has a displacement q_1 while the massless joint between the dampers has a displacement q_2. In such machinery, the rigid body of mass m denotes a machine while the rest of the system models a vibration absorber that is installed between the machine and the supporting ground in order to reduce the effect of vibrations induced for example by a force of excitation $E_0 \sin w_0 t$ (E_0, $w_0 \in \mathbb{R}$). The machine slide yields a total friction force $\tau(\dot{q}_3)$. Here we postulate the following relation between τ and the velocity $\dot{q}_3$:

$$\begin{cases} \text{if} \quad \dot{q}_3 > 0 \quad \text{then} \quad \tau = -a, \\ \text{if} \quad \dot{q}_3 < 0 \quad \text{then} \quad \tau = b, \\ \text{if} \quad \dot{q}_3 = 0 \quad \text{then} \quad \tau \in [-a, b], \end{cases}$$

with $a, b > 0$. Equivalently, we write

$$\tau \in \Gamma(-\dot{q}_3),$$

where Γ is the set-valued function

$$u \longmapsto \Gamma(u) = \begin{cases} -a & \text{if} \quad u < 0, \\ [-a, b] & \text{if} \quad u = 0, \\ b & \text{if} \quad u > 0. \end{cases}$$

The equations of motion are

$$\begin{cases} 0 = -k_1 q_1 + k_2(q_3 - q_1), \\ 0 = -c_1 \dot{q}_2 + c_2(\dot{q}_3 - \dot{q}_2), \\ m\ddot{q}_3 = -k_2(q_3 - q_1) - c_2(\dot{q}_3 - \dot{q}_2) - mg + E_0 \sin w_0 t + \tau(\dot{q}_3). \end{cases}$$

This system can be written under the form

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \in f(t) + F(t, q(t), \dot{q}(t)),$$

with

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_1 + c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{pmatrix}, \qquad K = \begin{pmatrix} k_1 + k_2 & 0 & -k_2 \\ 0 & 0 & 0 \\ -k_2 & 0 & k_2 \end{pmatrix},$$

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} 0 \\ 0 \\ E_0 \sin w_0 t - mg \end{pmatrix}, \qquad F(t, q(t), \dot{q}(t)) = \begin{pmatrix} 0 \\ 0 \\ \Gamma(-\dot{q}_3(t)) \end{pmatrix}.$$

The matrix $M + C + K$ is symmetric and positive definite. Moreover the matrices M, C, and K satisfy the assumptions of Corollary 4.5. Indeed, it is easy to remark that

$$\ker(M) = \ker(M^T), \qquad \ker(M + C) = \ker(M + C^T), \qquad \ker(M + K) = \ker(M + K^T),$$

since the matrices M, $M + C$, and $M + K$ are symmetric. Moreover, it is clear that

$$\operatorname{def}(M) > \operatorname{def}(M + C) > 0 \quad \text{with} \quad \ker(M + C) \subset \ker(M).$$

On the other hand, we have

$$\ker(M + C) \oplus \ker(M + K) = \ker(M) \quad \text{with} \quad \ker(M + C) \perp \ker(M + K),$$

$$\ker(M + C) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \qquad \ker(M + K) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

and

$$\ker(M) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Hence, we apply Corollary 4.5 to construct the orthogonal matrix $U \in \mathbb{R}^{3\times 3}$ and the matrix $S \in \mathbb{R}^{3\times 1}$ as follows:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad S = U \begin{pmatrix} -K_{11}^{-1} K_{13} \\ -C_{22}^{-1} C_{23} \\ I \end{pmatrix},$$

respectively, where $K_{11} = k_1 + k_2$, $K_{13} = -k_2$, $C_{22} = c_1 + c_2$ and $C_{23} = -c_2$. This implies that

$$S = \begin{pmatrix} k_2/(k_1+k_2) \\ c_2/(c_1+c_2) \\ 1 \end{pmatrix}.$$

Then we obtain $S^TMS = m$, $S^TCS = c_1c_2/(c_1+c_2)$ and $S^TKS = k_1k_2/(k_1+k_2)$. Finally, setting $q = S\tilde{q}$, we see that $\tilde{q}$ $(= q_3)$ is the solution of

$$m\ddot{\tilde{q}}(t) + \frac{c_1c_2}{c_1+c_2}\,\dot{\tilde{q}}(t) + \frac{k_1k_2}{k_1+k_2}\,\tilde{q}(t) \in E_0 \sin w_0 t - mg + \Gamma(-\dot{\tilde{q}}(t)). \tag{5.1}$$

Remark 5.2: From the mechanical point of view, the system described in this example can be immediately reduced to (5.1) by setting $1/k := 1/k_1 + 1/k_2$ and $1/c := 1/c_1 + 1/c_2$. But here the goal is just to illustrate simply the mathematical reduction technique presented in Section 4. A more complicated and realistic example (Example 5.3) is given below.

Setting $y = (\tilde{q}, \dot{\tilde{q}})$, we obtain from (5.1) the first order system

$$\dot{y}(t) \in \Phi(t, y(t)),$$

where

$$\Phi(t, y(t)) = \begin{pmatrix} y_2 \\ -\frac{1}{m}\frac{c_1c_2}{c_1+c_2}\,y_2 - \frac{1}{m}\frac{k_1k_2}{k_1+k_2}\,y_1 + \frac{1}{m}\,E_0 \sin\omega_0 t - g + \frac{1}{m}\,\Gamma(-y_2(t)) \end{pmatrix}.$$

The system is now studied on the time interval $[0, T]$, with the initial conditions $y(0) = 0$, i.e. $\tilde{q}(0) = \dot{\tilde{q}}(0) = 0$. The existence of a solution follows from Theorem 1.1. Moreover,

$$\Gamma(x) = \partial h_{a,b}(x),$$

where $h_{a,b}$ is the convex function

$$h_{a,b}(x) = \begin{cases} -ax & \text{if } x < 0, \\ bx & \text{if } x \geq 0. \end{cases}$$

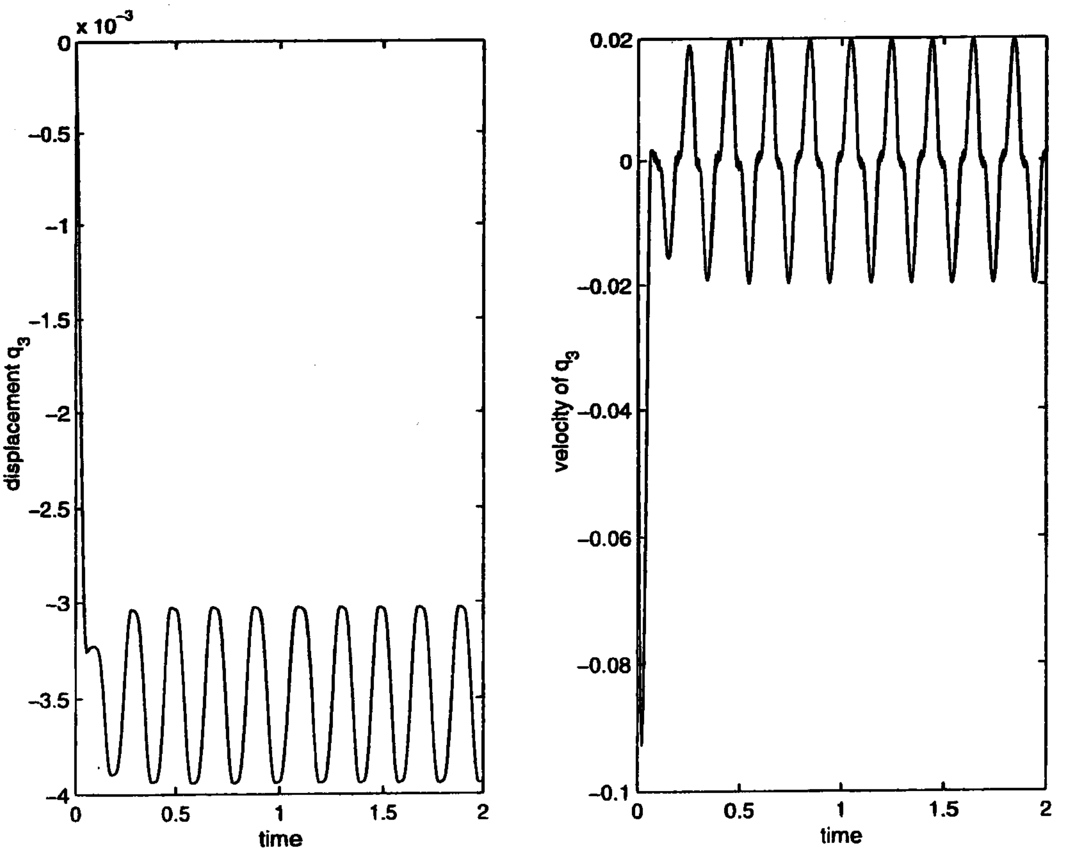


Fig. 5.2

It results that Γ is monotone. We have

$$\Phi(t, y(t)) = \theta(t, y(t)) - \beta(y(t)),$$

with

$$\theta(t, y(t)) = \begin{pmatrix} y_2 \\ -\frac{1}{m}\frac{c_1 c_2}{c_1 + c_2}\, y_2 - \frac{1}{m}\frac{k_1 k_2}{k_1 + k_2}\, y_1 + \frac{1}{m}\, E_0 \sin \omega_0 t - g \end{pmatrix},$$

and

$$\beta(y(t)) = \begin{pmatrix} 0 \\ -(1/m)\, \Gamma(-y_2(t)) \end{pmatrix}.$$

The application θ is Lipschitz continuous uniformly for all $t \in [0, T]$ and β is monotone. The monotonicity of β follows from the one of Γ. We have indeed

$$\begin{aligned}(\beta(z) - \beta(w))^T (z - w) &= (1/m)\, (((-\Gamma(-z_2)) - (-\Gamma(-w_2)))^T (z_2 - w_2), \\ &= (1/m)\, ((\Gamma(-z_2) - \Gamma(-w_2))^T (-z_2 - (-w_2)), \\ &\geq 0.\end{aligned}$$

The uniqueness of the solution follows. The Euler method was applied to the problem with the following data:

m	k_1	k_2	c_1	c_2	E_0	ω_0	a	b	$q_3(0)$	$\dot{q}_3(0)$
25	25×10^4	10×10^4	2500	2000	50	10π	29.43	22.075	0	0

The displacement q_3 of the machine and its velocity $\dot{q}_3$ are depicted in Fig. 5.2. Stick-slip phenomena appears clearly.

In Figs. 5.3, 5.4 we further illustrate the model by means of some additional numerical simulations. We present, in each case, the data chosen and the graph representing the displacement q_3 as well as the phase portrait.

m	k_1	k_2	c_1	c_2	E_0	a	b	$q_3(0)$	$\dot{q}_3(0)$
1	100	25	6	4	0	1	1	2	0

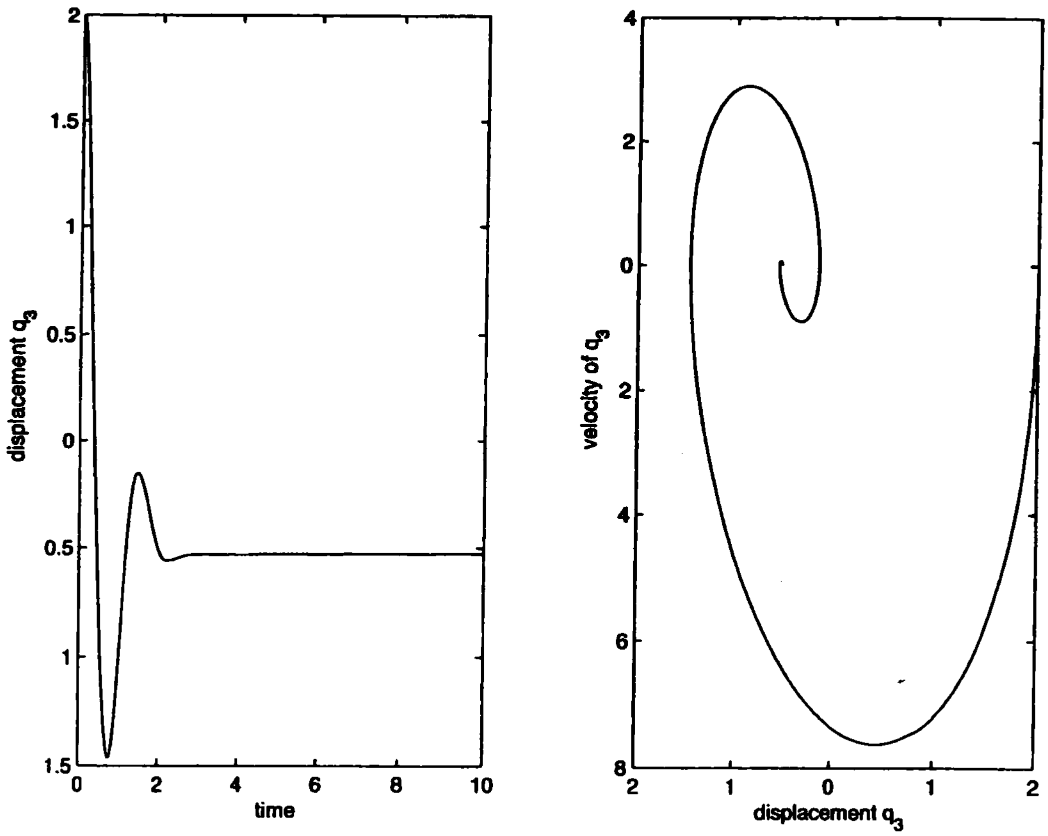


Fig. 5.3

m	k_1	k_2	c_1	c_2	E_0	a	b	$q_3(0)$	$\dot{q}_3(0)$
1	100	25	0	4	0	12	12	2	0

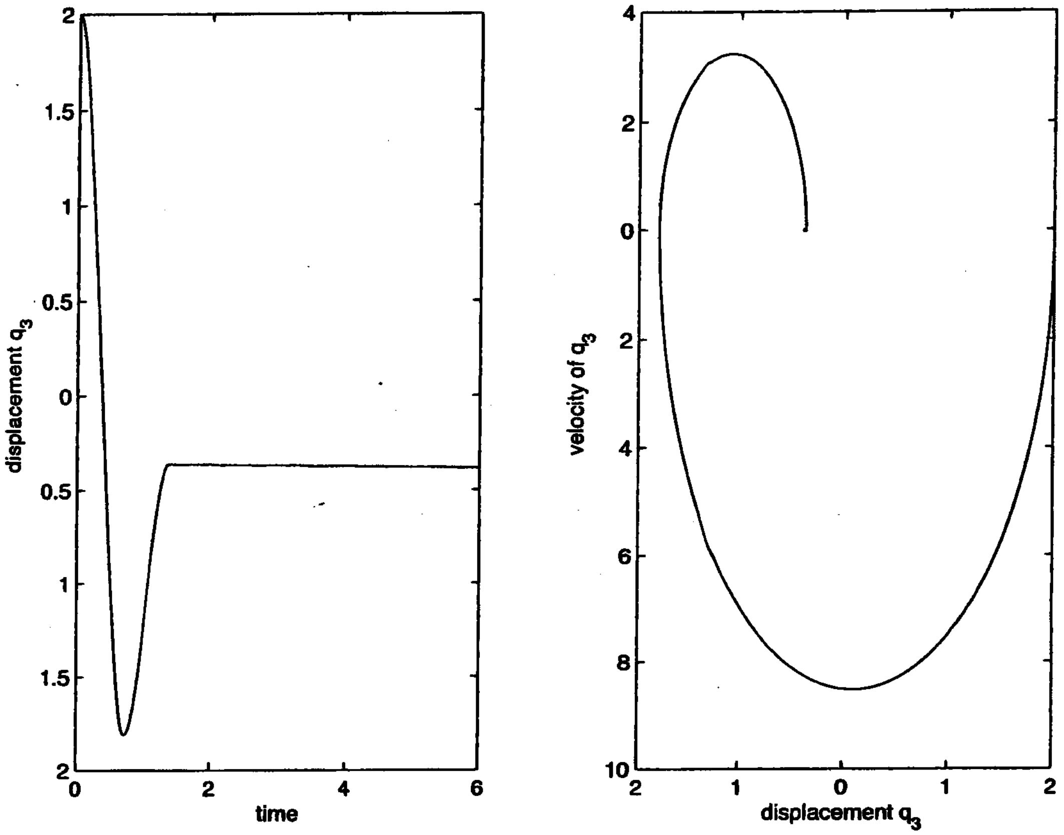


Fig. 5.4

Example 5.3: Fig. 5.5. depicts a model of a shock absorber supported vehicle traveling over a road. The shock absorber involves a linear spring with a positive spring constant k_1 and a linear viscous damper with a positive damping coefficient c_1. The tire is modeled through a mass m. The rigid body of mass M denotes the vehicle. The mass M is topped by a system constituted by a chain of springs and dampers that can be used to model the spinal column of a driver. Here three parts of the column are considered. It is clear that a complete system of vertebrae could be formulatedin a similar way. Here B denotes the mass supported by the spinal column. The road is modeled through the function h. The displacement coordinates are q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7. Let us denote by $\mathcal{N}$ the normal reaction force occurring as soon as the tire is in contact with the road. It is clear that $\mathcal{N} \geq 0$. On the other hand, we must impose the nonpenetration condition $q_0 \geq h$. If $q_0 > h$, then the tire and the road are not in contact so that $\mathcal{N} = 0$. Otherwise, if $\mathcal{N} > 0$ then the tire and the road are in contact and, consequently $q_0 = h$. This normal contact force-displacement relation is depicted in Fig. 5.6.

It is known that the relations

$$\begin{cases} q_0 \geq h\,, & \mathcal{N} \geq 0\,, \\ q_0 > h \Longrightarrow \mathcal{N} = 0\,, \\ \mathcal{N} > 0 \Longrightarrow q_0 = h\,, \end{cases}$$

are equivalent to the set-valued relation

$$\mathcal{N} \in -\partial\Psi_{C(t)}(q_0)\,,$$

where $\Psi_{C(t)}$ denotes the indicator function of the convex set

$$C(t) = \{v \in \mathbb{R} : v \geq h(t)\}\,,$$

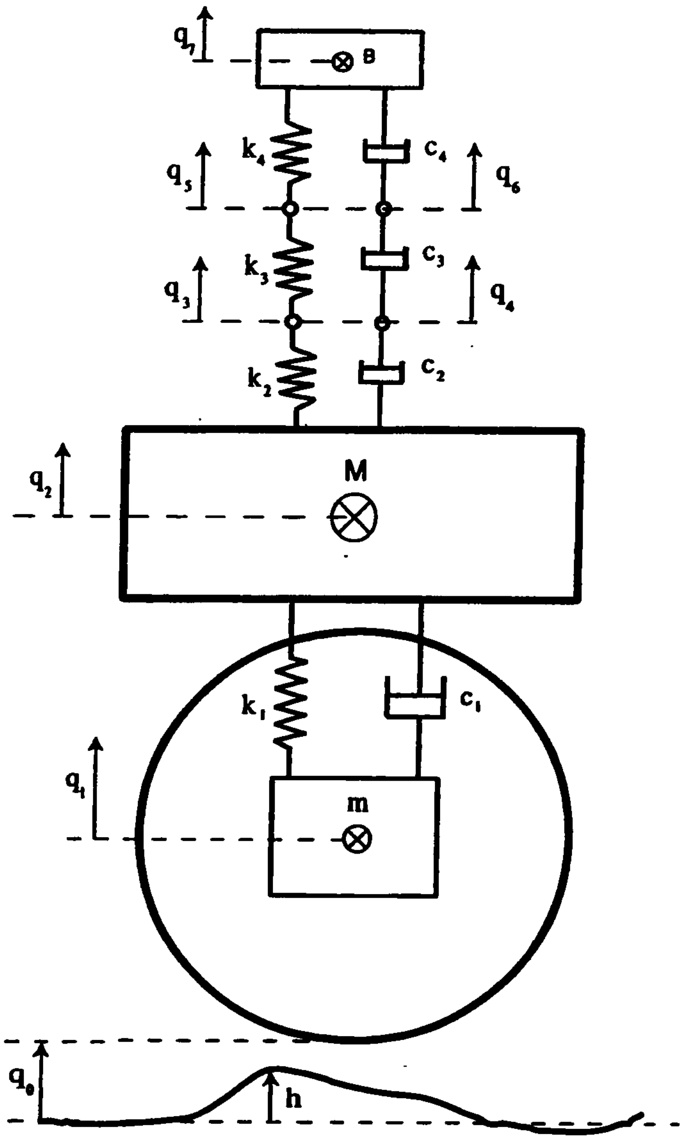


Fig. 5.5

that is

$$\Psi_{C(t)}(z) = \begin{cases} 0 & \text{if} \quad z \in C(t), \\ +\infty & \text{otherwise} . \end{cases}$$

The deformation of the tire is here neglected, so that $q_0 = q_1$. Consequently, we may assume that the transmitted part of the normal reaction $\mathcal{N}$ through the tire spring which is applied on the mass m is equal to $\mathcal{N}$. Therefore, the equa-

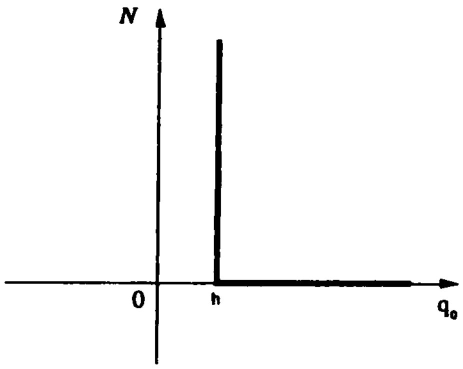


Fig. 5.6. The normal contact force-displacement graph

tions of motion are

$$\begin{cases} m\ddot{q}_1 = & k_1(q_2-q_1)+c_1(\dot{q}_2-\dot{q}_1)+\mathcal{N}-mg\,, \\ M\ddot{q}_2 = & -k_1(q_2-q_1)-c_1(\dot{q}_2-\dot{q}_1)+k_2(q_3-q_2)+c_2(\dot{q}_4-\dot{q}_2)-Mg\,, \\ 0 = & -k_2(q_3-q_2)+k_3(q_5-q_3)\,, \\ 0 = & -c_2(\dot{q}_4-\dot{q}_2)+c_3(\dot{q}_6-\dot{q}_4)\,, \\ 0 = & -k_3(q_5-q_3)+k_4(q_7-q_5)\,, \\ 0 = & -c_3(\dot{q}_6-\dot{q}_4)+c_4(\dot{q}_7-\dot{q}_6)\,, \\ B\ddot{q}_7 = & -k_4(q_7-q_5)-c_4(\dot{q}_7-\dot{q}_6)-Bg\,. \end{cases}$$

We get the system

$$M\ddot{q}(t)+C\dot{q}(t)+Kq(t)\in f(t)+F(t,q(t),\dot{q}(t))\,,$$

where

$$M=\begin{pmatrix} m&0&0&0&0&0&0\\ 0&M&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&B \end{pmatrix},\qquad C=\begin{pmatrix} c_1&-c_1&0&0&0&0&0\\ -c_1&c_1+c_2&0&-c_2&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-c_2&0&c_2+c_3&0&-c_3&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-c_3&0&c_3+c_4&-c_4\\ 0&0&0&0&0&-c_4&c_4 \end{pmatrix},$$

$$K=\begin{pmatrix} k_1&-k_1&0&0&0&0&0\\ -k_1&k_1+k_2&-k_2&0&0&0&0\\ 0&-k_2&k_2+k_3&0&-k_3&0&0\\ 0&0&0&0&0&0&0\\ 0&0&-k_3&0&k_3+k_4&0&-k_4\\ 0&0&0&0&0&0&0\\ 0&0&0&0&-k_4&0&k_4 \end{pmatrix},$$

and

$$q=\begin{pmatrix} q_1\\ q_2\\ q_3\\ q_4\\ q_5\\ q_6\\ q_7 \end{pmatrix},\qquad f(t)=\begin{pmatrix} -mg\\ -Mg\\ 0\\ 0\\ 0\\ 0\\ -Bg \end{pmatrix},\qquad F(t,q(t),\dot{q}(t))=\begin{pmatrix} -\partial\Psi_{C(t)}(q_1)\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}.$$

The matrix $M+C+K$ is symmetric and positive definite. Moreover the matrices M, C and K satisfy the conditions of Corollary 4.5. Indeed, it is easy to remark that

$$\ker(M)=\ker(M^T)\,,\qquad \ker(M+C)=\ker(M+C^T)\,,\qquad \ker(M+K)=\ker(M+K^T)\,,$$

since the matrices M, $M+C$ and $M+K$ are symmetric. Moreover, it is clear that

$$\operatorname{def}(M)>\operatorname{def}(M+C)>0 \quad\text{with}\quad \ker(M+C)\subset\ker(M)\,.$$

On the other hand, we have

$$\ker(M+C)\oplus\ker(M+K)=\ker(M) \quad\text{with}\quad \ker(M+C)\perp\ker(M+K)\,,$$

$$\ker(M+C)=\operatorname{span}\left\{\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix}\right\},\qquad \ker(M+K)=\operatorname{span}\left\{\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\end{pmatrix}\right\},$$

and

$$\ker(M) = \operatorname{span}\left\{\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\end{pmatrix}\right\}.$$

Here the matrix $U \in \mathbb{R}^{7\times 7}$ is given by

$$U = \begin{pmatrix}0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\1&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&1\end{pmatrix}.$$

Then we obtain

$$U^TMU = \begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&m&0&0\\0&0&0&0&0&M&0\\0&0&0&0&0&0&B\end{pmatrix},$$

$$U^TCU = \begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&c_2+c_3&-c_3&0&-c_2&0\\0&0&-c_3&c_3+c_4&0&0&-c_4\\0&0&0&0&c_1&-c_1&0\\0&0&-c_2&0&-c_1&c_1+c_2&0\\0&0&0&-c_4&0&0&c_4\end{pmatrix},$$

$$U^TKU = \begin{pmatrix}k_2+k_3&-k_3&0&0&0&-k_2&0\\-k_3&k_3+k_4&0&0&0&0&-k_4\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&k_1&-k_1&0\\-k_2&0&0&0&-k_1&k_1+k_2&0\\0&-k_4&0&0&0&0&k_4\end{pmatrix}.$$

Moreover, the matrix $S \in \mathbb{R}^{7\times 3}$ is given by

$$S = U\begin{pmatrix}-K_{11}^{-1}K_{13}\\-C_{22}^{-1}C_{23}\\I_{33}\end{pmatrix},$$

where

$$K_{11} = \begin{pmatrix}k_2+k_3&-k_3\\-k_3&k_4+k_3\end{pmatrix},\qquad K_{13} = \begin{pmatrix}0&-k_2&0\\0&0&-k_4\end{pmatrix},$$

$$C_{22} = \begin{pmatrix}c_2+c_3&-c_3\\-c_3&c_3+c_4\end{pmatrix},\qquad C_{23} = \begin{pmatrix}0&-c_2&0\\0&0&-c_4\end{pmatrix}.$$

This implies that

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{(k_4+k_3)\,k_2}{k_4k_2+k_4k_3+k_3k_2} & \dfrac{k_3k_4}{k_4k_2+k_4k_3+k_3k_2} \\ 0 & \dfrac{(c_3+c_4)\,c_2}{c_2c_3+c_2c_4+c_3c_4} & \dfrac{c_3c_4}{c_2c_3+c_2c_4+c_3c_4} \\ 0 & \dfrac{k_3k_2}{k_4k_2+k_4k_3+k_3k_2} & \dfrac{(k_2+k_3)\,k_4}{k_4k_2+k_4k_3+k_3k_2} \\ 0 & \dfrac{c_3c_2}{c_2c_3+c_2c_4+c_3c_4} & \dfrac{(c_2+c_3)\,c_4}{c_2c_3+c_2c_4+c_3c_4} \\ 0 & 0 & 1 \end{pmatrix}.$$

It results that

$$S^TMS = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & B \end{pmatrix},$$

$$S^TCS = \begin{pmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1 + \dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} & -\dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} \\ 0 & -\dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} & \dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} \end{pmatrix},$$

$$S^TKS = \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + \dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} & -\dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} \\ 0 & -\dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} & \dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} \end{pmatrix}.$$

Finally, setting $q = S\bar{q}$, we see that $\bar{q}$ is the solution of

$$\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & B \end{pmatrix} \ddot{\bar{q}} + \begin{pmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1 + \dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} & -\dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} \\ 0 & -\dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} & \dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} \end{pmatrix} \dot{\bar{q}}$$

$$+ \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + \dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} & -\dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} \\ 0 & -\dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} & \dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} \end{pmatrix} \bar{q} \in \begin{pmatrix} -mg \\ -Mg \\ -Bg \end{pmatrix} + \begin{pmatrix} -\partial\Psi_{C(t)}(\bar{q}_1) \\ 0 \\ 0 \end{pmatrix}.$$

Note here that S^TCS and S^TKS are singular and that $\bar{q}_1 = q_1 = q_0$, $\bar{q}_2 = q_2$ and $\bar{q}_3 = q_7$. The contact is supposed elastic so that the recovery coefficient takes value 1 and q_1 satisfies the impact law

$$\dot{q}_1(t_+) - \dot{h}(t_+) = -(\dot{q}_1(t_-) - \dot{h}(t_-)),$$

for all $t \in [0, T]$ such that $q_1 = 0$ (see for instance [18]). Setting now

$$\tilde{q} = \begin{pmatrix} \bar{q}_1 - h \\ \bar{q}_2 \\ \bar{q}_3 \end{pmatrix},$$

and assuming that h is Lipschitz continuous and $\dot{h}$ has bounded variations, we obtain the system

$$\ddot{\bar{q}} + \begin{pmatrix} \frac{c_1}{m} & -\frac{c_1}{m} & 0 \\ -\frac{c_1}{M} & \frac{c_1}{M} + \frac{c_4c_2c_3}{M(c_4c_2+c_2c_3+c_3c_4)} & -\frac{c_4c_2c_3}{M(c_4c_2+c_2c_3+c_3c_4)} \\ 0 & -\frac{c_4c_2c_3}{B(c_4c_2+c_2c_3+c_3c_4)} & \frac{c_4c_2c_3}{B(c_4c_2+c_2c_3+c_3c_4)} \end{pmatrix} \dot{\bar{q}}$$

$$+ \begin{pmatrix} \frac{k_1}{m} & -\frac{k_1}{m} & 0 \\ -\frac{k_1}{M} & \frac{k_1}{M} + \frac{k_4k_2k_3}{M(k_4k_2+k_2k_3+k_3k_4)} & -\frac{k_4k_2k_3}{M(k_4k_2+k_2k_3+k_3k_4)} \\ 0 & -\frac{k_4k_2k_3}{B(k_4k_2+k_2k_3+k_3k_4)} & \frac{k_4k_2k_3}{B(k_4k_2+k_2k_3+k_3k_4)} \end{pmatrix} \bar{q}$$

$$+ \begin{pmatrix} \ddot{h} + \frac{c_1}{m}\dot{h} + \frac{k_1}{m}h + g \\ -\frac{c_1}{M}\dot{h} - \frac{k_1}{M}h + g \\ g \end{pmatrix} \in -\partial\psi_K(\bar{q}), \tag{5.2}$$

where $K = \mathbb{R}_+ \times \mathbb{R}$. Note that

$$\partial\psi_K(\bar{q}) = \begin{pmatrix} \partial\ \psi_{\mathbb{R}_+}(\bar{q}_1) \\ 0 \\ 0 \end{pmatrix}.$$

System (5.2) is considered with the initial conditions

$$\bar{q}(0) = \bar{q}_0 \in \mathbb{R}_+, \qquad \dot{\bar{q}}(0_+) = \bar{q}_1 \in T_{\mathbb{R}_+}(\bar{q}_0),$$

and the impact law, as used in [18],

$$\dot{\bar{q}}(t_+) = -\dot{\bar{q}}_N(t_-) + \dot{\bar{q}}_T(t_-),$$

for all $t \in [0, T]$ such that $\bar{q}(t) \in \partial K$, i.e. $\bar{q}_1(t) = 0$. The last formulation of the impact law follows from the fact that here

$$\dot{\bar{q}}_N = \begin{pmatrix} \dot{\bar{q}}_1 - \dot{h} \\ 0 \\ 0 \end{pmatrix}, \qquad \dot{\bar{q}}_T = \begin{pmatrix} 0 \\ \dot{\bar{q}}_2 \\ \dot{\bar{q}}_3 \end{pmatrix}.$$

It is now easy to remark that all assumptions required in Theorem 1.2 are here satisfied. Therefore, the existence of a solution in the sense of Theorem 1.2 is ensured. To solve Problem (5.2) a Yosida approximant $(i_d - P_{C(t)})/\lambda$ of $\partial\psi_{C(t)}$ with λ small has been considered (see also (1.8)). Note here that we have

$$\frac{x - P_{C(t)}x}{\lambda} = \frac{x - \max\{x, h(t)\}}{\lambda}.$$

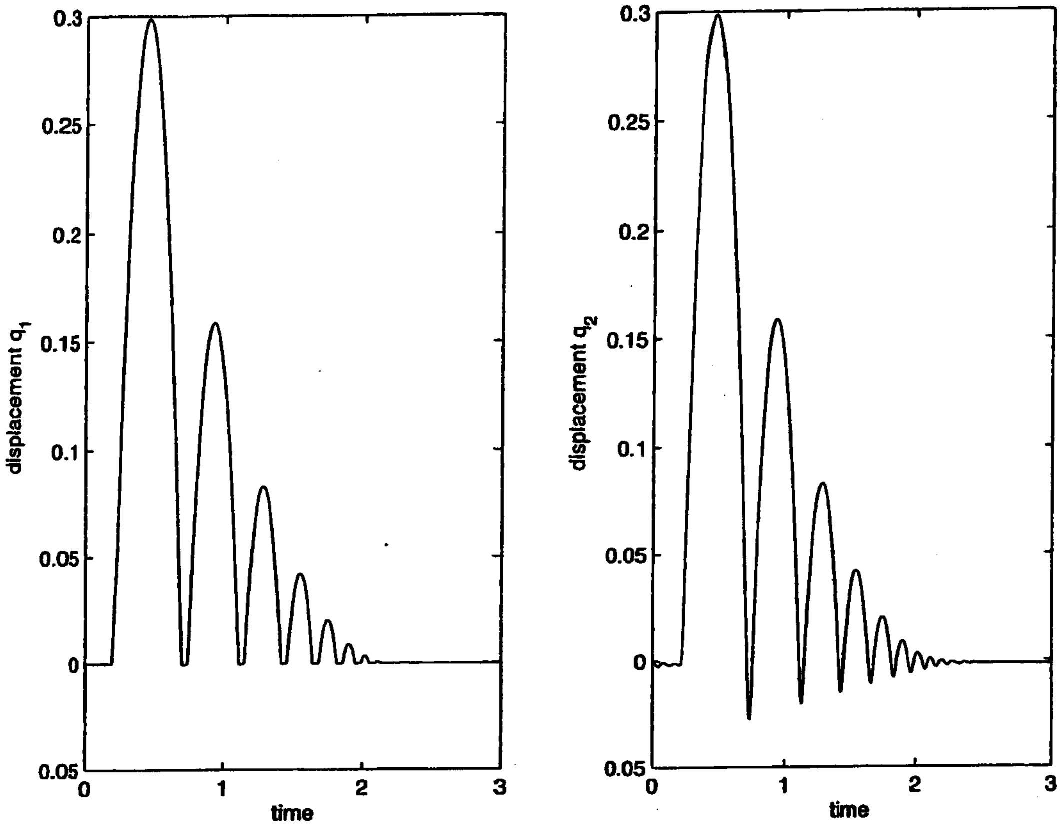
The following technical data were considered:

M	m	B	k_1	k_2	k_3	k_4	c_1	c_2	c_3	c_4
1460	35	25	95×10^5	20×10^5	15×10^5	10^6	21700	20000	15000	10000

and displacements q_1, q_2 and q_7 have been simulated for the following type of road:

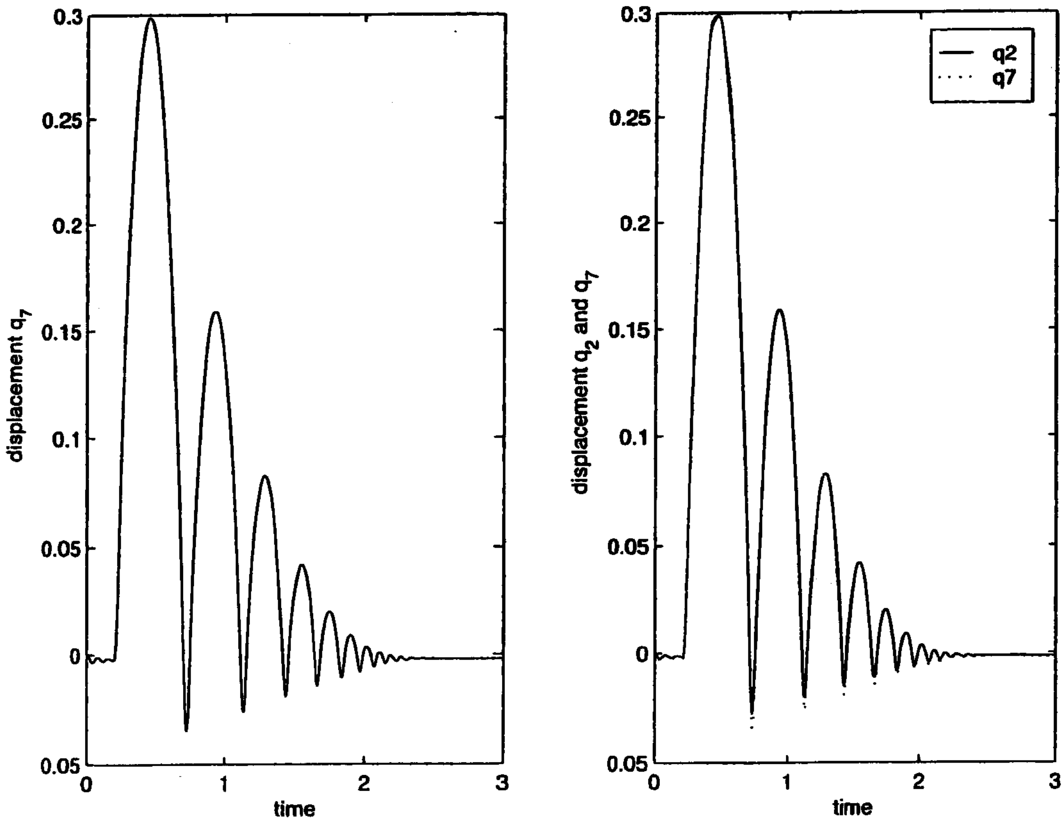
$$\text{Figs. 5.7, 5.8:} \begin{cases} \max\{0,\, 0.05\sin(10\pi t)\} & \text{if} \quad 0.2 < t < 0.3\,, \\ 0 & \text{otherwise}. \end{cases}$$

The difference between q_2 and q_7 cannot be really distinguished for the previous data. For this aim, let us end this section with data leading to displacements q_2 and q_7 whose relative changes are appreciable (see Fig. 5.9.). The data that we consider are:



displacement q_1
displacement q_2
time
time
0.3
0.25
0.2
0.15
0.1
0.05
0
0.05
0
1
2
3

Fig. 5.7



displacement q_7
displacement q_2 and q_7
q2
q7
time
time
0.3
0.25
0.2
0.15
0.1
0.05
0
0.05
0
1
2
3

Fig. 5.8

M	m	B	k_1	k_2	k_3	k_4	c_1	c_2	c_3	c_4
1460	35	25	35×10^5	20000	15000	10000	21700	2000	1500	1000

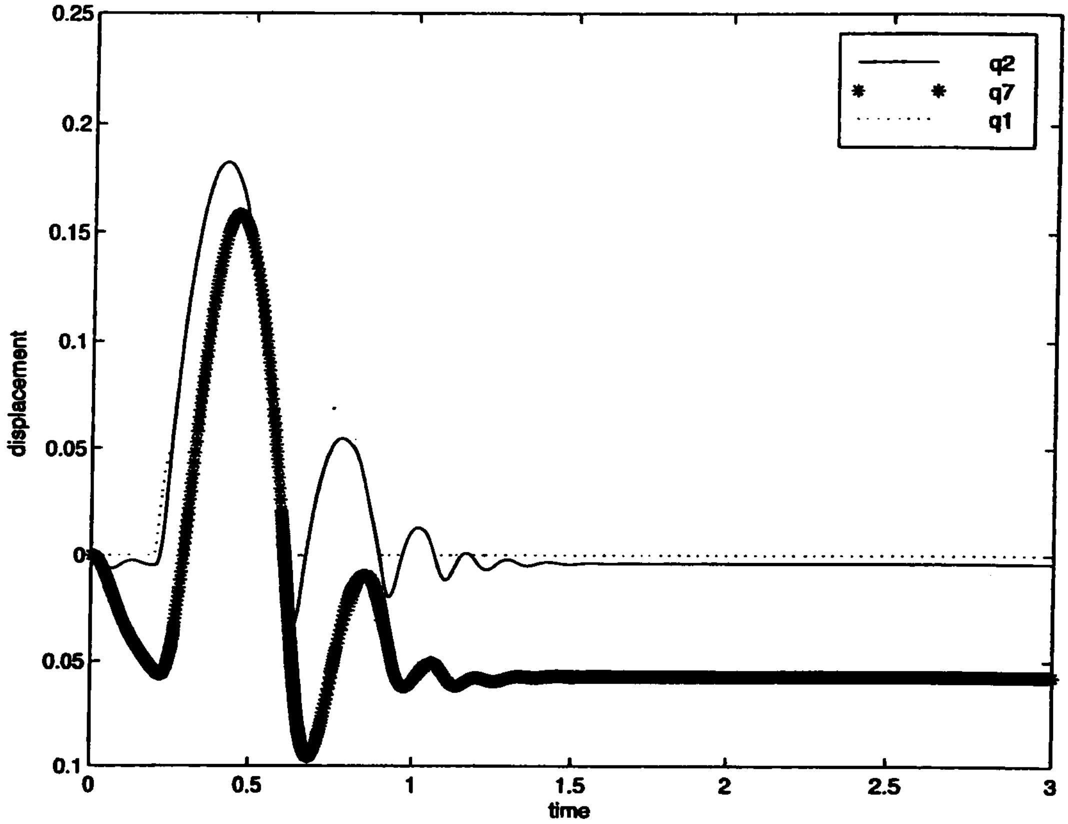


Fig. 5.9

Example 5.4: Reduction methods are of great interest in the mathematical treatment of models of skeletal muscle. The approach discussed in Section 2 is illustrated here. The system displayed in Fig. 5.8. is formed by a model of a fiber of some skeletal muscle and a device of mass m used to simulate the fiber response to external forces. The symbols T_0 and T_N denote the tendinous fiber parts. A linear model for the force across the tendinous part is $(i = 0, N)$

$$F_{T_i} = k_i\delta_i + c_i\dot{\delta}_i\,, \tag{5.3}$$

where δ_i denotes the relative elongation of the element T_i and k_i, c_i are positive constants. The symbols T_i $(i = 1, \ldots, N-1)$ denote the Z-disks marking the boundaries of the muscle fiber. The linear model (5.3) is also considered for $i = 1, \ldots, N-1$. The symbols F_i $(i = 1, \ldots, N-1)$ are used to represent the fibril substructure. The springs of constant K_i $(i = 1, \ldots, N-1)$ are introduced to model the i-th sarcomere within the fiber. The symbols B_i $(i = 1, \ldots, N-1)$ denote the cross-bridges. The force across the cross-bridges is defined by the relation $(i = 1, \ldots, N-1)$

$$F_{B_i} = \gamma_i\lambda_i + \theta_i\dot{\lambda}_i\,,$$

where λ_i is the relative elongation of the element B_i and γ_i, θ_i are positive constants. The "boxes" C_i $(i = 1, \ldots, N-1)$ are pure contractile elements. These elements are active component in the fiber model. The force output of the i-th-contractile machinery is denoted by $\delta_{C_i}(\theta_i)$, where θ_i is the relative elongation of the element C_i. For further details about the functional characteristics and models of muscle elements, we refer the reader to [13]. The machine slide yields a total friction force that can be written as follows:

$$\tau \in \Gamma(-\dot{q}_N)\,,$$

where Γ is defined as in example 5.1.

For example, in the case $N = 2$, the dynamic of the model is described by the system

$$M\ddot{Q}(t) + C\dot{Q}(t) + KQ(t) \in F(Q(t))\,,$$

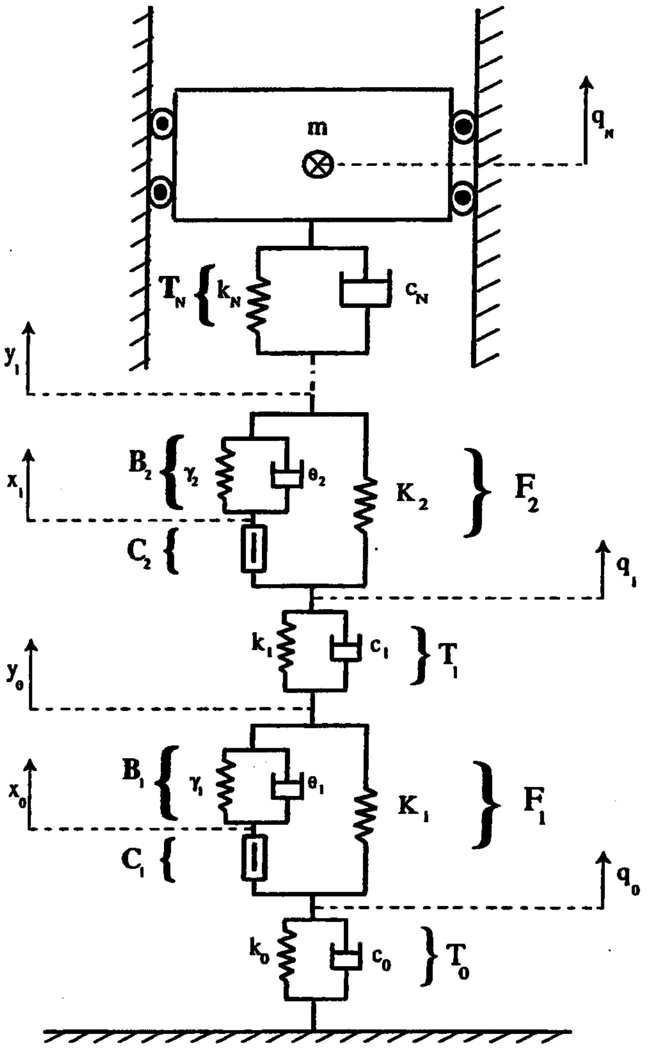


Fig. 5.10

where

$$M = \begin{pmatrix} 0_{6\times 6} & 0_{6\times 1} \\ 0_{1\times 6} & m \end{pmatrix},$$

$$K = \begin{pmatrix} k_0 + K_1 & 0 & -K_1 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & -\gamma_1 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_1 - K_1 & \gamma_1 + k_1 + K_1 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & -k_1 & k_1 + K_2 & 0 & -K_2 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2 & -\gamma_2 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_2 - K_2 & \gamma_2 + K_2 + k_2 & -k_2 \\ 0 & 0 & 0 & 0 & 0 & -k_2 & k_2 \end{pmatrix},$$

$$C = \begin{pmatrix} c_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_1 & -\theta_1 & 0 & 0 & 0 & 0 \\ 0 & -\theta_1 & \theta_1 + c_1 & -c_1 & 0 & 0 & 0 \\ 0 & 0 & -c_1 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta_2 & -\theta_2 & 0 \\ 0 & 0 & 0 & 0 & -\theta_2 & \theta_2 + c_2 & -c_2 \\ 0 & 0 & 0 & 0 & 0 & -c_2 & c_2 \end{pmatrix}, \quad Q = \begin{pmatrix} q_0 \\ x_0 \\ y_0 \\ q_1 \\ x_1 \\ y_1 \\ q_2 \end{pmatrix}, \quad F(Q) = \begin{pmatrix} F_{C_1}(x_0 - q_0) \\ -F_{C_1}(x_0 - q_0) \\ 0 \\ F_{C_2}(x_1 - q_1) \\ -F_{C_2}(x_1 - q_1) \\ 0 \\ \Gamma(-\dot{q}_2) \end{pmatrix}.$$

Setting $X_1 = q_0$, $X_2 = x_0$, $X_3 = y_0$, $X_4 = q_1$, $X_5 = x_1$, $X_6 = y_1$, $X_7 = q_2$, $X_8 = \dot{q}_0$, $X_9 = \dot{x}_0$, $X_{10} = \dot{y}_0$, $X_{11} = \dot{q}_1$, $X_{12} = \dot{x}_1$, $X_{13} = \dot{y}_1$, $X_{14} = \dot{q}_2$, we rewrite the system as the first order differential inclusion

$$E\dot{X} \in AX + G(X),$$

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where

$$E = \begin{pmatrix} I_{7\times 7} & 0_{7\times 6} & 0_{7\times 1} \\ 0_{6\times 7} & 0_{6\times 6} & 0_{6\times 1} \\ 0_{1\times 7} & 0_{1\times 6} & m \end{pmatrix}, \qquad A = \begin{pmatrix} 0_{7\times 7} & I_{7\times 7} \\ -K & -C \end{pmatrix}, \qquad G = \begin{pmatrix} 0_{7\times 1} \\ F_{C_1}(X_2 - X_1) \\ -F_{C_1}(X_2 - X_1) \\ 0 \\ F_{C_2}(X_5 - X_4) \\ -F_{C_2}(X_5 - X_4) \\ 0 \\ \Gamma(-X_{14}) \end{pmatrix}.$$

Let us consider now our problem with the following data:

m	k_0	c_0	K_1	γ_1	θ_1	k_1	c_1	K_2	γ_2	θ_2	k_N	c_N
1	2	2	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1

Applying the method discussed in Section 2 with $\lambda = 1$, we check that $\hat{E}_1 := (E - A)^{-1}\, E$ has a normal Jordan form $\hat{E}_1 = TJT^{-1}$ where

$$J = \begin{pmatrix} W & 0 \\ 0 & N \end{pmatrix}, \quad W = \begin{pmatrix} \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the notation of Section 2, we see that

$$g = T^{-1}(A - E)^{-1}\, \mathrm{e}^{-t}\, G(\mathrm{e}^t T v(t))\,.$$

Let us recall that $v := T^{-1}u$ and $u = \mathrm{e}^{-t}Q$. The explicit computation of g shows that the first system in (2.5) consists in two differential inclusions and five ordinary differential equations. On the other hand, the second system in (2.5) is formed by one differential equation and six algebraic equations.

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Frictional Contact of a Nonlinear Spring

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Frictional Contact of a Nonlinear Spring

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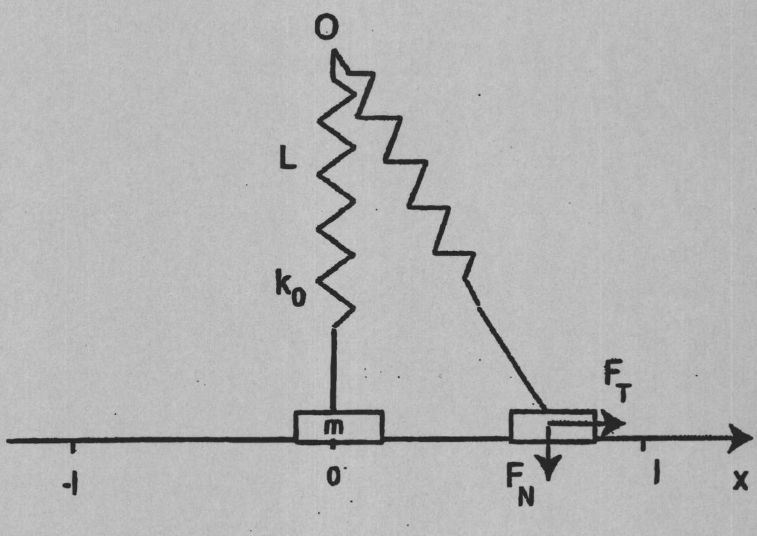
Descriptif

L'étude concerne, dans ce travail, la description et l'analyse d'un problème dynamique avec frottement pour un système contenant un ressort comprimé qui se comporte comme s'il a une constante de raideur négative sur une partie de sa phase d'extension.

On considère un ressort comprimé positionné verticalement dont l'extrémité supérieure est fixe alors que l'autre extrémité est fixée à une masse m qui est susceptible d'osciller le long d'un rail horizontal. Le frottement de la masse avec le rail est modélisé par la loi de Coulomb avec un coefficient de frottement réel fixe. Le système est illustré par la figure ci-dessous. Il se comporte comme un oscillateur non linéaire.



Quand la masse est perturbée de sa position verticale ($x = 0$), le ressort se détend vers sa longueur au repos L_0. Quand on tire sur le ressort au delà de sa longueur au repos, il se comprime. Soient $x = x(t)$ la position du centre de masse de m qui représente aussi l'extrémité inférieure du ressort, L la longueur du ressort en compression, k_0 la constante de raideur du ressort et $x = \pm l$ les points sur l'axe des x où le ressort est à sa longueur au repos. Il résulte ainsi trois points critiques pour le système $x = 0$, $x = -l$ et $x = +l$ autour desquels se constituent trois régions d'adhérence de la masse au rail. Le phénomène est gouverné par une équation différentielle avec un second membre dicontinu.

Deux autres problèmes associés sont étudiés : pour un coefficient de frottement variant de sa valeur statique vers sa valeur dynamique et pour un coefficient de frottement dépendant de la vitesse de glissement de la masse sur la rail.

Dans la dernière partie de ce travail, on décrit l'algorithme numérique utilisé (Euler et Runge-Kutta) puis on donne plusieurs simulations numériques pour différentes valeurs des paramètres.

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Frictional Contact of a Nonlinear Spring

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Abstract—We describe and analyze a frictional problem for a system with a compressed spring which behaves as if it has a spring constant that is negative over a part of its extension range. As a result, the problem has three critical points. The friction is modeled by the Coulomb law. We show that there are three separate stick regions for some values of the parameters, centered on the critical points. We model three other versions of the process. Then we describe a numerical scheme for the models and present a number of computer simulations. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Nonlinear spring, Negative spring constant, Friction.

1. INTRODUCTION

We model, analyze, and simulate the process of frictional contact of a mass which moves on a rail under the influence of a compressed spring. The mechanical behavior of the system is that of a spring which has a negative spring constant over a range of its displacements. This behavior, which is similar to that of the "Duffing Oscillator", introduces a strong nonlinearity into the model. Our main interest is the motion of the mass when it is subject to friction.

The mechanical device, without considering frictional contact, was proposed recently in [1] as a practical means for controlling and stabilizing car suspensions. He showed that a low energy active suspension control can be realized by maintaining the control within the range of displacements where the constant is negative. The mathematical analysis of the frictionless problem in [2] shows that the system has three critical points: two stable and one unstable one.

In this paper, we investigate the problem of friction between the mass and the rail and model it with the Coulomb law. The problem has an unusual structure, and indeed, we show that there may exist three separate stick zones centered on the critical points. If the mass is at rest in any one of these zones, it remains at rest, i.e., it is stuck, because of the frictional resistance force. Thus, these regions are the steady states of the system.

Although the mechanical setting is simple, the problem is interesting and has properties that are not obvious. It gives insight into the behavior of frictional contact that is easy to analyze and simulate.

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The model is derived in Section 2. In Section 3, we analyze the problem and obtain the characterization of the stick zones in terms of the system parameters. We also show, as expected, that the energy dissipates, and therefore, in the absence of external forces, each trajectory ends in one of the stick zones. Then we obtain an estimate on the initial velocity needed to cross the stick zone which is centered on the origin. We present three variants of the problem in Section 4. First, we model the case when the friction coefficient has different values depending on whether the mass is pushed towards or pulled away from the rail. Secondly, we describe the model when the static friction coefficient is larger than the dynamic one. Finally, we model the frictional problem when the spring is under tension. Then, there is only one stick region, and the system behaves as a nonlinear spring with positive spring constant. In Section 5, we describe a numerical algorithm for the problem based on the Runge-Kutta scheme, and present the results of some of our simulations. Section 6 provides a short summary.

2. THE MODEL

We model the dynamic behavior of a mechanical system which consists of a vertically positioned compressed spring attached to a horizontally moving mass. The system, depicted in Figure 1, behaves as a nonlinear oscillator, similarly to the so-called "Duffing Oscillator".

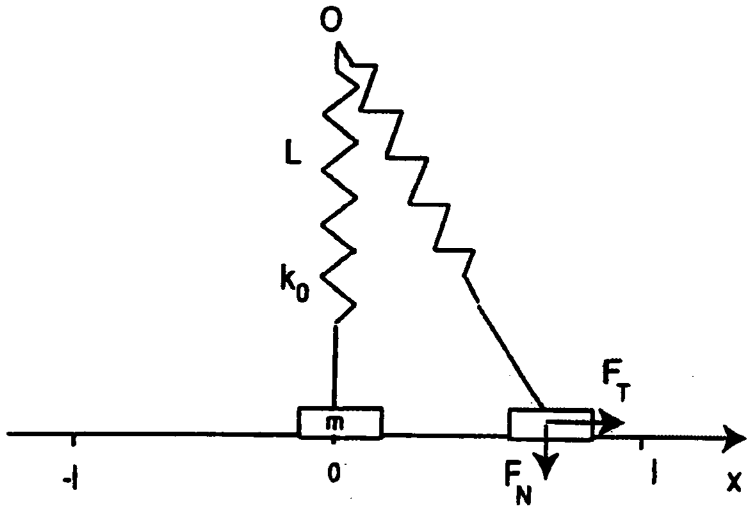

Figure 1. The setting.

A vertically positioned compressed spring is pinned at the pivot O and has a rigid body of mass m attached to its lower end. The mass is attached to a rail, and can move horizontally along the x-axis. When the body is perturbed from the vertical position ($x = 0$), the spring expands to its natural length L_0. When it is stretched beyond its natural length, it contracts. Let $x = x(t)$ denote the position of the center of mass of the body, which is also the end of the spring; L the compressed length of the spring; k_0 the spring constant. Next, let $x = \pm l$ be the points on the x-axis where the spring is at its natural length, thus, $L_0^2 = l^2 + L^2$.

It follows from the geometry in Figure 1 that the force the spring exerts on the body has a horizontal component F_T and a vertical component F_N, which at position x are given by

$$F_T = -kx\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right)$$

and

$$F_N = kL\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right).$$

Here, $k = k_0/m$ and the forces are scaled with m. It follows now from Newton's second law that in the absence of applied forces or damping the motion of the body is determined by

$$x'' + kx\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right) = 0. \tag{2.1}$$

When simple horizontal damping and external forces are taken into account, the equation reads

$$x'' + cx' + kx\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right) = h, \tag{2.2}$$

where the coefficient of damping c and the force h are scaled with m.

Full analysis of this problem can be found in [2]. It is shown there that the system has three critical points $x = 0$ and $x = \pm l$. The first is unstable and the other two are stable equilibrium points.

The problem has the basic structure of the "Duffing Oscillator" which has been discussed in the literature, see, e.g., [3,4] and references therein. However, in the Duffing Oscillator case the force is given by

$$F_D = kx - \delta x^3,$$

for some $\delta > 0$, whereas, in our case it is given by F_T.

We remark that if a damper with damping coefficient c_D is added to the system in parallel with the spring, the resulting equation is

$$x'' + c_D\frac{x^2 x'}{x^2 + L^2} + kx\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right) = 0.$$

This is an unusual equation, and will be investigated elsewhere.

In this paper, we extend the analysis in [2] to include the friction force which arises from the contact with the rail. Now, the body moves horizontally under the influence of the tangential spring force F_T and the frictional resistance force F_{fr}, and the total horizontal force acting on it is

$$f_T = F_T + F_{\mathrm{fr}}. \tag{2.3}$$

The friction force depends on the total vertical force f_N exerted by the mass on the rail, given by

$$f_N = F_N - g = kL\left(1 - \frac{L_0}{\sqrt{L^2 + x^2}}\right) - g, \tag{2.4}$$

where g is the weight (we recall that the scaling is such that $m = 1$). Next, let $x = \pm l_*$ be the points where f_N changes its direction from pointing down ($f_N < 0$) to pointing up ($f_N > 0$). If $kL < g$, such points do not exist and $f_N < 0$ for all x. Otherwise, $\pm l_*$ are the roots of the equation $f_N(l_*) = 0$, thus,

$$l_* = \sqrt{\frac{L_0^2 k^2 L^2}{(kL - g)^2} - L^2}. \tag{2.5}$$

It is easy to verify that $l < l_*$. Below, we assume that $kL > g$, since the other case is simpler to analyze.

We model the frictional resistance of the rail by the classical Coulomb law. Let μ be the coefficient of friction, assumed to be a positive constant. Then we have two cases: either the body is at rest and the tangential force satisfies $|f_T| < \mu|f_N|$, where $\mu|f_N|$ is the so-called *friction bound*, in which case the friction force F_{fr} exactly opposes the tangential force, i.e., $F_{\mathrm{fr}} = -F_T$. Or the body is in motion, the friction force is $|F_{\mathrm{fr}}| = \mu|F_N - g|$ and acts in opposite direction to the motion. Thus,

$$F_{\mathrm{fr}} = -\mu|F_N - g|\ \mathrm{sgn}(x'). \tag{2.6}$$

Here, we used the following definition of the sgn function:

$$\operatorname{sgn}(r) = \begin{cases} 1, & r > 0, \\ 0, & r = 0, \\ -1, & r < 0. \end{cases}$$

The Coulomb condition can be written as follows:

$$\text{if } x' = 0 \text{ and } |F_T| \le \mu|F_N - g|, \qquad \text{then } F_{\text{fr}} = -F_T; \tag{2.7}$$

$$\text{if } x' \ne 0 \text{ or } |F_T| > \mu|F_N - g|, \qquad \text{then } x'' = F_T - \mu|F_N - g| \operatorname{sgn}(x'). \tag{2.8}$$

It follows from (2.7) that in the so-called *stick region*

$$x'' = 0, \tag{2.9}$$

and therefore, when the mass is in this region and has zero velocity, it will remain motionless.

To complete the model, we assume that x satisfies the initial conditions

$$x(0) = x_0, \qquad x'(0) = v_0. \tag{2.10}$$

To summarize, the problem of *the horizontal motion with friction of a mass attached to a vertically compressed spring* consists of finding the displacement function $x : [0,T] \to R$ such that (2.7), (2.8), and (2.10) hold.

More generally, when in addition a horizontal force h acts on the mass, the problem may be written as

$$\text{if } x' = 0 \text{ and } |F_T + h| \le \mu|F_N - g|, \qquad \text{then } F_{\text{fr}} = -(F_T + h); \tag{2.11}$$

$$\text{if } x' \ne 0 \text{ and } |F_T + h| > \mu|F_N - g|, \qquad \text{then } x'' = F_T + h - \mu|F_N - g| \operatorname{sgn}(x'). \tag{2.12}$$

Our interest lies in the case when the driving force is periodic,

$$h = h(t) = A\cos(\omega t),$$

where the amplitude A and the frequency ω are given.

3. STICK REGIONS AND ENERGY DISSIPATION

This section deals with the stick regions and the behavior of the solutions.

First, we investigate the stick regions of the problem. These are the regions where the equality $F_{\text{fr}} = -F_T$ holds when $x' = 0$. From this condition and from $|F_{\text{fr}}| = \mu|F_N - g|$, we can find the boundary points of these regions.

We show that there are two possibilities.

(i) There exists one point $x = l_3$, such that $l < l_3 < l_*$, and an extended stick region which includes the origin and the two equilibrium points $x = \pm l$,

$$S_0 = \{-l_3 \le x \le l_3\}.$$

(ii) There exist three stick regions

$$S_0 = \{-l_1 \le x \le l_1\}, \quad S_+ = \{l_2 \le x \le l_3\}, \quad \text{and} \quad S_- = \{-l_2 \le x \le -l_3\}.$$

The boundary points of the stick regions satisfy

$$0 < l_1 < l_2 < l < l_3 < l_*.$$

Thus, S_0 has the origin at its center, S_+ has the point $x = l$ at its center, and S_- the point $x = -l$.

The ls are obtained by solving the following nonlinear equations. When $0 < x < l$, they satisfy

$$k(\mu L - x)\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right) = \mu g, \tag{3.1}$$

since $F_T > 0$ and $|F_N - g| = -F_N + g$. First, we note that the smallest solution has to satisfy $\mu L < l_1$, since the term in the second brackets on the left-hand side is negative. Now, let

$$\Psi(r) = \left(1 - \frac{L_0}{\sqrt{r^2 + L^2}}\right), \qquad z_\pm = z_\pm(r) = \frac{\mu g}{k(\mu L \pm r)}.$$

Then, condition (3.1) reads

$$\Psi(x) = \frac{\mu g}{k(\mu L - x)} = z_-(x). \tag{3.2}$$

On the interval $\mu L < x < l$, the function $z_-(x)$ is negative, increasing and satisfies

$$\lim_{r \to \mu L} z_-(r) = -\infty, \qquad z_-(l) = -\frac{\mu g}{k(l - \mu L)}.$$

Also, $\Psi(0) = 1 - L_0 L^{-1}$ and $\Psi(l) = 0$. As can be seen from Figure 2, equation (3.2) may have two solutions, $x = l_1$ and $x = l_2$, or one solution $x = l_1 = l_2$, or no solutions.

The case of one solution means that $l_1 = l_2$, and effectively, this and the case of no solutions form Case (i) above. Then the interval $[-l, l]$ lies in S_0.

When equation (3.2) has two distinct solutions $\mu L < l_1 < l_2 < l$, we obtain Case (ii) above.

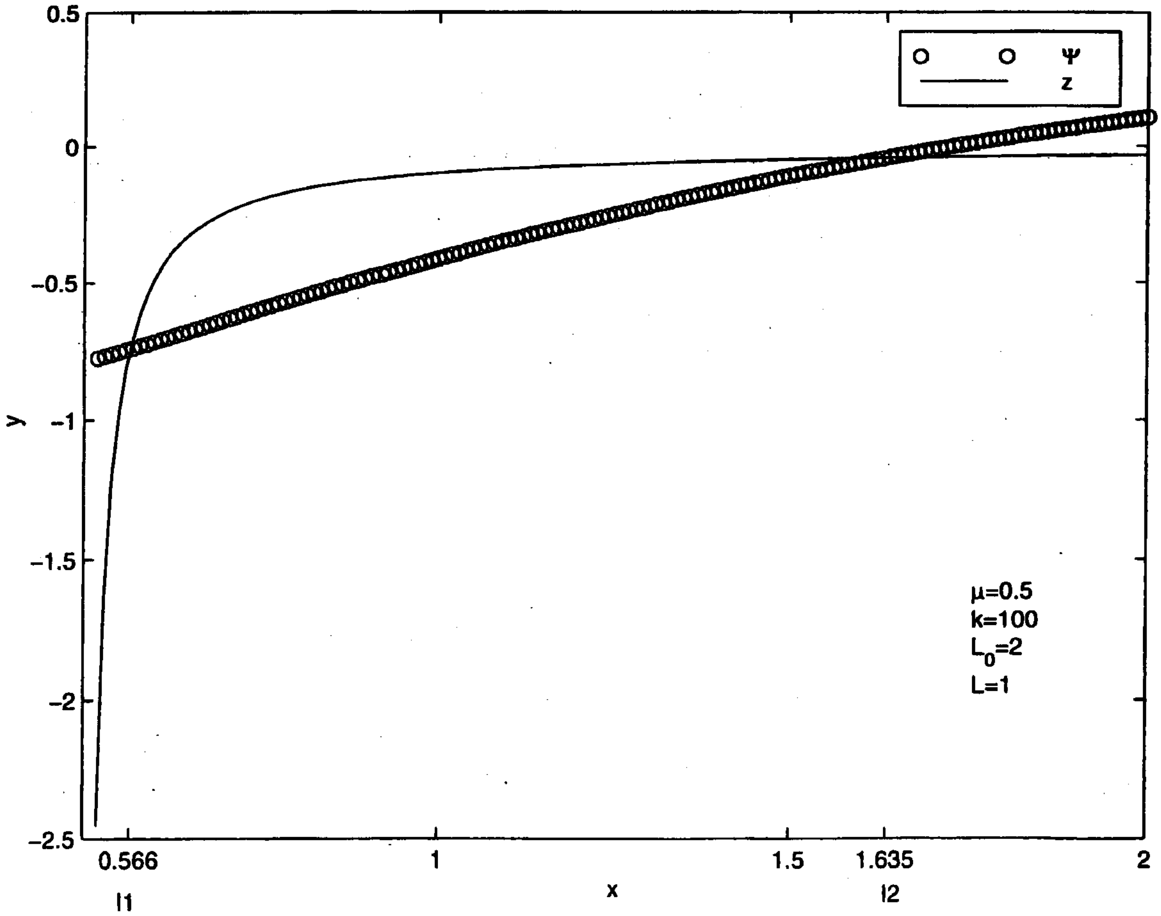

Figure 2. The points l_1 and l_2.

A sufficient condition for the nonexistence of two roots or one root of (3.2) is obtained when the minimum of Ψ is larger than the maximum of z_- over $\mu L \le x \le l$. Thus,

$$1 - \frac{L_0}{L\sqrt{1+\mu^2}} \ge -\frac{\mu g}{k(l-\mu L)}. \tag{3.3}$$

A precise condition for the existence or nonexistence of solutions may be obtained by solving for the single point $x^* = l_1 = l_2$ such that

$$\Psi(x^*) = z_-(x^*) \quad \text{and} \quad \Psi'(x^*) = z'_-(x^*).$$

This is the point where Ψ touches z_- tangentially, and there can be only one such point since Ψ is convex on $\mu L \le x \le l$ and z_- is concave.

Next, we investigate the solutions for $l < x \le l_*$. Then $F_N > 0$ and $|F_N - g| = g - F_N$ by the definition of l_*. Thus, $x = l_3$ satisfies

$$\Psi(x) = z_+(x) = \frac{\mu g}{k(\mu L + x)}. \tag{3.4}$$

$\Psi(x)$ is now positive and increasing with $\Psi(l) = 0$ and $\Psi(l_*) = g/kL$, by the definition of l_*. The right-hand side of (3.4) is positive, decreasing and

$$z_+(l) = \frac{\mu g}{k(\mu L + l)} > 0 = \Psi(l).$$

Then there exists one and only one solution $x = l_3$, since

$$\Psi(l_*) = \frac{g}{kL} > z_+(l_*) = \frac{g}{k(L + l_*/\mu)}.$$

Finally, for $l_* < x$, we have $|F_N - g| = F_N - g$ and it is easy to see that the condition $|F_T| = -F_T = \mu|F_N - g| = F_N - g$ reads $k(x - \mu L)\Psi(x) = -\mu g$, and since $x > \mu L$ and $\Psi > 0$, it follows that there are no solutions.

This establishes the claims above. We stress the unusual structure of the problem when there are three separate stick regions.

We obtain now an energy estimate on the solutions of (2.7), (2.8), and (2.10). Let $x = x(t)$, for $0 \le t \le T$, be a solution of the problem. Let $E = E(t)$ be the total energy of the system, i.e.,

$$E(t) = \frac{1}{2}(x'(t))^2 + \frac{1}{2}k(x(t))^2 - kL_0\sqrt{(x(t))^2 + L^2} + kLL_0, \tag{3.5}$$

scaled so that $E = 0$ when $x = 0$ and $x' = 0$. Then

$$E(t) = E(0) - \mu \int_0^t \left| kL\left(1 - \frac{L_0}{\sqrt{x^2(\tau) + L^2}}\right) - g \right| |x'(\tau)|\, d\tau. \tag{3.6}$$

It follows, as expected, that the frictional force causes energy dissipation. And as long as $x \ne l$ and $x' \ne 0$, the frictional energy dissipation term is nonzero. Moreover, it follows from (3.6) that the trajectories in the phase plane remain in bounded sets. The points $x = \pm l$ are the attractors of the frictionless system. However, the system with friction has the sets $S_0, S_\pm$ as steady states. Indeed, any trajectory in the phase plane which reaches the x-axis in any one of the sets $S_0, S_\pm$ ends there.

We conclude that the set

$$\Omega = (S_0 \cup S_- \cup S_+) \times \{0\} \tag{3.7}$$

in the phase plane is the set of all limit points of the system. Actually, we have a stronger result: for any solution $x = x(t)$, there exists a time $t = t_x$, which depends on the solution, such that $x(t) = x_\omega = \text{const.} \in \Omega$ for all $t_x < t$. The solution reaches its steady state in finite time.

To show (3.6), we multiply equation (2.8) by x' and integrate over $0 \le \tau \le t$ for $0 \le t \le T$. Using straightforward manipulations, we obtain (3.6).

Next, we provide an estimate on the initial velocity v_0 needed for the mass starting at $x_0 = -l_1$ to cross the stick region S_0, assuming that Case (ii) holds. Clearly, we need to guarantee that at the time t^* when the mass reaches the position $x(t^*) = l_1$, its velocity is positive, that is, $x'(t^*) > 0$.

We assume that $x(0) = x_0 = -l_1$, $x'(0) = v_0$ and are interested in the case $x' > 0$, thus (2.8), after an integration over $[0, t]$, can be written as

$$x'(t) = v_0 + k \int_0^t (\mu L - x(\tau))\, \Psi(x(\tau))\, d\tau - \mu g t.$$

Here we used the fact that $|F_N - g| = -F_N + g$. Clearly, it is sufficient to obtain the estimates for

$$v_0 \ge -k \int_0^{t^*} (\mu L - x(\tau))\, \Psi(x(\tau))\, d\tau + \mu g t^*.$$

We note that the integral is nonnegative and we estimate it from below. First, we note that the travel time t^* is necessarily longer than that without any forces, thus $t^* \ge 2l_1/v_0$. Next let t_1 and t_2 be the times such that $x(t_1) = -(1/2)\mu L$ and $x(t_2) = (1/2)\mu L$. Then $t_2 - t_1 > (1/2)t^* \ge \mu L/v_0$. Now, for $t_1 \le t \le t_2$, we have $|\mu L - x(t)| \ge (1/2)\mu L$. Next, for $-\mu L \le x \le \mu L$,

$$|\Psi(x(\tau))| = \left| 1 - \frac{L_0}{\sqrt{x^2(\tau) + L^2}} \right| \ge \frac{L_0}{L\sqrt{1+\mu^2}} - 1.$$

Collecting the estimates above, we obtain

$$v_0 > \frac{1}{2} k \mu L \left(\frac{L_0}{L\sqrt{1+\mu^2}} - 1 \right) \int_{t_1}^{t_2} dt + \frac{2}{v_0} \mu g l_1.$$

Noting that the integral is estimated from below by $t_2 - t_1 \ge \mu L/v_0$, we finally obtain the following sufficient condition for crossing the stick zone S_0:

$$v_0^2 \ge k(\mu L)^2 \left(\frac{L_0}{L\sqrt{1+\mu^2}} - 1 \right) + 2\mu g l_1. \tag{3.8}$$

Clearly, this estimate is not optimal, and better estimates may be obtained using a more refined analysis. One may obtain similar estimates for the other stick regions.

4. RELATED PROBLEMS

In this short section, we describe three related problems.

First, we model the case when the friction coefficient depends on whether the mass is pushed against the rail or is pulled from it. Let μ_1 be the friction coefficient when $f_N < 0$ and μ_2 when $f_N > 0$. We define the friction coefficient function by

$$\mu(x) = \begin{cases} \mu_1, & x \le l_*, \\ \mu_2, & x > l_*. \end{cases}$$

Then, the dynamic problem consists of (2.7) and (2.8) where μ has been replaced with $\mu(x)$, together with (2.10).

The discussion and results above carry over to this case with minor, rather obvious, modifications.

The second version of the problem is obtained when the friction coefficient jumps from the static value μ_0 to a smaller dynamic value μ_d. The problem now is to find a function $x : [0,T] \to \mathbb{R}$ such that

$$\text{if } x' = 0 \text{ and } |F_T| \le \mu_0|F_N - g|, \qquad \text{then } F_{\text{fr}} = -F_T; \tag{4.1}$$

$$\text{if } x' \ne 0 \text{ or } |F_T| > \mu_0|F_N - g|, \qquad \text{then } x'' = F_T - \mu_d|F_N - g|\,\text{sgn}(x'), \tag{4.2}$$

$$x(0) = x_0, \qquad x'(0) = v_0. \tag{4.3}$$

This problem has the same structure as the one with constant friction coefficient. The points $\pm l$ and $\pm l_*$ are the same. The difference is in the size of the stick regions. The conditions defining the stick sets S_0, S_-, S_+ are as in Section 3 with μ_0 replacing μ. On the other hand, the sufficient condition for transit, (3.8), holds with μ_d replacing μ. The energy estimate (3.6) holds with μ_d replacing μ as well.

Furthermore, we may consider the problem with slip-dependent friction coefficient. Let $\mu_d = \mu_d(|x'|)$ be a positive smooth function such that

$$0 < \mu_* \le \mu(r) \le \mu^* \le \mu_0, \qquad r \in \mathbf{R}_+,$$

where μ_* and μ^* are constants. Let β_f be the graph

$$\beta_f(r) = \begin{cases} [\mu_d(0), \mu_0], & r = 0, \\ \mu_d(r), & r > 0. \end{cases}$$

Then the modified problem is to find two functions $x : [0,T] \to \mathbf{R}$ and $\mu : \mathbf{R}_+ \to \mathbf{R}_+$ such that μ is a selection out of the graph β_f and

$$\text{if } x' = 0 \text{ and } |F_T| \le \mu_0|F_N - g|, \qquad \text{then } F_{\text{fr}} = -F_T; \tag{4.4}$$

$$\text{if } x' \ne 0 \text{ or } |F_T| > \mu_0|F_N - g|, \qquad \text{then } x'' = F_T - \mu(|x'|)|F_N - g|\,\text{sgn}(x'), \tag{4.5}$$

$$x(0) = x_0, \qquad x'(0) = v_0. \tag{4.6}$$

This case covers (4.1)–(4.3) when the μ_d is a constant and the vertical segment in β_f is missing.

General contact problems with slip-dependent friction coefficient were considered recently in [5–7].

The final version of the problem which we describe is obtained when the spring is under tension and not under compression. Thus, the stretched length L satisfies $L > L_0$. Then the origin is the unique equilibrium point, and it is stable. It is easy to see that the force acting on the mass has the same tangential and normal components

$$F_T = -kx\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right), \qquad F_N = kL\left(1 - \frac{L_0}{\sqrt{x^2 + L^2}}\right).$$

However, now the expression in the brackets is positive, since $L_0 < \sqrt{L^2 + x^2}$. It is straightforward to show that there is only one stick zone

$$S_0 = \{-\mu L \le x \le \mu L\}.$$

This problem is simpler than the one in Section 2.

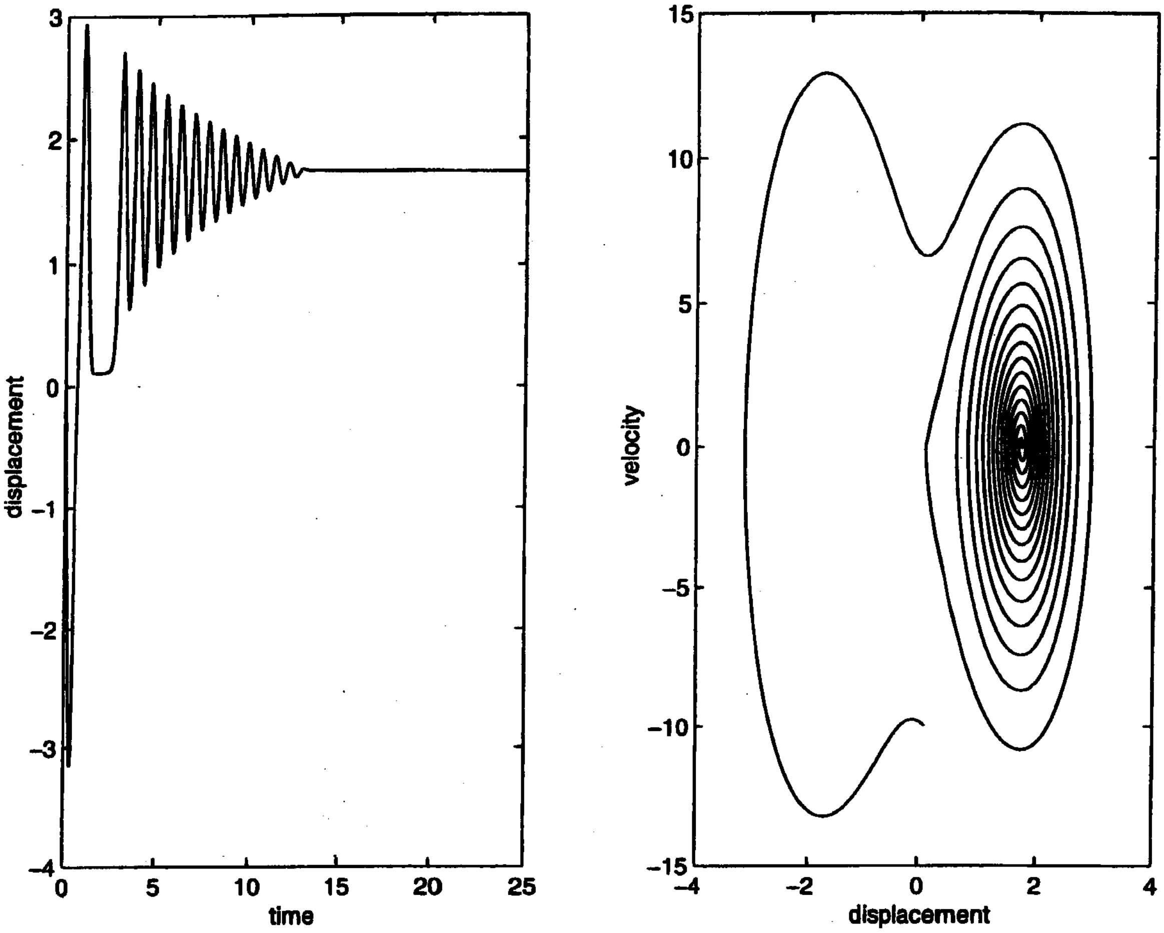

Figure 3. Decay to a steady state and the trajectory in phase plane.

5. NUMERICAL ALGORITHM AND SIMULATIONS

In this section, we describe a numerical algorithm for the problem (2.10)–(2.12). Then we present a number of computer simulations. We use both the Euler method and a Runge-Kutta method to obtain numerical approximations of the solutions.

Our problem consists of a differential equation with a discontinuous right-hand side. Let I represent the stick region(s), i.e., either $I = S_0$ or $I = S_0 \cup S_+ \cup S_-$ (cf. Section 3). Using Filippov's regularization [8], we reformulate our problem as the following initial value problem for a first-order differential inclusion, where $y = (x, x') = (y_1, y_2)$.

Find an absolutely continuous function $y : [0,T] \to \mathbb{R}^2$ *such that*

$$\text{if } y(t) = (y_1, 0) \text{ and } y_1 \in I, \text{ then } \begin{cases} y_1'(t) = 0, \\ y_2'(t) = 0; \end{cases}$$

$$\begin{aligned} &\text{else } y_1'(t) = y_2(t), \\ &\quad y_2'(t) = -ky_1(t)\left(1 - \frac{L_0}{\sqrt{y_1^2(t) + L^2}}\right) + h(t) \\ &\qquad - \mu \left| kL\left(1 - \frac{L_0}{\sqrt{y_1^2(t) + L^2}}\right) - g\right| \operatorname{Sgn}(y_2(t)), \end{aligned} \tag{5.1}$$

for almost all t in $[0, T]$. Here Sgn is the multivalued function

$$\operatorname{Sgn}(x) = \begin{cases} -1, & x < 0, \\ [-1, 1], & x = 0, \\ 1, & x > 0, \end{cases}$$

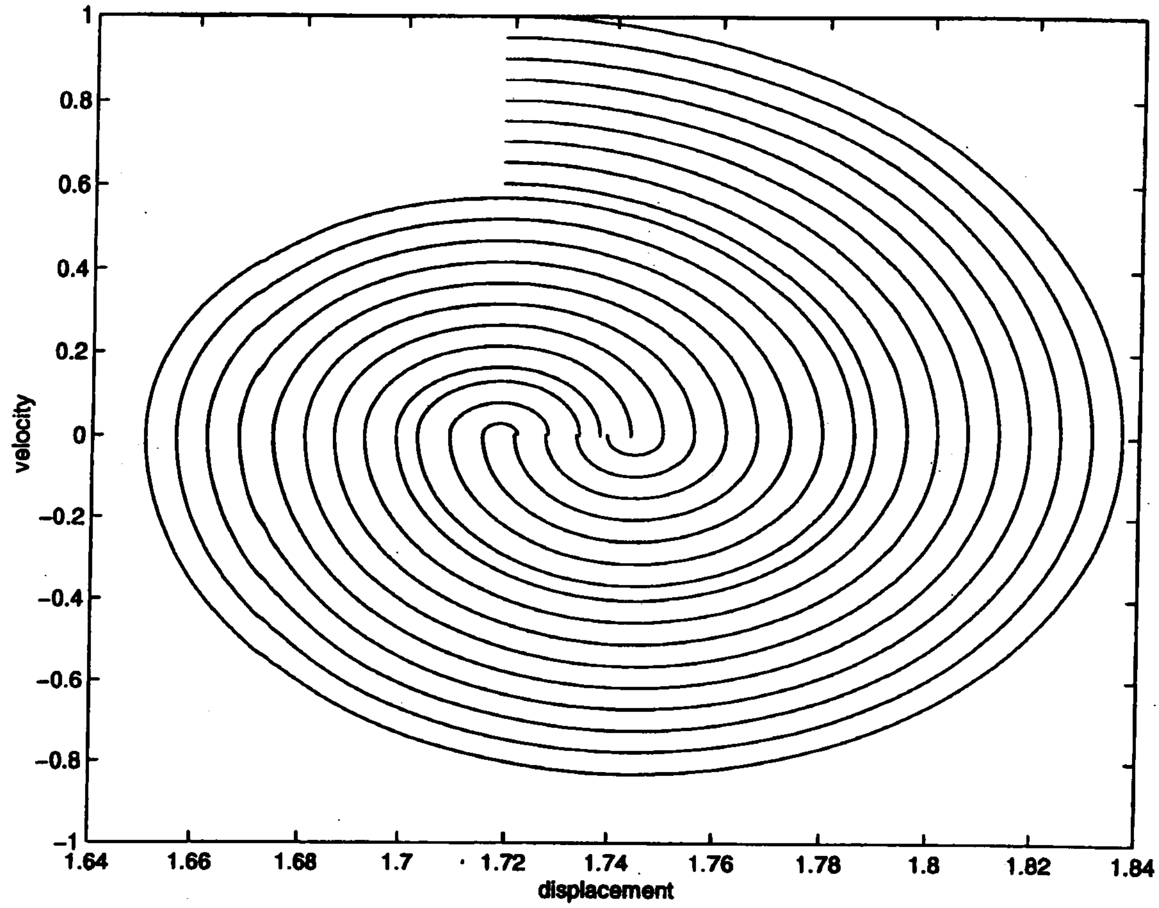

Figure 4. Trajectories starting at l_2.

which is the set-valued extension of the "sgn" function used in (2.11),(2.12). Thus, the problem is in the form of a differential inclusion, say $y' \in \Phi(t, y)$.

For numerical purposes, we replace the differential inclusion $y' \in \Phi(t, y)$ on $[0, T]$ by a sequence of discrete inclusion on the subintervals $t_0 = 0 < t_1 < t_2 < \cdots < t_N = T$ for $N > 0$, with a constant step-size $\Delta t = T/N$. Let a_i, $b_i \in \mathbf{R}$ for $i = 0, \ldots, r$, with $a_r \neq 0$ and $|a_0| + |b_0| > 0$. We are given the starting values $y_j \in \mathbf{R}^n$ for $j = 0, \ldots, r-1$, and the corresponding starting selections $\xi_j \in \Phi(t_i, y_i)$ out of the graph, for $i = 0, \ldots, r-1$. These may be computed by a linear p-step method with $p < r$ or by a one-step method.

Then, for $j = r, \ldots, n$, we compute y_j from

$$\frac{1}{h} \sum_{i=0}^{r} a_i y_{j-r+i} = \sum_{i=0}^{r} b_i \xi_{j-r+i}, \qquad \text{with } \xi_i \in \Phi\left(t_i, y_i\right).$$

When $b_r \neq 0$, the method is implicit.

The following convergence result for these methods can be found in [8,9].

THEOREM 5.1. *Let $D \in \mathbf{R}^n$ and $\Phi : D \times [0, T] \longrightarrow \mathcal{P}(\mathbf{R}^n)$ be a set-valued map. Let the following assumptions be satisfied.*

(i) *Φ is a nonempty closed and convex-valued function.*
(ii) *Φ is upper semicontinuous in $D \times [0, T]$ and verifies*

$$\|\zeta\| \leq c(1 + \|x\|),$$

for all $\zeta \in \Phi(x, t)$, $t \in [0, T]$ and $x \in D$ with constant $C \geq 0$.

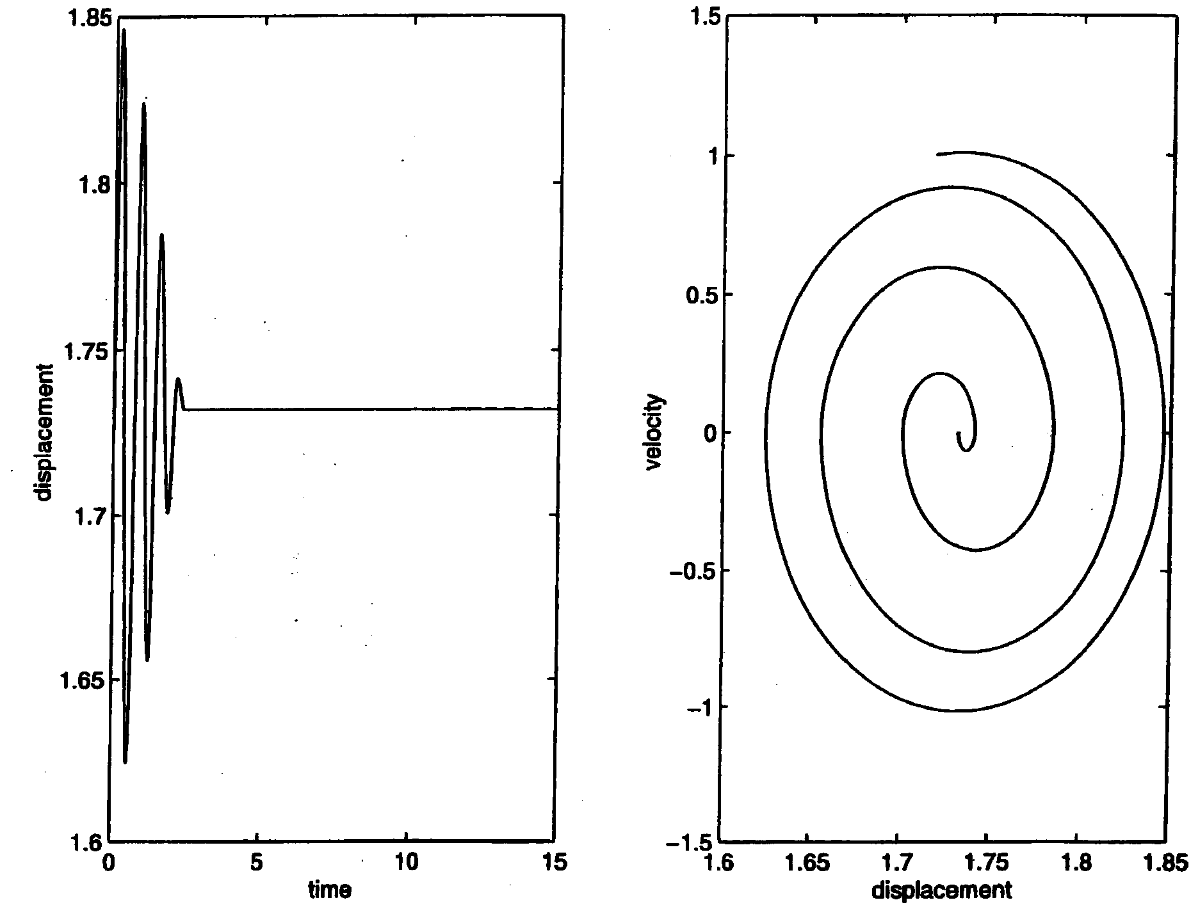

Figure 5. Motion with force $h = \cos(10t)$.

(iii) *All the zeros* λ *of the polynomial* $\sum_{i=0}^{r} a_i\lambda^i$ *have absolute value* $|\lambda| < 1$ *except for the simple zero* $\lambda = 1$.

(iv) *The following consistency conditions are satisfied:*

$$\sum_{i=0}^{r} a_i = 0, \qquad \sum_{i=0}^{r} i a_i = \sum_{i=0}^{r} b_i.$$

(v) *The coefficients* b_i *are nonnegative for* $i = 0, \ldots, r$.

(vi) *The starting values satisfy*

$$\left\|y_{j+1}^n - y_j^n\right\| \le M\Delta t, \qquad (j = 0, \ldots, r-2).$$

(vii) *The approximations of the initial value* y_0 *satisfy* $\lim_{n\to+\infty} y_0^n = y_0$.

Then, the sequence $(y^n)_{n\in\mathbb{N}}$ *of piecewise linear and continuous interpolants of the grid values* $(y_0^n, \ldots, y_n^n)$ *contains a subsequence which converges uniformly to a solution of the initial value problem.*

REMARK. We note that the theorem establishes the convergence of the numerical approximations; however, it provides neither the order of convergence nor any qualitative properties of the limit functions.

Problem (5.1) can be written as

$$y_0^n = y_0,$$
$$y_{i+1}^n \in y_i^n + \Delta t \Phi\left(t_i^n, y_i^n\right).$$

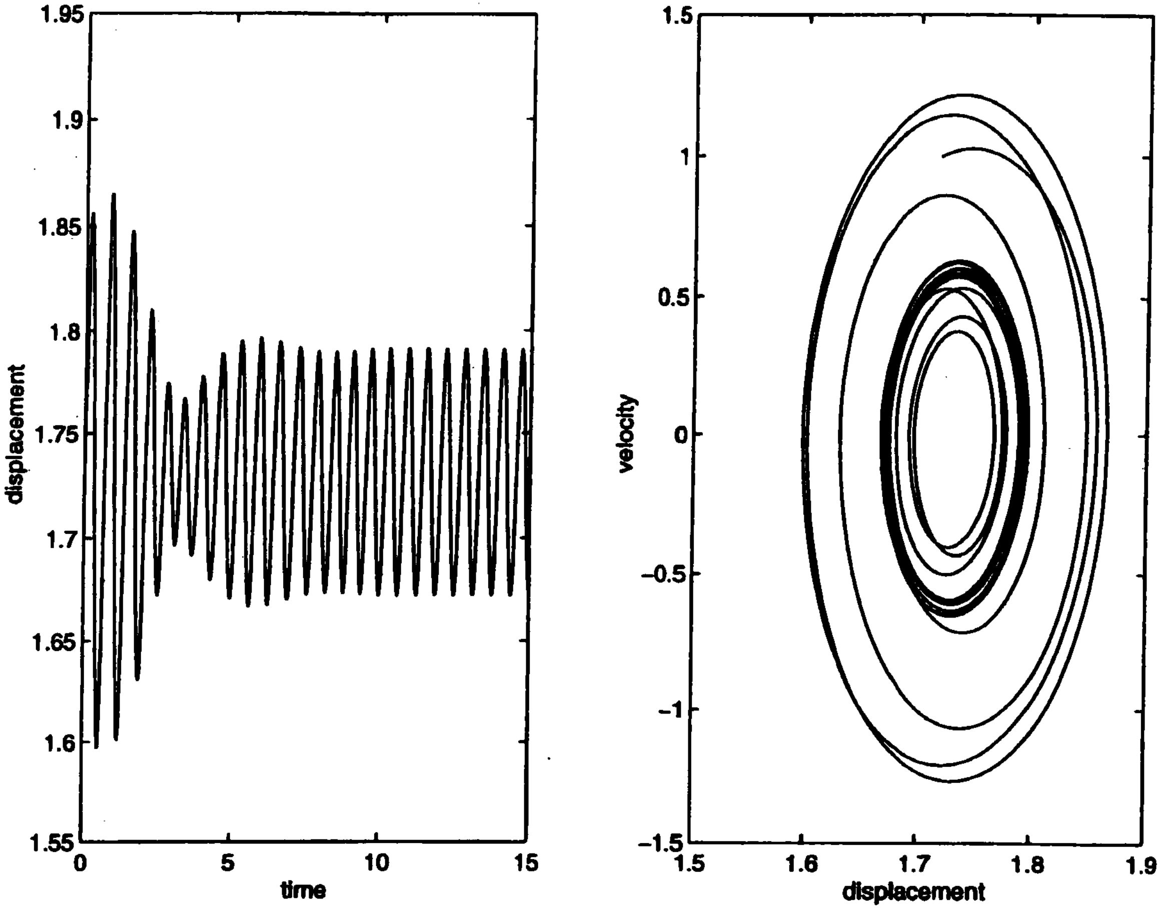

Figure 6. Eventual periodic oscillations.

In our numerical simulations, we used the explicit Euler method, in which the coefficients were chosen as

$$r = 1, \quad a_0 = -1, \quad b_0 = 1, \quad a_1 = 1, \quad b_1 = 0.$$

This scheme verifies the assumptions of the preceding theorem, and therefore, it is convergent. In problem (5.1), the set of multivalued points reduces to the one line $y_2 = 0$, which may be "neglected" during computations. However, starting with zero velocity one has to decide which value of the vertical segment of the graph Sgn to choose. In our case, since in the original problem the single-valued sgn function was used, we numerically choose 0.

In addition to the Euler method, we used a more consistent one, the fourth-order Runge-Kutta method. However, we found that there was no noticeable improvement in the accuracy of the solutions.

The classical fourth-order Runge-Kutta scheme applied to the differential inclusion yields

$$\begin{aligned}
&y_0^n = y_0, \\
&a_0^n \in \Phi\left(t_i^n, y_i^n\right), \\
&a_{1,i}^n \in \Phi\left(t_i^n + \frac{\Delta t}{2}, y_i^n + \frac{\Delta t}{2} a_{0,i}^n\right), \\
&a_{2,i}^n \in \Phi\left(t_i^n + \frac{\Delta t}{2}, y_i^n + \frac{\Delta t}{2} a_{1,i}^n\right), \\
&a_{3,i}^n \in \Phi\left(t_{i+1}^n, y_i^n + \Delta t a_{2,i}^n\right), \\
&y_{i+1}^n = y_i^n + \frac{\Delta t}{6}\left(a_{0,i}^n + 2a_{1,i}^n + 2a_{2,i}^n + a_{3,i}^n\right).
\end{aligned}$$

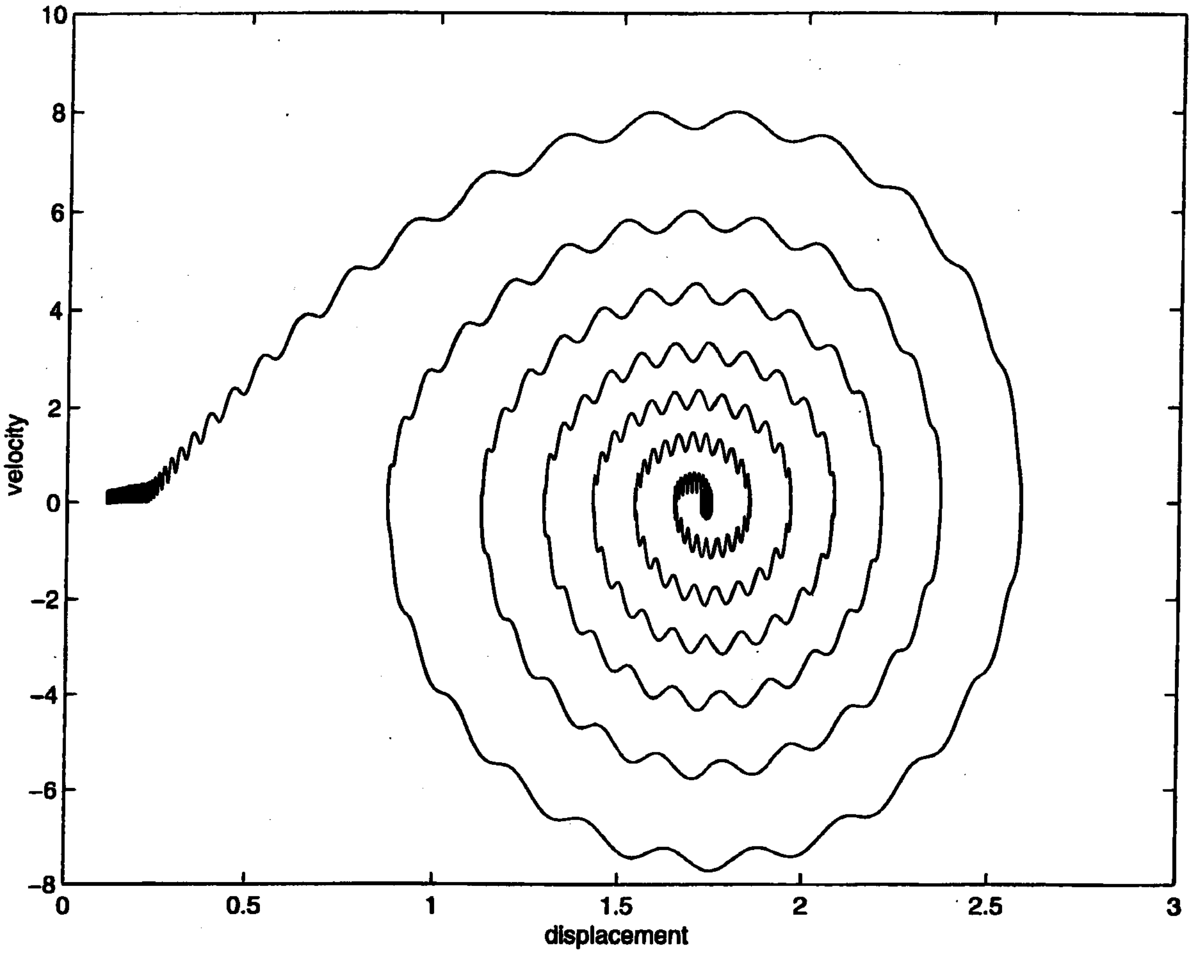

Figure 7. High frequency force $h = -40\cos(200t)$.

This scheme is often used to solve differential inclusion even though it does not verify the assumptions of the preceding theorem (see also [8] for further details).

We now present a number of numerical experiments showing the behavior of the solutions. We begin with a typical behavior depicted in Figure 3.

The initial conditions are $x_0 = 0.1098$, $v_0 = -10.0$. The position of the mass as a function of time and phase portrait are shown in Figure 3. As expected, the trajectory decays to the steady solution $x = l$. However, first it travels once around the origin and both points $x = \pm l$ and then it spirals inward to its steady state. In this simulation, the time interval is $0 \le t \le T = 25$ and the time step is $\Delta t = 0.001$, so the graph represents 2.5×10^4 time steps. Also, $\mu = 0.05$ and $k = 100$.

In Figure 4, we show the trajectories in phase plane starting at $x_0 = l_2 = 1.7181$ and initial velocities $v_0 = 0.6, 0.65, 0.7, \dots, 1.0$. The trajectories are clockwise and all end in S_+. Clearly, once the trajectory reaches the x-axis in the phase plane at a point in Ω, it stops there. In this simulation, the time interval is $0 \le t \le T = 10$, the time step is $\Delta t = 0.001$, $\mu = 0.1$, and $k = 100$.

In Figure 5, we show the motion of the mass, given by (2.10)–(2.12), when a horizontal periodic force $h = \cos(10t)$ acts on it. Clearly, the force is insufficient to overcome the friction. Both the motion in time and the trajectory in the phase plane are depicted. In this simulation, the time is $T = 15$, $\Delta t = 0.0001$, $\mu = 0.1$, and $k = 100$.

The phase plane trajectory with force $h = 2\cos(10t)$ is depicted in Figure 6. Here, the force is sufficient to move the mass periodically, after a period of adjustment. In this simulation, $x_0 = l_2$, $v_0 = 1$, $T = 15$, and the time step is $\Delta t = 0.0001$, and $\mu = 0.1$, $k = 100$.

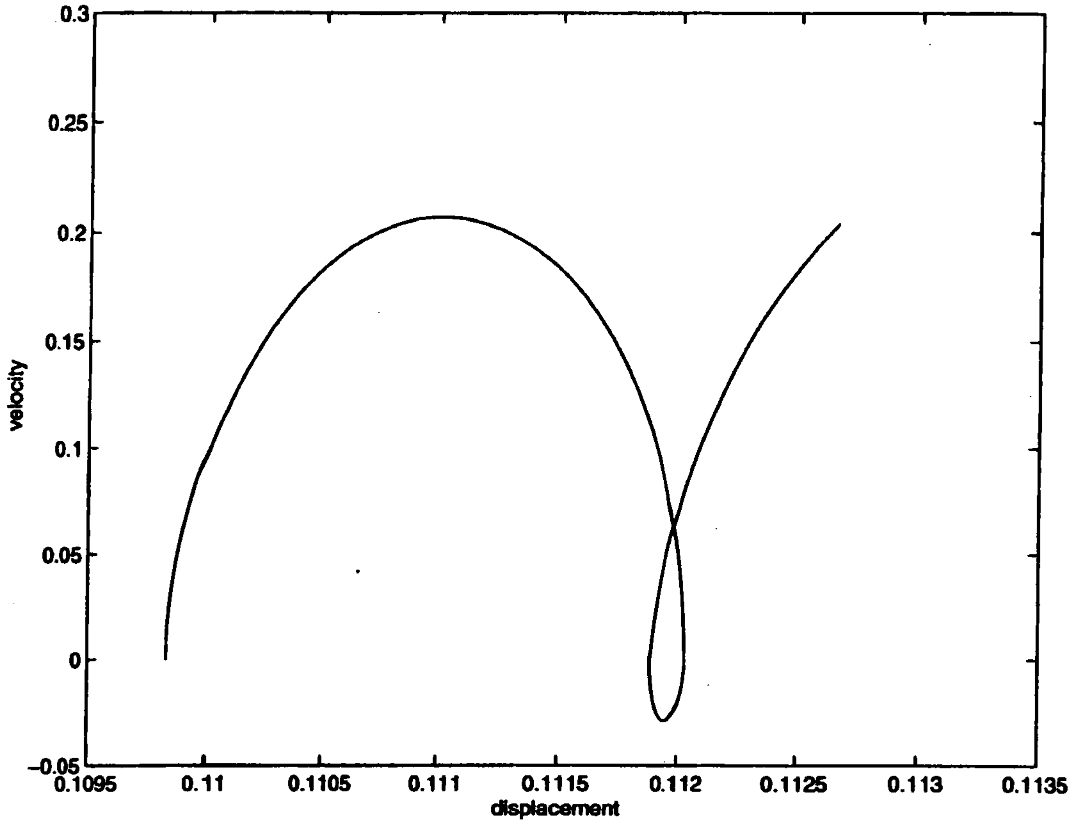

Figure 8. The initial oscillation.

Finally, in Figure 7, we show the trajectory with high frequency force $h = -40\cos(200t)$ in the case of different static and dynamic friction coefficients, here $\mu_s = 0.4$, $\mu_d = 0.2$. In Figure 8, we show the expanded view of the motion at the beginning to show the steep oscillations.

In this simulation, $x_0 = l_1$, $v_0 = 0$, $T = 10$, $\Delta t = 0.0001$, and $k = 100$.

6. CONCLUSIONS

We present a model for the motion accompanied by friction of a mass acted upon by a compressed spring. The problem can be formulated as a system of ordinary differential inclusions.

We derive some of the properties of the model and its solutions. We establish that there exist three separate stick regions in the model for certain values of the system parameters. These regions are centered on the three critical points of the frictionless problem. Otherwise, there exists one extended stick zone which includes all three points. Each of the stick zones is a set of steady solutions. We also present three related versions of the model.

The model is simulated numerically using the Euler and the Runge-Kutta methods. The simulations show the behavior of the solutions, which in the absence of applied forces approach the stick regions in finite time.

The question of existence and uniqueness of solutions is open. Moreover, there seems to be some interest in more detailed analysis and simulations of the main problem and of its variants.

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A Dynamic Model with Friction and Adhesion with Applications to Rocks

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A Dynamic Model with Friction and Adhesion with Applications to Rocks

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Descriptif

On présente un modèle général pour les problèmes dynamiques de contact avec frottement et adhérence entre un matériau viscoélastique et une fondation rigide. Ceci peut modéliser l'interaction entre des couches rocheuses. Vu le phénomène d'adhérence, la relation contraintres-déplacements normaux est non monotone et non convexe ; ce qui conduit à une formulation faible présentant une inéquation hémivariationnelle.

On considère un milieu continu viscoélastique occupant un domaine Ω de $\mathbb{R}^m$ ($m = 2, 3$), et dont la frontière Γ, supposée suffisamment régulière, est divisée en trois parties disjointes Γ_D, Γ_N et Γ_C. On suppose que, pendant l'intervalle de temps $[0, T]$, des forces volumiques $\boldsymbol{f}_A$ agissent dans Ω, que la partie Γ_D est encastrée dans une structure fixe, que des forces surfaciques $\boldsymbol{f}_N$ s'appliquent sur Γ_N. La partie Γ_C de la frontière représente la partie potentielle de contact du matériau avec une fondation indéformable se trouvant à une distance g. Le phénomène d'adhérence est supposé obéir à une relation générale qui lie les contraintes normales aux déplacements normaux sous la forme suivante :

$$-\sigma_n(u_n, \cdot) \in \mathcal{P}_n(u_n, \cdot) \quad \text{sur} \quad \Gamma_C.$$

Pour presque tout $x \in \Gamma_C$, $\mathcal{P}_n(\cdot\,, x)$ est un graphe tel que

$\mathcal{P}_n(\cdot, x) = 0$ sur $]-\infty, -g(x)]$,

$\mathcal{P}_n(-g(x), x) = [-p^*(x), 0]$,

$\mathcal{P}_n(\cdot\,, x)$ est une fonction de Lipschitz strictement monotone sur $]-g(x), 0]$,

$\mathcal{P}_n(0, x) = 0$,

$\mathcal{P}_n(\cdot\,, x)$ est une fonction de Lipschitz strictement monotone sur $[0, \infty[$.

La fonction p^* modélise le seuil maximal que la réaction normale σ_n peut atteindre avant la séparation complète des deux surfaces adhérentes. La portion $[0, \infty[$ du graphe représente la compliance normale des surfaces. Ce graphe n'est pas convexe ce qui conduit à une formulation du problème en une inéquation hémivariationnelle.

Le problème mécanique considéré ici se formule de la manière suivante :

Problème P : Trouver le champ des déplacements $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^m$, le champ des contraintes $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{R}_s^{m\times m}$ et la température $\theta : \Omega \times [0,T] \longrightarrow \mathbb{R}$ tels que

$$\sigma_{ij} = a_{ijkl}u_{k,l} + b_{ijkl}u'_{k,l} \quad \text{dans} \quad \Omega \times (0,T), \tag{1}$$

$$\boldsymbol{u}\text{"} - \operatorname{Div}\sigma = \boldsymbol{f}_B \quad \text{dans} \quad \Omega \times (0,T), \tag{2}$$

$$\boldsymbol{u} = 0 \quad \text{sur} \quad \Gamma_D \times (0,T), \tag{3}$$

$$\sigma \cdot \boldsymbol{n} = \boldsymbol{f}_N \quad \text{sur} \quad \Gamma_N \times (0,T), \tag{4}$$

$$-\sigma_n(u_n,\cdot) \in \mathcal{P}_n(u_n,\cdot) \quad \text{sur} \quad \Gamma_C \times (0,T), \tag{5}$$

$$|\sigma_\tau| \le \mu\,(-\sigma_n)_+ \quad \text{sur} \quad \Gamma_C \times (0,T), \tag{6}$$

$$\boldsymbol{u}'_\tau \neq 0 \quad \text{and} \quad \sigma_n < 0 \Longrightarrow \quad \frac{\boldsymbol{u}'_\tau}{|\boldsymbol{u}'_\tau|} = -\frac{\sigma_\tau}{\mu|\sigma_n|} \quad \text{sur} \quad \Gamma_C \times (0,T), \tag{7}$$

$$\boldsymbol{u}(\cdot\,,0) = \boldsymbol{u}_0, \qquad \boldsymbol{u}'(\cdot\,,0) = \boldsymbol{v}_0 \quad \text{dans} \quad \Omega. \tag{8}$$

On note par $\mathbb{R}_s^{m\times m}$ l'espace des tenseurs symétriques du second ordre sur $\mathbb{R}^m$. Le "prime" au dessus d'une quantité représente sa dérivée temporelle, $\boldsymbol{n}$ est la normale unitaire sortante à Ω et $\sigma \cdot \boldsymbol{n}$ est le vecteur des contraintes de Cauchy. u_n, u'_τ, σ_n et σ_τ représentent respectivement le déplacement normal, la vitesse tangentielle et les contraintes normales et tangentielles. Le réel μ représente le *coefficient de frottement.* Les relations (6)–(7) représentent la loi de frottement de Coulomb modifiée vu la prise en compte du phénomène d'adhérence.

On commence tout d'abord par donner une interprétation mécanique de chacune des équations et des termes cités dans le problème P. On insiste tout particulièrement sur les conditions aux limites considérées sur la partie de contact potentielle Γ_C pour chacunes des inconnues. On introduit ensuite les hypothèses utilisées suivies d'une formulation faible du problème P. Le système est formulé ensuite en terme d'opérateurs. Un résultat d'existence est alors établi en utilisant une méthode de régularisation suivie d'estimations *a priori* puis de passages à la limite.

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A Dynamic Model with Friction and Adhesion with Applications to Rocks

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Dynamic frictional contact with adhesion of a viscoelastic body and a foundation is formulated as a hemivariational inequality. This may model the dynamics of rock layers. The normal stress–displacement relation on the contact boundary is nonmonotone and nonconvex because of the adhesion process. A sequence of regularized problems is considered, the necessary a priori estimates are obtained, and the existence of a weak solution for the hemivariational inequality is established by passing to the limit as the regularization parameter vanishes.

Key Words: dynamic frictional contact; rocks; hemivariational inequality; adhesion; normal compliance; viscoelastic body.

1. INTRODUCTION

The aim of this paper is to study a dynamic contact problem involving the unilateral phenomena of coupled adhesion and friction. The setting we employ and the result we obtain are very general, but our particular interest lies in the frictional contact between rocks which involves adhe-

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sion or bonding. Adhesion and friction are highly nonlinear processes due to the nonmonotone stress–strain relationship which contains vertical jumps that correspond to abrupt stiffness changes. To accommodate such stress–strain laws, the theory of generalized gradients of Clarke [2] has been recently extended and applied in contact mechanics by Panagiotopoulos [15]. This approach allows for the rigorous formulation of mathematical models for these phenomena through variational and hemivariational inequalities, which we use in this work.

Contact problems involving both adhesion and friction effects have been studied mostly in special cases: in problems involving constitutive relations with uncoupled shear and normal stress, or in problems with given normal stress. However, interactions between normal and tangential contact forces are often present in problems arising in applications, such as in contact of rocks. A general static problem of frictional contact with adhesion of rocks has been recently studied in [5]. There, a model for the process has been developed and the existence of its weak solutions established by using the theory of hemivariational inequalities. Here we extend their results to the dynamic case.

In this paper we establish the existence of weak solutions for a specific problem. However, the constitutive relation which we employ is not convex, and this approach can be extended to other dynamic problems in mechanics with nonmonotone and nonconvex constitutive relations.

General problems of adhesion were considered by Frémond and coworkers in [3, 4, 16] where the model was derived from thermodynamical considerations. Friction, however, was not taken into account. There, a bonding field was introduced to describe the adhesion and an equation for its evolution was derived. A one-dimensional, quasi-static, and frictionless contact problem with adhesion, using the bonding field, has been investigated in [6]. The quasi-static problem with friction and adhesion, using the bonding field, has been modeled and investigated recently in [17, 18].

Recent results on dynamic frictional problems without adhesion can be found in [1, 9–11, 13] and in the references therein.

We use a graph to model the contact. It describes the adhesion and allows for interpenetration of surface asperities, as in the *normal compliance condition*, see, e.g., [1, 7, 8, 10, 13, 19]. The graph has a vertical segment related to the sudden debonding when all the bonds are severed. This leads to the use of the generalized subgradient theory, since the graph is not convex. The rest of the paper is structured as follows. The classical model, its weak formulation, and the statement of our results are given in Section 2. The material is assumed to be viscoelastic and linear, for the sake of simplicity. We employ the normal compliance condition for the compressive part of the contact, and model the adhesion with a graph which has a vertical segment at the yield point where debonding takes place. In Section 3, we consider a sequence of approximate problems in which the vertical segment in the adhesion condition is replaced with a tilted segment. This approximation may be useful in constructing numerical algorithms for the problem. We use the recent theory of [9] to obtain the existence of the unique solution for each approximate problem. A priori estimates on the approximate solutions are derived in Section 4. Using these estimates allows us to pass to the limit and obtain a solution of the original problem.

It may be of interest to investigate the dynamic problem when the adhesion is modeled by the bonding function, following Frémond, instead of having one graph for contact and adhesion.

2. CLASSICAL MODEL, WEAK FORMULATION AND RESULTS

In this section, we present the physical setting and formulate the model as a system of differential equations and initial and boundary conditions. Then we introduce a weak formulation, state the assumptions on the data and our main result. Because of adhesion, the contact condition is nonconvex and, therefore, the problem is formulated as a hemivariational inequality (see, e.g., [14] and references therein). For the sake of simplicity, the bulk material is assumed to be linear; the nonlinear effects arise from the contact with the foundation.

The physical setting is depicted in Fig. 1. A viscoelastic body, the rock, occupies (in its reference configuration) the region Ω in $\mathbb{R}^m$ $(m = 2, 3)$. Its boundary is divided into three disjoint parts. On Γ_D the body is clamped; known tractions act on Γ_N; and on Γ_C the body may contact a foundation. We assume that the foundation is soft, of the Winkler type, or is rigid but has a layer of deformable asperities. The reference configuration is assumed to be stress-free and the process isothermal.

Let $f_B = (f_{B1}(x,t), \dots, f_{Bm}(x,t))$ be the (dimensionless) density of applied body forces acting in Ω and let $f_N = (f_{N1}(x,t), \dots, f_{Nm}(x,t))$ be the

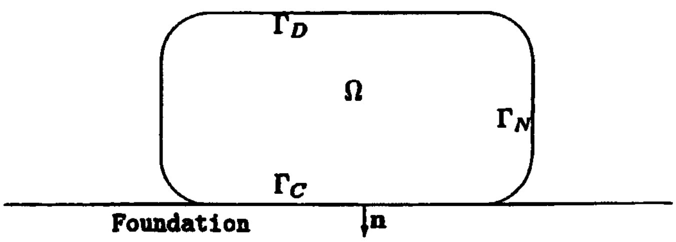

FIG. 1. The physical setting; Γ_C is the contact surface.

tractions applied on Γ_N. For the sake of simplicity, we assume that the density of the material is constant equal to 1. Let $u = (u_1(x,t), \dots, u_m(x,t))$ and $\sigma = (\sigma_{ij}(x,t))$ for $i, j = 1, \dots, m$, represent the dimensionless displacement vector and stress tensor, at location x and time t, respectively. The equations of motion take the (dimensionless) form

$$u'' - \operatorname{Div} \sigma = f_B \qquad \text{in } \Omega_T. \tag{2.1}$$

Here and below, $i, j = 1, \dots, m$; the repeated index convention is employed; the prime represents the time derivative; the portion of a subscript prior to a comma indicates a component and the portion after the comma refers to a partial derivative. We use the Kelvin–Voight stress–strain relation

$$\sigma_{ij} = a_{ijkl} u_{k,l} + b_{ijkl} u'_{k,l} \qquad \text{in } \Omega_T. \tag{2.2}$$

Here, $a = (a_{ijkl})$ and $b = (b_{ijkl})$ are the tensors of elastic and of viscosity coefficients, respectively. This relation holds within linearized elasticity, and we assume small displacements and strains.

The initial conditions are

$$u(\cdot, 0) = u_0, \qquad u'(\cdot, 0) = v_0 \qquad \text{in } \Omega. \tag{2.3}$$

To describe the boundary conditions, we introduce the unit outward normal $n = (n_1, \dots, n_m)$ on Γ. We assume that Γ is Lipschitz, hence n exists at almost every point. We then let $\sigma_n = \sigma_{ij} n_i n_j$ and $u_n = u \cdot n$ be the normal components of σ and u on Γ, and let $\sigma_\tau = \sigma \cdot n - \sigma_n n$, $u_\tau = u - u_n n$ be the tangential vectors. We use the following boundary conditions:

$$u = 0 \qquad \text{on } \Gamma_D, \tag{2.4}$$

$$\sigma \cdot n = f_N \qquad \text{on } \Gamma_N. \tag{2.5}$$

We turn to consider the conditions on the potential contact surface Γ_C, which is where our main interest lies. Physically, the contact surface is assumed to be covered with adhesive material, such as liquid glue, or there is a weak chemical bonding between the materials. This implies that for small tensile contact force there is resistance to separation. Let $g > 0$ be the *bond length*, and then $u_n = -g$ denotes the maximal distance for which bonding still holds, and let $p^* > 0$ denote the *tensile yield limit*, i.e., the maximal tensile force that the bonds can support. For $-g < u_n \leq 0$, there is tensile traction $0 \leq \sigma_n \leq p^*$ on Γ_C. However, when the pulling force at a point exceeds the threshold $\sigma_n = p^*$, the surfaces debond, the connections snap, and the contact at the point is lost. When the normal traction is negative, i.e., compressive, the penetration of the body's surface asperities into the outer surface of the foundation takes place. This represents a foundation with soft surface or the deformation of surface asperities. We assume a general relationship between the normal stress and normal displacement

$$-\sigma_n(u_n,\cdot) \in \mathscr{P}_n(u_n,\cdot) \qquad \text{on } \Gamma_C. \tag{2.6}$$

Here, for almost every $x \in \Gamma_C$, the graph $\mathscr{P}_n(\cdot,x)$ is such that

$$\begin{aligned}
&\mathscr{P}_n(\cdot,x) = 0 \text{ on } (-\infty, -g(x)],\\
&\mathscr{P}_n(-g(x),x) = [-p^*(x),0],\\
&\mathscr{P}_n(\cdot,x) \text{ is an increasing Lipschitz function on } (-g(x),0],\\
&\mathscr{P}_n(0,x) = 0,\\
&\mathscr{P}_n(\cdot,x) \text{ is an increasing Lipschitz function on } [0,\infty).
\end{aligned} \tag{2.7}$$

The portion of the graph on $[0,\infty)$ represents the normal compliance of the surfaces (see, e.g., [1, 7, 8, 10, 13, 19] and references therein). The graph is nonconvex, which leads to a hemivariational inequality formulation of the problem.

We note that the dependence of $\mathscr{P}_n$ on x is via g, and below we denote $\mathscr{P}_n(\cdot,x)$ by $\mathscr{P}_n$. A possible choice of the graph, depicted in Fig. 2, is

$$\mathscr{P}_n(\xi,x) = \begin{cases} \frac{1}{\varepsilon}\xi & \text{if } \xi \geq 0,\\ \alpha\xi & \text{if } -g(x) < \xi \leq 0,\\ [-\alpha g(x),0] & \text{if } \xi = -g(x),\\ 0 & \text{if } \xi < -g(x). \end{cases} \tag{2.8}$$

Here, $\alpha > 0$ is the slope for $-g(x) < \xi \leq 0$, and in the normal compliance portion of the graph the penetration of the foundation is penalized with the coefficient $1/\varepsilon$, for ε positive and small. Then, $p^*(x) = \alpha g(x)$ is the tensile yield limit.

The following graph has been used in [5], where the contact was between two deformable bodies,

$$\mathscr{P}_n(\xi,x) = \begin{cases} 0 & \text{if } |\xi| > g(x),\\ \alpha\xi & \text{if } |\xi| \leq g(x),\\ [-\alpha g(x),0] & \text{if } \xi = -g(x),\\ [0,\alpha g(x)] & \text{if } \xi = g(x). \end{cases} \tag{2.9}$$

Similar graphs can be found in [14] (and the references therein).

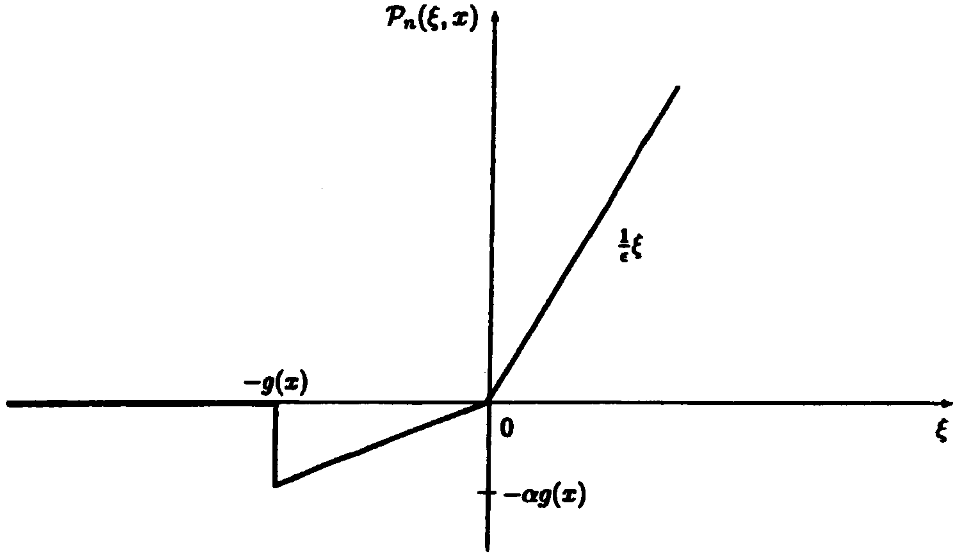

FIG. 2. Normal stress–displacement relationship (2.8).

We note that a different approach to modeling adhesion can be found in [3, 4] (see also the references therein) where a new dependent variable, the *bonding function*, which describes the ratio of active bonds at each point on the surface, was introduced. A differential equation for this variable was derived from a virtual power argument. The steady problem was analyzed in [16].

We turn to the tangential frictional contact condition. The usual Coulomb friction law is

$$|\sigma_\tau| \leq \mu |\sigma_n| \qquad \text{on } \Gamma_C,$$
$$u'_\tau \neq 0 \quad \Rightarrow \quad \frac{u'_\tau}{|u'_\tau|} = -\frac{\sigma_\tau}{\mu|\sigma_n|}.$$

Here μ is the friction coefficient. By convention, $\sigma_\tau = 0$ when there is no contact ($\sigma_n = 0$) and u'_τ remains undetermined. In the case of adhesion, this condition needs to be modified, since when σ_n is positive the body is pulled away from the foundation and we assume that there is no friction. Therefore, we use the following friction law:

$$|\sigma_\tau| \leq \mu(-\sigma_n)_+ \qquad \text{on } \Gamma_C,$$
$$u'_\tau \neq 0 \text{ and } \sigma_n < 0 \quad \Rightarrow \quad \frac{u'_\tau}{|u'_\tau|} = -\frac{\sigma_\tau}{\mu|\sigma_n|}. \tag{2.10}$$

When the tangential stress is less than the limiting value $\mu(-\sigma_n)_+$, the boundary sticks to the foundation: the part of the boundary where it takes place is called the *stick zone*; when the tangential stress reaches its limiting value, the boundary slips: this is the so-called *slip zone*. The slip is opposite to the shear stress σ_τ.

The classical formulation of the *dynamic viscoelastic frictional contact problem with adhesion* is to find a function u such that (2.1)–(2.6) and (2.10) hold.

It is well known that, generally, there are no classical solutions for the problem because of the regularity ceiling related to possible jumps in the velocity. Therefore, we turn to the weak or variational formulation of the problem. To this end we introduce the following Hilbert spaces:

$$E = \{w \in H^1(\Omega)^m: w = 0 \text{ on } \Gamma_D\}, \tag{2.11}$$

$$V = \{\eta \in H^1(\Omega): \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\}, \tag{2.12}$$

$$H = L^2(\Omega), \qquad H^m = (L^2(\Omega))^m, \qquad \mathbb{E} = L^2(0,T;E), \qquad \mathbb{V} = L^2(0,T;V). \tag{2.13}$$

Below, we use $\|\cdot\|_E$ and $\|\cdot\|_V$ to denote the norms of E and V, respectively, and $|\cdot|_H$ and $|\cdot|_{H^m}$ denote the norms of H and H^m. Also, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E' and E, or V' and V, where the meaning is evident from the context.

We now describe the assumptions on the data.

The coefficients of elasticity and viscosity satisfy

$$\begin{aligned}
&a_{ijkl} \in L^\infty(\Omega), \qquad b_{ijkl} \in L^\infty(\Omega),\\
&a_{ijkl} = a_{jikl}, \qquad a_{ijkl} = a_{klij}, \qquad a_{ijkl} = a_{ijlk},\\
&a_{ijkl}\chi_{ij}\chi_{kl} \ge \alpha_1|\chi_{ij}|^2 \quad \text{for all symmetric tensors } \chi = (\chi_{ij});\\
&b_{ijkl} = b_{jikl}, \qquad b_{ijkl} = b_{klij}, \qquad b_{ijkl} = b_{ijlk},\\
&b_{ijkl}\chi_{ij}\chi_{kl} \ge \alpha_2|\chi_{ij}|^2 \quad \text{for all symmetric tensors } \chi = (\chi_{ij}).
\end{aligned} \tag{2.14}$$

Here α_1 and α_2 are positive constants.

The body forces satisfy

$$f_B \in \mathbb{E}'. \tag{2.15}$$

The friction coefficient satisfies

$$\mu: \Gamma_C \to (0, +\infty) \qquad \text{and} \qquad 0 < \mu_* \le \mu \le \mu^* \qquad \text{a.e. on } \Gamma_C, \tag{2.16}$$

where μ_* and μ^* are constants.

The boundary and initial data satisfy

$$f_N \in L^2\big(0,T;\big(L^2(\Gamma_N)\big)^m\big); \tag{2.17}$$

$$u_0 \in E, \qquad v_0 \in H^m. \tag{2.18}$$

We conclude this section with a brief description of a hemivariational formulation of the problem, similar to the one in [5]. For almost every $x \in \Gamma_C$, let $\beta_n(\cdot, x)$ be the function given by

$$\beta_n(\xi, x) = \mathscr{P}_n(\xi, x) \qquad \text{if } \xi \neq -g,$$
$$\beta_n(-g(x), x) = 0.$$

The graph of β is the one depicted in Fig. 2, but without the vertical segment at $x = -g$. Let $\varphi_n(\cdot, x)$ be the function $\varphi_n(\xi, x) = \int_0^\xi \beta_n(s, x)\, ds$ for $\xi \in \mathbf{R}$. We define the functional Φ_n: $L^1(T_C) \to \mathbf{R}$ as

$$\Phi_n(z) = \int_{\Gamma_C} \varphi_n(z(x), x)\, d\Gamma,$$

where $d\Gamma$ denotes the surface measure on Γ_C. This definition makes sense only when $\varphi_n(z(\cdot), \cdot) \in L^1(T_C)$, and Φ_n is a Lipschitz continuous function, but is not necessarily convex.

Now, we may write the contact condition (2.6) as

$$-\sigma_n \in \partial\Phi_n(u_n) \qquad \text{on } \Gamma_C,$$

where $\partial\Phi_n$ represents the generalized subdifferential of Φ_n in the sense of Clarke (see, e.g., [14]). For this reason, the problem is formulated as a hemivariational inequality.

The generalized subdifferential in the sense of Clarke is a generalization of the usual subdifferential of a convex function. The latter is the set of all subgradients of the convex function at each point: when the function is differentiable at a point, its subdifferential contains only the gradient, and when it is not differentiable, the subdifferential contains the slopes of all the supporting lines (i.e., the lines which lie below the graph and touch it at the point only). In the case of a nonconvex function, the generalized subdifferential may contain vertical finite segments, too.

Let $\varphi_T(\eta; z) = \mu|\eta|\,|z|$ and define the functional Φ_T by

$$\Phi_T(\eta; z) = \int_{\Gamma_C} \varphi_T(\eta; z)\, d\Gamma,$$

provided the integral exists. Then, we may rewrite the friction condition (2.10) as

$$-\sigma_T \in \partial_z\Phi_T(\sigma_n; u_T) \qquad \text{on } \Gamma_C,$$

where $\partial_z \Phi_T(\eta; z)$ is the subdifferential of Φ_T with respect to its second variable. Since Φ_T is convex, this is the usual subdifferential.

3. WEAK FORM OF THE PROBLEM

In this section, we derive an abstract form of the problem. To that end let $p^+: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$p^+(\xi) = \begin{cases} \mathscr{P}_n(\xi) & \text{if } \xi > 0, \\ 0 & \text{if } \xi \leq 0. \end{cases}$$

It is Lipschitz continuous and monotone increasing, and there exists a constant $K > 0$ such that

$$|p^+(\xi_1) - p^+(\xi_2)| \leq K|\xi_1 - \xi_2| \qquad \text{for } \xi_1, \xi_2 \in \mathbb{R}. \tag{3.1}$$

Thus, (2.10) can be written as

$$|\sigma_\tau| \leq \mu p^+(u_n), \qquad v_\tau \neq 0 \quad \Rightarrow \quad \frac{v_\tau}{|v_\tau|} = \frac{-\sigma_\tau}{\mu p^+(u_n)}, \tag{3.2}$$

where $v_\tau = u'_\tau$. Similarly, for almost every $x \in \Gamma_C$, let $p^-(\cdot, x): \mathbb{R} \to \mathbb{R}$ be the graph

$$p^-(\xi, x) = \begin{cases} 0 & \text{if } \xi > 0, \\ \mathscr{P}_n(\xi, x) & \text{if } \xi \leq 0. \end{cases}$$

As usual, derivation of the abstract problem involves integration by parts. Let $w \in \mathbb{E}$ and $v = u'$, then we integrate by parts in the balance of momentum equation (2.1); taking into account (2.4)–(2.6), we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega v' \cdot w \, dx \, dt \\
&\quad = \int_0^T \int_{\Gamma_C \cup \Gamma_N} (\sigma_\tau + \sigma_n n) \cdot w \, d\Gamma \, dt - \int_0^T \int_\Omega \sigma : \nabla w \, dx \, dt \\
&\qquad + \int_0^T \int_\Omega f_B \cdot w \, dx \, dt + \int_0^T \int_{\Gamma_N} f_N \cdot w \, d\Gamma \, dt \\
&\quad \in \int_0^T \int_{\Gamma_C} -\mathscr{P}_n(u_n, x) n \cdot w_n \, d\Gamma \, dt \\
&\qquad + \int_0^T \int_{\Gamma_C} \sigma_\tau \cdot w_\tau \, d\Gamma \, dt + \int_0^T \int_{\Gamma_N} f_N \cdot w \, d\Gamma \, dt \\
&\qquad - \int_0^T \int_\Omega \sigma : \nabla w \, dx \, dt + \int_0^T \int_\Omega f_B \cdot w \, dx \, dt.
\end{aligned}$$

Now, it follows from (3.2) that regardless of whether $w_\tau \neq 0$ or not, there exists an element $z \in L^\infty(0,T; L^\infty(\Gamma_C)^m)$ such that

$$\int_0^T \int_{\Gamma_C} \sigma_\tau \cdot w_\tau \, d\Gamma \, dt = -\int_0^T \int_{\Gamma_C} \mu p^+(u_n) z \cdot w_\tau \, d\Gamma, \tag{3.3}$$

and

$$\int_0^T \int_{\Gamma_C} z \cdot w_\tau \, d\Gamma \leq \int_0^T \int_{\Gamma_C} (|v_\tau + w_\tau| - |v_\tau|) \, d\Gamma. \tag{3.4}$$

Thus, there exists $z \in L^\infty(0,T; L^\infty(\Gamma_C)^m)$ satisfying (3.4) such that

$$\int_0^T \int_\Omega v' \cdot w \, dx \, dt + \int_0^T \int_{\Gamma_C} \mathscr{P}_n(u_n, x) n \cdot w_n \, d\Gamma \, dt$$

$$+ \int_0^T \int_{\Gamma_C} \mu z p^+(\mu_n) \cdot w_\tau \, d\Gamma$$

$$+ \int_0^T \int_\Omega \sigma : \nabla w \, dx \, dt \ni \int_0^T \int_{\Gamma_N} f_N \cdot w \, d\Gamma \, dt + \int_0^T \int_\Omega f_B \cdot w \, dx \, dt. \tag{3.5}$$

Therefore, using (2.2), we define the viscosity, elasticity, and normal compliance operators A, B, P^+: $E \to E'$, respectively, by

$$\langle Au, w \rangle = \int_\Omega a_{ijkl} u_{k,l} w_{i,j} \, dx, \tag{3.6}$$

$$\langle Bu, w \rangle = \int_\Omega b_{ijkl} u_{k,l} w_{i,j} \, dx, \tag{3.7}$$

$$\langle P^+(u), w \rangle = \int_{\Gamma_C} p^+(u_n) w_n \, d\Gamma, \tag{3.8}$$

for all u, $w \in E$. It follows from (2.14) that there exists $\eta > 0$ such that, for all $u \in E$,

$$\langle Au, u \rangle \geq \eta \left(\|u\|_E^2 - |u|_{H^m}^2 \right), \tag{3.9}$$

$$\langle Bu, u \rangle \geq \eta \left(\|u\|_E^2 - |u|_{H^m}^2 \right). \tag{3.10}$$

We note that the operators A, B, and P^+ extend, in a natural way, to operators defined on $\mathbb{E}$ into $\mathbb{E}'$. With a slight abuse of notation, we use below the same symbol to denote both the original operators and their extensions.

Next, let $f \in \mathbb{E}'$ be given by

$$\langle f, w \rangle_{\mathbb{E}', \mathbb{E}} = \int_0^T \int_\Omega f_B \cdot w \, dx \, dt + \int_0^T \int_{\Gamma_N} f_N \cdot w \, d\Gamma \, dt. \tag{3.11}$$

Finally, let $\mathscr{P}(\mathbb{E}')$ be the set of all subsets of $\mathbb{E}'$. We consider the friction operator Q mapping $\mathbb{E}$ into $\mathscr{P}(\mathbb{E}')$, defined as follows: $v^* \in Q(v) \subseteq \mathbb{E}'$ means that there exists $z \in L^\infty(0,T; L^\infty(\Gamma_C)^m)$ satisfying

$$\int_0^T \int_{\Gamma_C} z \cdot w_T \, d\Gamma \, dt \leq \int_0^T \int_{\Gamma_C} (|v_T + w_T| - |v_T|) \, d\Gamma \, dt, \qquad (3.12)$$

such that

$$\langle v^*, w \rangle = \int_0^T \int_{\Gamma_C} \mu p^+(u_n) z \cdot w_T \, d\Gamma \, dt \qquad \forall w \in E. \qquad (3.13)$$

We have now all the ingredients needed to state the weak formulation of the problem and our main result in this work.

THEOREM 3.1. *Let* (2.7), (2.14)–(2.18) *hold. Then there exists a triplet* $\{\xi, u, v\}$ *such that*

$$\xi \in L^\infty(0,T; L^\infty(\Gamma_C)), \qquad v \in \mathbb{E}, \qquad v' \in \mathbb{E}', \qquad (3.14)$$

$$\xi(x,t) \in p^-(u_n(x,t), x) \qquad \text{a.e. on } \Gamma_C \times (0,T), \qquad (3.15)$$

$$v' + Bv + Au + P^+(u) + \Lambda_n^* \xi + Q(v) \ni f, \qquad (3.16)$$

$$u(t) = u_0 + \int_0^t v(s) \, ds \qquad \text{a.e. } t \in (0,T), \qquad (3.17)$$

$$v(0) = v_0. \qquad (3.18)$$

Here, γ_n is the map from E into $L^2(\Gamma_C)$ defined by $\gamma_n u = u_n$, γ_n^* is its adjoint map, and

$$\Lambda_n^* \xi = \int_0^T \int_{\Gamma_C} \gamma_n^* \xi \, d\Gamma \, dt.$$

We note that ξ represents the tension due to adhesion, P^+ represents the compressive part of the normal contact traction, and Q represents the friction.

4. APPROXIMATE PROBLEM

In this section, we consider a regularized version of the problem where the vertical segment of the adhesion part in the graph $\mathscr{P}_n$ is replaced by segments with decreasing slopes. We use the results of [9] to show that each one of the approximate problems has a unique solution.

Let $\delta > 0$ and let $p_\delta^-(\cdot, x)$: $\mathbb{R} \to \mathbb{R}$ be, for $p^*(x) = \alpha g(x)$ and almost every $x \in \Gamma_C$, the piecewise linear approximation of $p^-(\cdot, x)$ given by

$$p_\delta^-(\xi, x) = \begin{cases} -\frac{1}{\delta} p^*(x)(\xi + g(x) + \delta) & \text{if } -\delta - g(x) \le \xi \le -g(x), \\ p^-(\xi, x) & \text{otherwise.} \end{cases}$$

Thus $p_\delta^-(\cdot, x) = p^-(\cdot, x)$ except on the interval $[-\delta - g(x), -g(x)]$, where $p_\delta^-(\cdot, x)$ is a linear function. Clearly, p_δ^- is Lipschitz continuous and there exists $K_\delta > 0$ such that

$$|p_\delta^-(\xi_1, x) - p_\delta^-(\xi_2, x)| \le K_\delta |\xi_1 - \xi_2| \qquad \forall \xi_1, \xi_2 \in \mathbb{R}, \quad (4.1)$$

where $K_\delta \to \infty$ as $\delta \to 0^+$. The function $\mathscr{P}_n^\delta(\xi, x)$ and the modified part p_δ^- are depicted in Fig. 3.

We associate with the function p_δ^- the operator P_δ^-: $E \to E'$, given by

$$\langle P_\delta^-(u), w \rangle = \int_{\Gamma_C} p_\delta^-(u_n(x), x) w_n(x) \, d\Gamma, \quad (4.2)$$

for all $u, w \in E$. The operator P_δ^- extends naturally to an operator from $\mathbb{E}$ into $\mathbb{E}'$.

The following nonlinear evolution inclusion is the abstract form of the approximate problem.

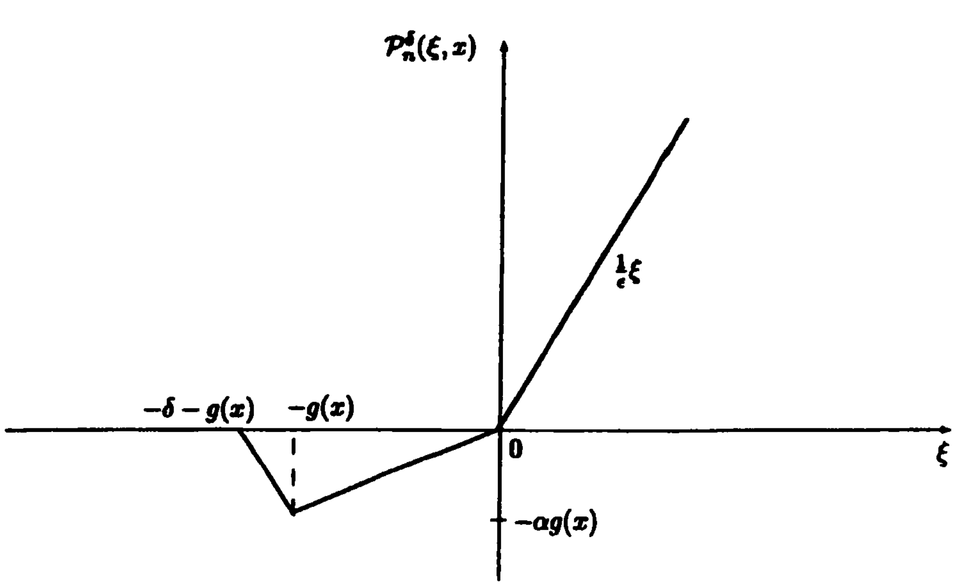

FIG. 3. The approximate function $\mathscr{P}_n^\delta(\xi, x)$.

PROBLEM $\mathscr{P}_\delta$. Find a pair $\{u_\delta, v_\delta\}$ such that

$$v_\delta \in \mathbb{E}, \qquad v_\delta(0) = v_0, \qquad v'_\delta \in \mathbb{E}', \tag{4.3}$$

$$v'_\delta + Bv_\delta + Au_\delta + P^+(u_\delta) + P_\delta^-(u_\delta) + Q(v_\delta) \ni f, \tag{4.4}$$

$$u_\delta(t) = u_0 + \int_0^t v_\delta(s)\, ds \qquad \text{a.e. } t \in (0,T). \tag{4.5}$$

We now establish the existence of the solution $\{u_\delta, v_\delta\}$ of the approximate problem $\mathscr{P}_\delta$, for each $\delta > 0$, and obtain a priori estimates independent of δ.

We remark that the approximate problem has some interest on its own. It has better mathematical properties than the idealized problem, and, indeed, the solution is more regular and is unique. For this reason it may be used as a basis for numerical approximations of the problem.

To prove the existence and uniqueness of the solution for Problem (4.3)–(4.5), we need the following two results due to Lions [12] and Simon [20], respectively.

THEOREM 4.1. *Let $p \geq 1$, $q > 1$, and let $W \subseteq U \subseteq Y$ be Banach spaces with compact inclusion map i: $W \to U$ and continuous inclusion map i: $U \to Y$. Then the set*

$$S_R = \{u \in L^p(0,T;W) \colon u' \in L^q(0,T;Y),$$
$$\|u\|_{L^p(0,T;W)} + \|u'\|_{L^q(0,T;Y)} < R\},$$

is precompact in $L^p(0,T;U)$.

THEOREM 4.2. *Let $q > 1$ and W, U, and Y be as in Theorem 4.1. Then the set*

$$S_{RT} = \{u \colon \|u(t)\|_W + \|u'\|_{L^q(0,T;Y)} \leq R, t \in [0,T]\},$$

is precompact in $C(0,T;U)$.

In order to use Theorems 4.1 and 4.2, we introduce a Banach space U such that $E \subseteq U$, the embedding $E \to U$ is compact, and the trace map $U \to L^2(\Gamma_C)^m$ is continuous. We denote by $\|\cdot\|_U$ the norm on U.

For technical reasons, we change the independent variable and use $y(t)e^{\lambda t} = v(t)$, for $\lambda \geq 0$. Then, Problem (4.3)–(4.5) written in terms of y is

$$y \in \mathbb{E}, \qquad y(0) = v_0, \qquad y' \in \mathbb{E}', \tag{4.6}$$

$$y' + \lambda y + By + e^{-\lambda(\cdot)}Au + e^{-\lambda(\cdot)}P^+(u) + e^{+\lambda(\cdot)}P_\delta^-(u)$$
$$+ e^{-\lambda(\cdot)}Q(e^{\lambda(\cdot)}y) \ni f, \tag{4.7}$$

$$u(t) = u_0 + \int_0^t v(s)\, ds \qquad \text{a.e. } t \in (0,T). \tag{4.8}$$

We define the Banach space $\mathbf{X}$, endowed with the norm $\|\cdot\|_{\mathbf{X}}$, as follows:

$$\mathbf{X} = \{y \in \mathbf{E}: y' \in \mathbf{E}'\}, \qquad \|y\|_{\mathbf{X}} = \|y\|_{\mathbf{E}} + \|y'\|_{\mathbf{E}'}. \tag{4.9}$$

Let also $\mathscr{P}(\mathbf{X}')$ be the set of all subsets of the dual space $\mathbf{X}'$.

PROPOSITION 4.3. *The operator* $Q_\lambda: \mathbf{X} \to \mathscr{P}(\mathbf{X}')$ *defined by* $Q_\lambda(y) = e^{-\lambda(\cdot)}Q(e^{\lambda(\cdot)}y)$ *is pseudomonotone and bounded.*

The proof of this proposition is accomplished through the following lemmas.

LEMMA 4.4. *If* $v^k \to v$ *weakly in* $\mathbf{X}$, *then* $v^k \to v$ *in* $L^2(0,T;(L^2(\Gamma_C))^m)$.

Proof. If v^k fails to converge to v in $L^2(0,T;L^2(\Gamma_C)^m)$, there exist an $\varepsilon > 0$ and a subsequence, still denoted by v_k, such that $\|v^k - v\|_{L^2(0,T;U)} \geq \varepsilon$. Then we can extract a further subsequence such that $v^k \to w$ strongly in $L^s(0,T;U)$, for some w. But the weak convergence of v^k to v in $\mathbf{X}$ implies the weak convergence of v^k to v in $L^2(0,T;U)$. Hence $w = v$, which contradicts the assumption that $\|v^k - v\|_{L^2(0,T;U)} \geq \varepsilon$.

LEMMA 4.5. *If* $y^k \to y$ *weakly in* $\mathbf{X}$, *then*

$$p^+(u_n^k) \to p^+(u_n) \qquad \text{in } L^2(0,T;L^2(\Gamma_C)). \tag{4.10}$$

Proof. It follows from (3.1) that

$$|p^+(u_n^k) - p^+(u_n)| \leq K|u_n^k - u_n|. \tag{4.11}$$

Now,

$$|u_n^k(t) - u_n(t)|_{L^2(\Gamma_C)} \leq \int_0^t |v_n^k(s) - v_n(s)|_{L^2(\Gamma_C)}\, ds$$

$$= \int_0^t e^{\lambda s}|y_n^k(s) - y_n(s)|_{L^2(\Gamma_C)}\, ds,$$

and using the Jensen inequality, we obtain

$$\|u_n^k - u_n\|^2_{L^2(0,T;L^2(\Gamma_C))} \leq C_{T\lambda}\int_0^T \int_0^T |y_n^k(s) - y_n(s)|^2_{L^2(\Gamma_C)}\, ds\, dt, \tag{4.12}$$

where $C_{T\lambda}$ is a positive constant which depends on T and λ. We deduce from Lemma 4.4 that $y_n^k \to y_n$ strongly in $L^2(0,T;L^2(\Gamma_C))$, and this together with (4.11) and (4.12) yield the result.

LEMMA 4.6. *Let* $y^k \to y$ *weakly in* $\mathbf{X}$ *and* $z^k \to z$ *weak* in* $L^\infty(0,T;L^\infty(\Gamma_C)^m)$. *Then*

$$\int_0^T \int_{\Gamma_C} \mu p^+(u_n^k) z^k \cdot \xi \, d\Gamma\, dt \to \int_0^T \int_{\Gamma_C} \mu p^+(u_n) z \cdot \xi \, d\Gamma\, dt, \tag{4.13}$$

for all $\xi \in L^2(0,T;L^2(\Gamma_C)^m)$.

Proof. We argue by contradiction. Let $\varepsilon > 0$. If (4.13) does not hold, then there exist $\xi \in L^2(0,T;L^2(\Gamma_C)^m)$ and two sequences $\{y^k\}$ and $\{z^k\}$ such that $y^k \to y$ weakly in $\mathbb{X}$, $z^k \to z$ weak* in $L^\infty(0,T;L^\infty(\Gamma_C)^m)$ and

$$\left| \int_0^T \int_{\Gamma_C} \mu p^+(u_n^k) z^k \cdot \xi \, d\Gamma \, dt - \int_0^T \int_{\Gamma_C} \mu p^+(u_m) z \cdot \xi \, d\Gamma \, dt \right| \geq 2\varepsilon. \quad (4.14)$$

Since $L^\infty(0,T;L^\infty(\Gamma_C)^m)$ is dense in $L^s(0,T;L^2(\Gamma_C)^m)$, we may assume that (4.14) holds for some $\xi \in L^\infty(0,T;L^\infty(\Gamma_C)^m)$ with ε in place of 2ε. However, it follows from Lemma 4.5 that

$$\int_0^T \int_{\Gamma_C} \mu p^+(u_n^k) z^k \cdot \xi \, d\Gamma \, dt \to \int_0^T \int_{\Gamma_C} \mu p^+(u_n) z \cdot \xi \, d\Gamma \, dt,$$

and so (4.14) cannot hold for all k. This contradiction proves the lemma.

Proof of Proposition 4.3. It is clear that Q_λ is bounded, and it is straightforward to show that $Q_\lambda(y)$ is convex. We now show that $Q_\lambda(y)$ is closed. Let W be a weakly open set in $\mathbb{X}'$ and let $W_\lambda = e^{\lambda(\cdot)}$. Assume that $y^k \to y$ weakly in $\mathbb{X}$, $Q_\lambda(y) \subseteq W$, and let $(y^k)^* \in Q_\lambda(y^k) \setminus W$ for all k. Then $v^k \to v$ weakly in $\mathbb{X}$, W_λ is a weakly open set in $\mathbb{X}'$ containing $Q(v)$, and $(v^k)^* = e^{\lambda(\cdot)}(y^k)^* \in Q(v^k) \setminus W_\lambda$ for all k. Next, let $\{z^k\}$ be a sequence in $L^\infty(0,T;L^\infty(\Gamma_C)^m)$, satisfying (3.12) and (3.13), such that, possibly for a subsequence, $z^k \to z$ weak* in $L^\infty(0,T;L^\infty(\Gamma_C)^m)$. It follows from Lemma 4.4 that z satisfies (3.12). Now, we obtain from Lemma 4.6 that $(v^k)^* \to v^*$ weakly in $\mathbb{E}'$, and thus

$$\langle v^*, w \rangle = \int_0^T \int_{\Gamma_C} \mu p^+(u_n) z \cdot w_T \, d\Gamma \, dt, \qquad w \in \mathbb{E}.$$

Then, by the definition of Q, we obtain that $v^* \in Q(v) \subseteq W_\lambda$. This is a contradiction to the assumption that $(v^k)^* \notin W_\lambda$, for all k. Hence $Q(v^k) \subseteq W_\lambda$ for all sufficiently large k. This argument also shows that $Q_\lambda(y)$ is closed.

It remains to verify the limit condition for pseudomonotone operators. To that end, assume that $y^k \to y$ weakly in $\mathbb{X}$ and let $(y^k)^* \in Q_\lambda(y^k)$, for all k. We show that if $w \in \mathbb{X}$, then

$$\liminf_{k\to\infty} \langle (y^k)^*, y^k - w \rangle \geq \langle y^*(w), y - w \rangle, \qquad y^*(w) \in Q_\lambda(y).$$

We choose a subsequence y^k (which depends on w) such that

$$\lim_{k\to\infty} \langle (y^k)^*, y^k - w \rangle = \liminf_{k\to\infty} \langle (y^k)^*, y^k - w \rangle.$$

Let $(v^k)^* = e^{\lambda(\cdot)}(y^k)^* \in Q(v^k)$ and let $z^k \in L^\infty(0,T;L^\infty(\Gamma_C)^m)$ be a related function satisfying (3.12) and (3.13), for all k. We extract a further subsequence, if necessary, such that

$$z^k \to z \qquad \text{weak* in } L^\infty\big(0,T;L^\infty(\Gamma_C)^m\big).$$

Then z satisfies (3.12) by Lemma 4.4. It follows from Lemma 4.6 that if we define $y^*(w)$ by

$$\langle y^*(w), b\rangle = \int_0^T \int_{\Gamma_C} e^{-\lambda t}\mu p^+(u_n) z \cdot b_T \, d\Gamma \, dt,$$

for $b \in \mathbb{E}$, we obtain

$$\begin{aligned}
&\liminf_{k\to\infty} \langle (y^k)^*, y^k - w\rangle \\
&\quad = \lim_{k\to\infty} \langle (y^k)^*, y^k - w\rangle \\
&\quad = \lim_{k\to\infty} \int_0^T \int_{\Gamma_C} e^{-\lambda t}\mu p^+(u_n^k) z^k \cdot \big(y_T^k - w_T\big)\, d\Gamma\, dt, \\
&\quad = \int_0^T \int_{\Gamma_C} e^{-\lambda t}\mu p^+(u_n) z \cdot \big(y_T - w_T\big)\, d\Gamma\, dt, \\
&\quad = \langle y^*(w), y - w\rangle.
\end{aligned}$$

This completes the proof of Proposition 4.3

LEMMA 4.7. *If $v^k \to v$ weakly in $\mathbf{X}$, then $P^+(u^k) \to P^+(u)$ in $\mathbb{E}'$.*

Proof. Let $w \in \mathbb{E}$. Using (3.1), we obtain

$$\begin{aligned}
&\big|\langle P^+(u^k) - P^+(u), w\rangle\big| \\
&\quad \le K \int_0^T \int_{\Gamma_C} |u_n^k - u_n|\,|w_n| \, d\Gamma\, dt, \\
&\quad \le K \int_0^T \left(\int_{\Gamma_C} |u_n^k - u_n|^2 \, d\Gamma\right)^{1/2} \left(\int_{\Gamma_C} |w_n|^2 \, d\Gamma\right)^{1/2} dt, \\
&\quad \le K \|u_n^k - u_n\|_{L^2(0,T;L^2(\Gamma_C))} \|w\|_{\mathbb{E}}.
\end{aligned}$$

Thus,

$$\|P^+(u^k) - P^+(u)\|_{\mathbb{E}'} \le K \|\gamma u^k - \gamma u\|_{L^2(0,T;L^2(\Gamma_C)^m)},$$

and the desired result follows from Lemma 4.4.

It is easy to check that for each $\lambda \geq 0$, the operator $y \mapsto e^{-\lambda(\cdot)}Au$ is monotone. In fact,

$$\begin{aligned}
&\langle e^{-\lambda(\cdot)}A(u^1 - u^2), y^1 - y^2 \rangle \\
&= \frac{1}{2}\int_0^T e^{-2\lambda}\frac{d}{dt}\langle A(u^1 - u^2), u^1 - u^2 \rangle \, dt \\
&= \frac{1}{2}e^{-2\lambda T}\langle A(u^1(T) - u^2(T)), u^1(T) - u^2(T) \rangle \\
&\quad + \lambda \int_0^T \langle A(u^1 - u^2), u^1 - u^2 \rangle e^{-2\lambda t} \, dt. \qquad (4.15)
\end{aligned}$$

Next, $y^k \to y$ weakly in $\mathbb{X}$ if and only if $v^k \to v$ weakly in $\mathbb{X}$ and Lemma 4.7 implies that the operator $y \mapsto e^{-\lambda(\cdot)}P^+(u)$ is completely continuous. Similar considerations show that the operator $y \mapsto e^{-\lambda(\cdot)}P_\delta^-(u)$ is completely continuous. Thus, if we let

$$\begin{aligned}
\mathscr{A}_\lambda y = \lambda y + By + e^{-\lambda(\cdot)}Au + e^{-\lambda(\cdot)}P^+(u) + e^{-\lambda(\cdot)}P_\delta^-(u) \\
+ e^{-\lambda(\cdot)}Q(e^{\lambda(\cdot)}y), \qquad (4.16)
\end{aligned}$$

then $\mathscr{A}_\lambda$ is a sum of pseudomonotone bounded operators. Consequently, $\mathscr{A}_\lambda : \mathbb{X} \to \mathscr{P}(\mathbb{X}')$ is pseudomonotone and bounded. The last three terms of (4.16) have the property that if v^* is either equal to or an element of any one of these terms, then

$$|\langle v^*, y \rangle| \leq C\|y\|_{\mathbb{U}}^2 + C,$$

where C is a constant independent of y and λ. Therefore, using the inequality, $\|y\|_{\mathbb{U}}^2 \leq \varepsilon\|y\|_{\mathbb{E}}^2 + C_\varepsilon|y\|_{H^m}^2$, which results from the compactness of the embedding of E into U, choosing ε small enough, and then choosing λ large enough, we find in addition that $\mathscr{A}_\lambda$ is coercive. Thus, by the existence theorem of [9], we conclude that the system (4.6)–(4.8) has a solution, and consequently, there exists a solution of Problem (4.3)–(4.5).

We have the following theorem.

THEOREM 4.8. *For each $\delta > 0$, there exists a unique solution of Problem $\mathscr{P}_\delta$.*

Proof. It remains to verify the uniqueness. Assume that v^1 and v^2 solve Problem $\mathscr{P}_\delta$. Let $u^i(t) = u_0 + \int_0^t v^i(s)\, ds$, let $(v^i)^* \in Q(v^i)$, and denote by z^i the element of $L^\infty(0,T; L^\infty(\Gamma_C)^m)$ that satisfies (3.12) and

(3.13), for $i = 1,2$. From (3.13) we obtain

$$\int_0^t \langle (v^1)^* - (v^2)^*, v^1 - v^2 \, ds$$

$$\geq \int_0^t \int_{\Gamma_C} \left(\mu p^+(u_n^1) z^1 - \mu p^+(u_n^2) z^2 \right) (v_T^1 - v_T^2) \, d\Gamma \, ds$$

$$\geq \int_0^t \int_{\Gamma_C} \mu z^1 (p^+(u_n^1) - p^+(u_n^2)) \cdot (v_T^1 - v_T^2) \, d\Gamma \, ds$$

$$+ \int_0^t \int_{\Gamma_C} \mu p^+(u_n^1)(z^1 - z^2) \cdot (v_T^1 - v_T^2) \, d\Gamma \, ds. \qquad (4.17)$$

Using (3.12) for z^1 and z^2, we find that the last term on the right-hand side is nonnegative. Therefore,

$$\int_0^t \langle (v^1)^* - (v^2)^*, v^1 - v^2 \rangle \, ds$$

$$\geq -C \int_0^t \| u^1(s) - u^2(s) \|_U \| v^1(s) - v^2(s) \|_U \, ds,$$

where C is a constant which may depend on z^1, μ, T, and K. Using the definitions of u^1 and u^2, in terms of v^1 and v^2, we may write

$$\int_0^t \langle (v^1)^* - (v^2)^*, v^1 - v^2 \, ds \geq -C \int_0^t \| v^1(s) - v^2(s) \|_U^2 \, ds. \quad (4.18)$$

From (4.4), (4.15), (4.18), (3.9), (3.10), and after adjusting the constant C to depend on δ, we obtain that

$$\frac{1}{2} | v^1(t) - v^2(t) |_{H^m}^2 + \eta \int_0^t \left(\| v^1(s) - v^2(s) \|_E^2 - | v^1(s) - v^2(s) |_{H^m}^2 \right) ds$$

$$- C \int_0^t \| v^1(s) - v^2(s) \|_U^2 \, ds \leq 0. \qquad (4.19)$$

Using the inequality $\| u \|_U \leq \varepsilon \| u \|_E + C_\varepsilon | u |_{H^m}$ for ε such that $0 < \varepsilon < \eta$, we find

$$| v^1(t) - v^2(t) |_{H^m}^2 + \int_0^t \| v^1(s) - v^2(s) \|_E^2 \, ds$$

$$\leq C \int_0^t | v^1(s) - v^2(s) |_{H^m}^2 \, ds,$$

where C depends on η, δ, z^1, μ, T, and K. Then, by Gronwall's inequality we find that $v^1 = v^2$, which proves the uniqueness of the solution, and therefore, the theorem.

5. ESTIMATES AND THE LIMIT

In this section, we prove Theorem 3.1. To that end, we establish estimates on the solutions of the approximate problems $\mathscr{P}_\delta$ leading to the following theorem.

THEOREM 5.1. *There exists a constant, C, independent of δ, such that*

$$|v_\delta(t)|_{H^m}^2 + \int_0^t \|v_\delta(s)\|_E^2\, ds + \|u_\delta(t)\|_E^2 + \int_{\Gamma_C} \Phi(u_{\delta n}(\cdot, t))\, d\Gamma \le C. \quad (5.1)$$

Proof. To simplify the notation, we omit the subscript δ in this proof. We apply (4.4) to v and integrate from 0 to t. We consider the resulting nonlinear terms first,

$$\begin{aligned}\int_0^t \langle P^+(u), v\rangle\, ds &= \int_0^t \int_{\Gamma_C} p^+(u_n(x,s))v_n(x,s)\, d\Gamma\, ds\\ &= \int_{\Gamma_C} \int_0^t p^+(u_n(x,s))v_n(x,s)\, ds\, d\Gamma\\ &= \int_{\Gamma_C} \Phi(u_n(x,t)) - \Phi(u_{0n}(x))\, d\Gamma,\end{aligned}$$

where Φ is the indefinite integral of p^+, i.e., $d\Phi/dt = p^+$. Therefore,

$$\int_0^t \langle P^+(u), v\rangle\, ds \ge \int_{\Gamma_C} \Phi(u_n(t,x))\, d\Gamma - C. \quad (5.2)$$

Here and below, C denotes a generic constant which is independent of δ. Then,

$$\begin{aligned}\int_0^t \langle P_\delta^-(u), v\rangle\, ds &= \int_0^t \int_{\Gamma_C} p_\delta^-(u_n(x), x)v_n(x)\, d\Gamma\, ds,\\ &\ge -C - \int_0^t \|v\|_U^2\, ds. \quad (5.3)\end{aligned}$$

Next, we consider the term involving $Q(v)$. Let v^* be the element of $Q(v)$ for which equality occurs in (4.4), and let $z \in L^\infty(0, T; L^\infty(\Gamma_C)^m)$ be the function satisfying (3.12) and (3.13); then

$$\int_0^t \langle v^*, v \rangle \, ds = \int_0^t \int_{\Gamma_C} \mu p^+(u_n) z \cdot v_T \, d\Gamma \, ds.$$

It follows from (3.12) that $\|z\|_{L^\infty(0,T;\, L^\infty(\Gamma_C)^m)} \leq 1$. Since p^+ is Lipschitz and equals zero at $\xi = 0$, we obtain that

$$\int_0^t \langle v^*, v \rangle \, ds \geq -C \int_0^t \|u\|_U \|v\|_U \, ds. \tag{5.4}$$

From estimates (5.2)–(5.4), (4.4), and (2.14), it results that

$$\begin{aligned} &\frac{1}{2}|v(t)|_{H^m}^2 + \alpha_2 \int_0^t \|v(s)\|_E^2 \, ds + \frac{\alpha_1}{2}\|u(t)\|_E^2 - C\int_0^t \|v\|_U^2 \, ds \\ &\quad + \int_{\Gamma_C} \Phi(u_n(t,x)) - \Phi(u_{0n}(x)) \, d\Gamma - C\int_0^t \|u(s)\|_U^2 \, ds \\ &\quad \leq C + C\int_0^t |v(s)|_{H^m}^2 \, ds + \frac{\alpha_1}{2}|u(t)|_{H^m}^2, \end{aligned} \tag{5.5}$$

where α_1 and α_2 are the positive constants appearing in (2.14). On the other hand, we have

$$|u(t)|_{H^m}^2 \leq C + \int_0^t |v(s)|_{H^m}^2 \, ds, \tag{5.6}$$

and

$$\int_0^t \|u(s)\|_U^2 \, ds \leq C + \int_0^t \int_0^s \|v(r)\|_U^2 \, dr \, ds. \tag{5.7}$$

Using now Gronwall's inequality, it follows from (5.5)–(5.7) that

$$\begin{aligned} &|v(t)|_{H^m}^2 + \int_0^t \|v(s)\|_E^2 \, ds + \|u(t)\|_E^2 + \int_{\Gamma_C} \Phi(u_n(t,x)) \, d\Gamma \\ &\quad - C\int_0^t \|v\|_U^2 \, ds \leq C. \end{aligned} \tag{5.8}$$

Finally, we use the compactness of the embedding of E into U and apply Gronwall's inequality again and obtain (5.1) from (5.8).

We use estimate (5.1) to pass to the limit when $\delta \to 0$ and thus obtain the existence of a solution for problem (3.15)–(3.18).

For each $\delta > 0$, let $\{u_\delta, v_\delta\}$ denote the unique solution of Problem (4.4)–(4.5). Using the estimate (5.1), (4.4), and the boundedness of the operators, Theorems 4.1 and 4.2 imply that there exists a subsequence of $\{u_\delta, v_\delta\}$ such that

$$v'_\delta \to v' \qquad \text{weakly in } \mathbb{E}', \tag{5.9}$$

$$v_\delta \to v \qquad \text{weakly in } \mathbb{E}, \tag{5.10}$$

$$u_\alpha \to u \qquad \text{strongly in } C(0,T;U), \tag{5.11}$$

$$u_{n\delta} \to u_n \qquad \text{strongly in } L^2(\Gamma_C \times (0,T)), \tag{5.12}$$

$$u_{n\delta}(x,t) \to u_n(x,t) \qquad \text{a.e. in } \Gamma_C \times (0,T), \tag{5.13}$$

$$v_\delta \to v \qquad \text{strongly in } L^2(0,T;U), \tag{5.14}$$

$$p_\delta^-(u_{\delta n}, \cdot) \to \xi \qquad \text{weak* in } L^\infty(\Gamma_C \times (0,T)). \tag{5.15}$$

Let v_δ^* denote the element of $Q(v_\delta)$ which yields equality in (4.4); thus,

$$v'_\delta + Bv_\delta + Au_\delta + P^+(u_\delta) + P_\delta^-(u_\delta) + v_\delta^* = f,$$

and let $z_\delta \in L^\infty(0,T; L^\infty(\Gamma_C)^m)$ be the function in the definition of $Q(v_\delta)$, (3.12), and (3.13). Furthermore, we may also assume that

$$z_\delta \to z \qquad \text{weak* in } L^\infty\big(0,T; L^\infty(\Gamma_C)^m\big). \tag{5.16}$$

Using (5.11), (5.16), and the definition of $Q(v_\delta)$ in (3.13), we conclude that

$$v_\delta^* \to v^* \qquad \text{weakly in } \mathbb{E}',$$

where

$$\langle v^*, w\rangle = \int_0^T \int_{\Gamma_C} \mu p^+(u_n) z \cdot w_T \, d\Gamma \, dt, \qquad w \in \mathbb{E},$$

and thus, $v^* \in Q(v)$. On the other hand, we obtain from (5.11) and (3.1) that

$$P^+(u_\delta) \to P^+(u) \qquad \text{strongly in } \mathbb{E}'.$$

Let now K be the set

$$K = \big\{\psi \in L^\infty(\Gamma_C \times (0,T)) : 0 \geq \psi(x,t) \geq -p^*(x) \text{ a.e. on } \Gamma_C \times (0,T)\big\}.$$

K is a closed and convex subset of $L^\infty(\Gamma_C \times (0,T))$, and from the definition of the function p_δ^-, it follows that $p_\delta^-(u_{\delta n}, \cdot) \in K$, for each δ. Therefore, we obtain from (5.15) that $\xi \in K$. Using (4.2), we have for $w \in \mathbb{E}$,

$$\langle P_\delta^-(u_\delta), w\rangle = \int_0^T \int_{\Gamma_C} p_\delta^-(u_{\delta n}(x,t), x) w_n(x,t) \, d\Gamma \, dt,$$

and using (5.15), we obtain that

$$\langle P_\delta^-(u_\delta), w\rangle \to \int_0^T \int_{\Gamma_C} \xi(x,t) w_n(x,t)\, d\Gamma\, dt.$$

Let us consider now $\xi(x,t)$. Suppose first that (x,t) is a point at which $u_n(x,t) \neq -g(x)$ and is also a point where $u_{\delta n}(x,t) \to u_n(x,t)$. Then $p_\delta^-(u_n(x,t),x) = p^-(u_n(x,t),x)$ for all δ sufficiently small. By the continuity of $p^-(x,\cdot)$ at such points, $p_\delta^-(u_{\delta n}(x,t),x) \to p^-(u_n(x,t),x)$. Consequently, if such points comprise a set S of positive measure, then for almost every point in S, $p^-(u_n(x,t),x) = \xi(x,t)$. On the other hand, the observation that ξ lies in K implies that even if $u_n(x,t) = -g(x)$, $\xi(x,t) \in p^-(u_n(x,t),x)$ almost everywhere. This completes the proof of Theorem 3.1.

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Analysis of a Quasistatic Viscoelastic Problem with Friction and Damage

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Descriptif

On considère dans cet article deux problèmes quasistatiques de contact d'un matériau ayant une loi de comportement viscoélastique avec une fondation indéformable. Les conditions aux limites de contact sont décrites par un modèle général de réaction amortie couplé avec la loi de frottement de Coulomb. Dans le second problème, on tient en plus compte de l'endommagement du matériau dû aux extensions et aux compressions. Il est formulé sous forme d'une inclusion différentielle faisant intervenir les déformations.

On considère un milieu continu viscoélastique occupant un domaine Ω de $\mathbb{R}^d$ ($d = 2, 3$) et dont la frontière Γ, supposée suffisamment régulière, est divisée en trois parties disjointes Γ_1, Γ_2 et Γ_3. On suppose que, pendant l'intervalle de temps $[0, T]$, la partie Γ_1 est encastrée dans une structure fixe, que des forces surfaciques $\mathbf{f}_2$ s'appliquent sur Γ_2 et que des forces volumiques $\mathbf{f}_0$ agissent dans Ω. Le premier problème quasistatique de contact qu'on se propose d'étudier se formule de la façon suivante :

Problème P : Trouver le champ des déplacements $u : \Omega \times [0, T] \longrightarrow \mathbb{R}^d$ et le champ des contraintes $\sigma : \Omega \times [0, T] \longrightarrow \mathbb{R}_s^{d\times d}$ tels que

$$\begin{aligned}
\sigma = \mathcal{A}(\varepsilon(\dot{u})) + G(\varepsilon(u)) \quad & \text{dans} \quad \Omega \times (0, T), \\
\operatorname{Div} \sigma + \mathbf{f}_0 = 0 \quad & \text{dans} \quad \Omega \times (0, T), \\
u = 0 \quad & \text{sur} \quad \Gamma_1 \times (0, T), \\
\sigma\nu = \mathbf{f}_2 \quad & \text{sur} \quad \Gamma_2 \times (0, T), \\
-\sigma_\nu = p_\nu(\dot{u}_\nu) \quad & \text{sur} \quad \Gamma_3 \times (0, T), \\
|\sigma_\tau| \leq p_\tau(\dot{u}_\nu) \quad & \text{sur} \quad \Gamma_3 \times (0, T), \\
|\sigma_\tau| < p_\tau(\dot{u}_\nu) \quad & \Longrightarrow \quad \dot{u}_\tau = 0, \\
|\sigma_\tau| = p_\tau(\dot{u}_\nu) \quad & \Longrightarrow \quad \sigma_\tau = -\lambda \dot{u}_\tau, \ \lambda \geq 0, \\
u(0) = u_0 \quad & \text{dans} \quad \Omega.
\end{aligned}$$

On note par $\mathbb{R}_s^{d\times d}$ l'espace des tenseurs symétriques du second ordre sur $\mathbb{R}^d$ et par $\varepsilon(u)$ le tenseur des petites déformations linéarisé. Le point au dessus d'une quantité représente sa dérivée temporelle, $\operatorname{Div} \sigma$ désigne la divergence de la fonction tensorielle

σ et ν la normale unitaire sortante à Ω. Le terme $\sigma\nu$ est le vecteur des contraintes de Cauchy, et u_ν, $\dot{u}_\tau$, σ_ν et σ_τ représentent respectivement le déplacement normal, la vitesse tangentielle, les contraintes normales et tangentielles. Les fonctionnelles p_ν et p_τ sont des positives données. La fonctionnelle p_ν représente la pénétration du corps dans la fondation, s'il y a contact, alors que la fonctionnelle p_τ désigne le seuil de frottement.

La première partie de ce travail porte sur l'analyse variationnelle du problème P. On commence par donner une interprétation mécanique des conditions aux limites considérées. On poursuit avec deux formulations variationnelles du problème P. La première est exprimée en terme de déplacements alors que la seconde est exprimée en terme de contraintes. On établit ensuite l'existence et l'unicité de la solution pour la première formulation faible puis un résultat d'équivalence entre les deux formulations via lequel on obtient donc l'existence et l'unicité de la solution pour la seconde formulation.

Le même problème tenant compte de l'endommagement du matériau se formule de la manière suivante :

Problème PE : Trouver le champ des déplacements $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$, le champ des contraintes $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{R}_s^{d\times d}$ et la fonction endommagement $\beta : \Omega \times [0,T] \longrightarrow \mathbb{R}$ tels que

$$\begin{aligned}
\sigma = \mathcal{A}(\varepsilon(\dot{u})) + G(\varepsilon(u),\beta) &\quad \text{dans} \quad \Omega \times (0,T),\\
\dot{\beta} - k\triangle\beta + \partial\psi_K(\beta) \ni \phi(\varepsilon(u),\beta) &\quad \text{dans} \quad \Omega \times (0,T),\\
\operatorname{Div}\sigma + \mathbf{f}_0 = 0 &\quad \text{dans} \quad \Omega \times (0,T),\\
\frac{\partial\beta}{\partial\nu} = 0 &\quad \text{sur} \quad \Gamma \times (0,T),\\
u = 0 &\quad \text{sur} \quad \Gamma_1 \times (0,T),\\
\sigma\nu = \mathbf{f}_2 &\quad \text{sur} \quad \Gamma_2 \times (0,T),\\
-\sigma_\nu = p_\nu(\dot{u}_\nu) &\quad \text{sur} \quad \Gamma_3 \times (0,T),\\
|\sigma_\tau| \le p_\tau(\dot{u}_\nu) &\quad \text{sur} \quad \Gamma_3 \times (0,T),\\
|\sigma_\tau| < p_\tau(\dot{u}_\nu) &\quad \Longrightarrow \quad \dot{u}_\tau = 0,\\
|\sigma_\tau| = p_\tau(\dot{u}_\nu) &\quad \Longrightarrow \quad \sigma_\tau = -\lambda\dot{u}_\tau,\ \lambda \ge 0,\\
u(0) = u_0, \quad \beta(0) = \beta_0 &\quad \text{dans} \quad \Omega.
\end{aligned}$$

On établit une formulation variationnelle du problème PE suivie d'un résultat d'existence et d'unicité.

Analysis of a Quasistatic Viscoelastic Problem with Friction and Damage *

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Abstract

We consider a model for quasistatic frictional contact between a viscoelastic body and a reactive foundation. The material constitutive relation is assumed nonlinear and viscoelastic. The contact is described by a general damped response condition and the associated version of the Coulomb law of dry friction. We derive a variational formulation for the problem and prove the existence of a unique weak solution. We also propose a dual formulation of the problem, in terms of the stress, and establish the equivalence between the two formulations. Then, we add the material damage which results from tension or compression as a dependent variable, and obtain the existence and uniqueness of the weak solution to the problem with damage. Finally, we derive a dual formulation and establish its equivalence to the primal problem.

Key words nonlinear viscoelastic constitutive law, frictional contact, the Coulomb friction law, damped response, variational inequality, fixed point, damage

1. Introduction

We model and analyze a process of quasistatic contact with friction between a viscoelastic body and a reactive foundation when the damage due to mechanical strain in the material is taken into account.

We investigated recently a number of quasistatic problems related to frictional contact. Indeed, a model for bilateral contact with friction was considered in [14], a model for contact with normal compliance and friction was analyzed in [12], and the problem of contact with damped response and directional friction was studied in [13]. In these papers friction was modeled with versions of the Coulomb law and the material was assumed to have nonlinear viscoelastic constitutive relation of the form

$$\sigma = \mathcal{A}(\varepsilon(\dot{u})) + G(\varepsilon(u)), \tag{1.1}$$

where u denotes the displacement field and σ and $\varepsilon(u)$ denote the stress and linearized strain tensor, respectively. Here $\mathcal{A}$ and G are nonlinear constitutive functions and the dot

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above represents the time derivative. The results obtained in [12, 13, 14] deal with the existence and uniqueness of weak solutions, i.e., solutions which satisfy variational formulations of the corresponding mechanical problems. The wear of the contacting surfaces was taken into account, however, the damage to the material arising from the strain was not included in these problems.

The aim of this paper is to extend the results obtained in [13] to a model which includes the material damage and a general contact condition.

We consider a contact condition with friction which generalizes the one used in [13]. We derive a variational formulation of the model and prove the existence and uniqueness of its solutions. We also obtain and study a dual formulation of the mechanical problem, which is in the form of a quasivariational inequality for the stress field. The importance of dual formulations of contact problems stems from the observation that contact stresses are the main variable of interest to the design engineer, while the displacements are of secondary interest. We prove the existence of a unique solution of the dual problem via an equivalence result.

Finally, we show that our mathematical approach may be applied to a contact problem when the mechanical damage of the material is taken into account. In many engineering applications the forces acting on the system vary periodically and, thus, the strain varies too. This may lead to the appearance and growth of microcracks which, in turn, lead to the reduction in the usefulness of the system. Recent models for mechanical damage which were derives from thermomechanical considerations can be found in Following Frémond et al. [6, 7]. There, they obtained the models and performed numerical simulations for concrete. Mathematical analysis of one-dimensional problems can be found in [4, 5]. Here we consider a general case, however, as we point out below, our source function is not allowed to become unbounded, which is the case when the material is completely damaged. And therefore, we may consider our results as local in time.

We consider a viscoelastic material with constitutive relation

$$\sigma = \mathcal{A}(\varepsilon(\dot{u})) + G(\varepsilon(u), \beta). \tag{1.2}$$

Here $\beta = \beta(x, t)$ represents the damage field which measures the decrease in the load bearing capacity of the material. Following Frémond et al. [6, 7], the effective elastic modulus of the material satisfies $E_{\text{eff}} = \beta E$, where E is the Young modulus of the damage-free material, and thus, $0 \leq \beta \leq 1$. The evolution of the microscopic cracks which cause the damage is governed by the differential inclusion

$$\dot{\beta} - k\triangle\beta + \partial\psi_K(\beta) \ni \phi(\varepsilon(u), \beta), \tag{1.3}$$

where k is a positive material constant; K denotes the set of admissible damage functions which satisfy $0 \leq \beta \leq 1$; $\partial\psi_K$ represents the subdifferential of the indicator function of K; and ϕ is a given constitutive function which describes the sources of damage in the system, resulting from tension and compression.

In Section 2 we introduce notation and preliminaries. In Section 3 the mechanical problem is stated and the contact boundary conditions discussed. In Section 4 we present the variational form of the model; the assumptions on the problem data are listed; a dual formulation is obtained and our main results are stated in theorems 4.1 and 4.2. Sections 5 and 6 are devoted to the proofs of the results. We use the Banach fixed point Theorem and as a result the existence of the unique solution is proved only under a smallness assumption on the contact functions. Finally, in Section 7 we consider the contact problem with damage and we use a variant of the proofs in Sections 5 and 6 to obtain the existence of a unique local weak solution to this problem. The local character of the solution is related to the fact that when the damage field β vanishes at a point the material there disintegrates and the model ceases to make sense.

2. Notation and preliminaries

We present in this section the notation we will use and preliminary material. For more details we refer the reader to [1, 9, 10] or [11].

We denote by $\mathbb{R}^{d\times d}_s$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$; "$\cdot$" and $|\cdot|$ represent the inner product and Euclidean norm on $\mathbb{R}^{d\times d}_s$ or $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let Γ_1 be a part of Γ such that $\operatorname{meas}\Gamma_1 > 0$. We denote by ν the unit outer normal on Γ and

$$H = \Big\{v = (v_i) \mid v_i \in L^2(\Omega)\Big\} = L^2(\Omega)^d,$$

$$H_1 = \Big\{v = (v_i) \mid v_i \in H^1(\Omega)\Big\} = H^1(\Omega)^d,$$

$$\mathcal{H} = \Big\{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega),\Big\} = L^2(\Omega)^{d\times d}_s,$$

$$\mathcal{H}_1 = \Big\{\tau \in \mathcal{H} \mid \tau_{ij,j} \in H\Big\}.$$

Here and below, $i, j = 1, 2, \dots, d$; the summation convention over repeated indices is adopted and a subscript that follows a comma indicates a partial derivative.

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products

$$\langle u, v\rangle_H = \int_\Omega u_i v_i \, dx, \qquad \langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx,$$

$$\langle u, v\rangle_{H_1} = \langle u, v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \qquad \langle \sigma, \tau\rangle_{\mathcal{H}_1} = \langle \sigma, \tau\rangle_{\mathcal{H}} + \langle \operatorname{Div}\sigma, \operatorname{Div}\tau\rangle_H,$$

respectively. Here $\varepsilon : H_1 \longrightarrow \mathcal{H}$ and $\operatorname{Div} : \mathcal{H}_1 \longrightarrow H$ are the *deformation* and the *divergence* operators, respectively, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

$$\operatorname{Div}\sigma = \sigma_{ij,j}.$$

The associated norms on H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively.

For every element $v \in H_1$ we denote by v its trace on Γ and we denote by V the closed subspace of H_1 given by

$$V = \Big\{ v \in H_1 \mid v = 0 \ \text{ on } \ \Gamma_1 \Big\}.$$

Since $\operatorname{meas}\Gamma_1 > 0$, Korn's inequality holds, thus

$$|\varepsilon(u)|_{\mathcal{H}} \geq C|u|_{H_1} \qquad \forall u \in V, \tag{2.1}$$

see, e.g., [8](p. 79). In (2.1) and in the sequel C represents a positive generic constant which may depend on Ω, Γ_1 and $\mathcal{A}$ and whose value may change from line to line. We define the inner product $\langle \cdot\,, \cdot \rangle_V$ on V by

$$\langle u, v \rangle_V = \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}. \tag{2.2}$$

It follows from (2.1) and (2.2) that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V. Therefore $(V, |\cdot|_V)$ is a real Hilbert space.

We denote by v_ν and v_τ the *normal* and *tangential* components of $v \in H_1$ on Γ given by $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu \nu$. We also denote by σ_ν and σ_τ the *normal* and *tangential* traces of $\sigma \in \mathcal{H}_1$. If σ is a regular (e.g., C^1) function we have $\sigma_\nu = (\sigma\nu)\cdot\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$ and

$$\langle \sigma, \varepsilon(v) \rangle_{\mathcal{H}} + \langle \operatorname{Div}\sigma, v \rangle_H = \int_\Gamma \sigma\nu \cdot v d\Gamma \qquad \forall v \in H_1. \tag{2.3}$$

Finally if X is one of the above Hilbert spaces, $C(0,T;X)$ and $C^1(0,T;X)$ represent the spaces of continuous and continuously differentiable functions from $[0,T]$ to X, with norms

$$|x|_{C([0,T];X)} = \max_{t\in[0,T]} |x(t)|_X, \quad |x|_{C^1([0,T];X)} = \max_{t\in[0,T]} |x(t)|_X + \max_{t\in[0,T]} |\dot{x}(t)|_X,$$

respectively. We use standard notation for the Sobolev spaces $W^{k,p}(0,T;X)$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product space endowed with the canonical inner product $\langle \cdot\,, \cdot \rangle_{X_1\times X_2}$ and norm $|\cdot|_{X_1\times X_2}$.

3. Problem statement

We now describe the model of the process and discuss the contact boundary conditions. The setting of the process is as follows. A viscoelastic body occupies the domain Ω and is acted upon by time-dependent volume forces and surface tractions and is in frictional contact with a rigid foundation, and as a result its mechanical state evolves on the time, over $[0,T]$, for $T > 0$. We assume that the boundary Γ of Ω is divided into three disjoint (measurable) parts Γ_1, Γ_2 and Γ_3 such that $\operatorname{meas}\Gamma_1 > 0$. The body is held fixed on $\Gamma_1 \times (0,T)$ and therefore the displacement field vanishes there. A volume force of density f_0 acts in $\Omega \times (0,T)$ and surface tractions of density f_2 are applied on $\Gamma_2 \times (0,T)$. We assume that volume forces and tractions vary slowly in time. Therefore the accelerations in the system are negligible, which leads to the quasistatic approximation of the process.

The solid is in frictional contact with a rigid obstacle on $\Gamma_3 \times (0,T)$. The contact is described with a generalized version of the normal damped response and the Coulomb friction law.

A classical formulation of the model for this process is the following.

Problem P. Find a displacement field $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ and a stress field $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{R}_s^{d\times d}$ such that

$$\sigma = \mathcal{A}(\varepsilon(\dot{u})) + G(\varepsilon(u)) \quad \text{in} \quad \Omega \times (0,T), \tag{3.1}$$

$$\operatorname{Div} \sigma + \mathbf{f}_0 = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{3.2}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0,T), \tag{3.3}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{on} \quad \Gamma_2 \times (0,T), \tag{3.4}$$

$$-\sigma_\nu = p_\nu(\dot{u}_\nu) \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{3.5}$$

$$\begin{aligned} &|\sigma_\tau| \le p_\tau(\dot{u}_\nu) \quad \text{on} \quad \Gamma_3 \times (0,T), \\ &|\sigma_\tau| < p_\tau(\dot{u}_\nu) \Longrightarrow \dot{u}_\tau = 0, \\ &|\sigma_\tau| = p_\tau(\dot{u}_\nu) \Longrightarrow \sigma_\tau = -\lambda \dot{u}_\tau, \ \lambda \ge 0, \end{aligned} \tag{3.6}$$

$$u(0) = u_0 \quad \text{in} \quad \Omega. \tag{3.7}$$

Here, (3.1) represents the nonlinear viscoelastic constitutive law. Equality (3.2) represents the equilibrium equations; (3.3) and (3.4) are the displacement and traction conditions; (3.7) represents the initial condition, where u_0 is given.

We describe now the contact conditions (3.5) and (3.6) where our interest lies. Here $\dot{u}_\nu$ and $\dot{u}_\tau$ represent the normal and tangential velocities, respectively, while σ_ν and σ_τ denote the normal and tangential stresses, respectively. The contact functions p_ν and p_τ are prescribed nonnegative functions. Equality (3.5) states a general dependence of the normal stress on the normal velocity. In the case when

$$p_\nu(r) = \kappa r \quad \text{with} \quad \kappa \ge 0, \tag{3.8}$$

the resistance of the obstacle to penetration is proportional to the normal velocity. This type of behavior was considered in [14] modeling the motion of a deformable body on sand or a granular material. We may also consider the case

$$p_\nu(r) = \kappa r_+ + p_0, \tag{3.9}$$

where $\kappa \ge 0$, $r_+ = \max\{0, r\}$ and $p_0 > 0$. This boundary condition models the contact with a normal damped response (see, e.g., [13]). Finally, in the case

$$p_\nu(r) = p_*, \tag{3.10}$$

with p_* prescribed, (3.5) and (3.6) lead to a simplified version of the Coulomb law of dry friction where the normal stress is prescribed (see for instance [3] or [11]).

The boundary condition (3.6) states that the tangential shear cannot exceed the maximal frictional resistance $p_\tau(\dot{u}_\nu)$. When strong inequality holds, the surface adheres to the foundation and it is in the so-called *stick* state, and when equality holds there is relative sliding, the so-called *slip* state. Therefore, at each time instant the contact surface Γ_3 is Divdivided into two zones: the stick zone and the slip zone. The boundaries of these zones are unknown a priori, are part of the problem and form free boundaries. If we choose

$$p_\tau = \mu p_\nu, \tag{3.11}$$

we obtain the usual Coulomb law of dry friction where $\mu \geq 0$ represents the coefficient of friction (see, e.g., [3] or [11]).

Recently a modified version of the Coulomb friction law has been derived in [15, 16] from thermodynamic considerations. It consists of using the friction law (3.6) with

$$p_\tau = \mu p_\nu (1 - \delta p_\nu)_+, \tag{3.12}$$

where δ is a small positive material constant related to the wear and hardness of the surface and $\mu \geq 0$ is the coefficient of friction.

A classical friction law which can be modeled by (3.6) is Tresca's friction law. It consists of choosing

$$p_\tau(r) = g_*, \tag{3.13}$$

where $g_* > 0$ is prescribed, and represents the friction bound, i.e., the magnitude of the limiting friction traction at which slip begins.

4. Weak formulation and statement of results

In this section we set Problem P in a variational form and state our main results.

To this end, we assume that the *viscosity operator*

$$\mathcal{A} : \Omega \times \mathbb{R}_s^{d\times d} \longrightarrow \mathbb{R}_s^{d\times d},$$

satisfies

$$\begin{array}{ll}
(a) & \text{there exists } L_{\mathcal{A}} > 0 \text{ such that} \\
 & |\mathcal{A}(\cdot,\varepsilon_1) - \mathcal{A}(\cdot,\varepsilon_2)| \leq L_{\mathcal{A}}|\varepsilon_1 - \varepsilon_2| \quad \forall \varepsilon_1,\varepsilon_2 \in \mathbb{R}_s^{d\times d}, \text{ a.e. in } \Omega, \\
(b) & \text{there exists } m > 0 \text{ such that} \\
 & (\mathcal{A}(\cdot,\varepsilon_1)) - \mathcal{A}(\cdot,\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m|\varepsilon_1 - \varepsilon_2|^2 \\
 & \forall \varepsilon_1,\varepsilon_2 \in \mathbb{R}_s^{d\times d}, \text{ a.e. in } \Omega, \\
(c) & x \longmapsto \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \quad \forall \varepsilon \in \mathbb{R}_s^{d\times d}, \\
(d) & x \longmapsto \mathcal{A}(x,0) \in \mathcal{H}.
\end{array} \tag{4.1}$$

The *elasticity operator*

$$G : \Omega \times \mathbb{R}_s^{d\times d} \longrightarrow \mathbb{R}_s^{d\times d}$$

satisfies

$$\begin{array}{ll}
(a) & \text{there exists } L_G > 0 \text{ such that} \\
 & |G(\cdot,\varepsilon_1) - G(\cdot,\varepsilon_2)| \leq L_G|\varepsilon_1 - \varepsilon_2| \quad \forall \varepsilon_1,\varepsilon_2 \in \mathbb{R}_s^{d\times d}, \text{ a.e. in } \Omega, \\
(b) & x \longmapsto G(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \quad \forall \varepsilon \in \mathbb{R}_s^{d\times d}, \\
(c) & x \longmapsto G(x,0) \in \mathcal{H}.
\end{array} \tag{4.2}$$

The contact functions

$$p_r : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+ \quad (r = \nu, \tau)$$

satisfy

$$\begin{aligned}
&(a) \quad \text{there exists } \ L_r > 0 \ \text{ such that} \\
&\qquad |p_r(\cdot\,,u_1) - p_r(\cdot\,,u_2)| \le L_r|u_1 - u_2| \quad \forall\, u_1,\, u_2 \in \mathbb{R}, \text{ a.e. on } \Gamma_3, \\
&(b) \quad x \mapsto p_r(x,u) \ \text{ is Lebesgue measurable on } \Gamma_3 \quad \forall\, u \in \mathbb{R}, \\
&(c) \quad x \longmapsto p_r(x,0) \in L^2(\Gamma_3).
\end{aligned} \tag{4.3}$$

We observe that assumptions (4.3) on p_ν and p_τ are fairly general. The main restriction arises from condition (a), which requires the functions to grow asymptotically at most linearly. The functions defined in (3.8)–(3.10) and (3.13) satisfy this condition. We also observe that if the functions p_ν and p_τ are related by (3.11) or (3.12) and p_ν satisfies the condition (4.3)(a), then p_τ also satisfies the condition (4.3)(a) with $L_\tau = \mu\, L_\nu$. We conclude that the results below are valid for the boundary value problem of each of these examples.

We also assume that the forces and tractions satisfy

$$\mathbf{f}_0 \in C(0,T;H), \qquad \mathbf{f}_2 \in C(0,T;L^2(\Gamma_2)^d), \tag{4.4}$$

and, finally, the initial displacement satisfies

$$u_0 \in V. \tag{4.5}$$

Next, we denote by $f(t)$ the element of Vgiven by

$$\langle f(t), v\rangle_V = \int_\Omega \mathbf{f}_0(t)\cdot v\, dx + \int_{\Gamma_2} \mathbf{f}_2(t)\cdot v\, d\Gamma \tag{4.6}$$

for all $v \in V$ and $t \in [0,T]$, and we note that conditions (4.4) imply

$$f \in C(0,T;V). \tag{4.7}$$

Let $j : V \times V \longrightarrow \mathbb{R}$ be the functional

$$j(v,w) = \int_{\Gamma_3} \Big(p_\nu(v_\nu)w_\nu + p_\tau(v_\nu)|w_\tau| \Big)\, d\Gamma. \tag{4.8}$$

It is straightforward to show that if $\{u, \sigma\}$ are sufficiently regular functions satisfying (3.2)–(3.6) then

$$\langle \sigma(t), \varepsilon(w) - \varepsilon(\dot u(t))\rangle_{\mathcal{H}} + j(\dot u(t), w) - j(\dot u(t), \dot u(t)) \ge \langle f(t), w - \dot u(t)\rangle_V \quad \forall w \in V \tag{4.9}$$

for all $t \in [0,T]$. Thus, by (3.1), (3.7) and (4.9) we obtain the following variational formulation of Problem P.

Problem P_V. Find a displacement field $u : [0,T] \longrightarrow V$ and a stress field $\sigma : [0,T] \longrightarrow \mathcal{H}$ such that

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + G(\varepsilon(u(t))) \qquad \forall t \in [0,T], \tag{4.10}$$

$$\langle \sigma(t), \varepsilon(w) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(\dot{u}(t), w) - j(\dot{u}(t), \dot{u}(t)) \geq \langle f, w - \dot{u}(t)\rangle_V \tag{4.11}$$

$$\forall w \in V,\ t \in [0,T],$$

$$u(0) = u_0. \tag{4.12}$$

Our main result, that we prove in the next section, is the following.

Theorem 4.1. *Assume that* (4.1)–(4.5) *hold. Then there exists $\alpha > 0$, which depends only on Ω, Γ_1 and $\mathcal{A}$, such that if $L_\nu + L_\tau < \alpha$, then Problem P_V has a unique solution $\{u, \sigma\}$ such that*

$$u \in C^1(0,T;V), \qquad \sigma \in C(0,T;\mathcal{H}_1).$$

The proof will be given in the next Section.

We conclude that, when the Lipschitz constants of the functions p_ν and p_τ are sufficiently small, Problem P has a unique weak solution $\{u, \sigma\}$. The question of estimating α is left open, and such estimates are likely to depend on the particular problem and setting.

We now establish a *dual formulation* of the mechanical problem P. To that end we define the set of admissible stress fields $\Sigma(t, v)$ by

$$\Sigma(t,v) = \Big\{\sigma \in \mathcal{H} \mid \langle \sigma, \varepsilon(w)\rangle_{\mathcal{H}} + j(v,w) \geq \langle f(t), w\rangle_V \quad \forall w \in V\Big\}, \tag{4.13}$$

for all $t \in [0,T]$ and all $v \in V$. Choosing $v = 2\dot{u}(t)$ and $v = 0$, both in V, in (4.11) we deduce that

$$\langle \sigma(t), \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(\dot{u}(t), \dot{u}(t)) = \langle f(t), \dot{u}(t)\rangle_V \tag{4.14}$$

for all $t \in [0,T]$, which implies that

$$\sigma(t) \in \Sigma(t, \dot{u}(t)), \quad \langle \tau - \sigma(t), \varepsilon(\dot{u}(t)\rangle_{\mathcal{H}} \geq 0 \quad \forall \tau \in \Sigma(t, \dot{u}(t)),\ t \in [0,T]. \tag{4.15}$$

Thus, by (3.1), (3.7) and (4.15) we obtain the following variational problem.

Problem P_D. Find a displacement field $u : [0,T] \longrightarrow V$ and a stress field $\sigma : [0,T] \longrightarrow \mathcal{H}$ such that

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + G(\varepsilon(u(t))) \qquad \forall t \in [0,T], \tag{4.16}$$

$$\sigma(t) \in \Sigma(t, \dot{u}(t)), \qquad \langle \tau - \sigma(t), \varepsilon(\dot{u}(t)\rangle_{\mathcal{H}} \geq 0 \quad \forall \tau \in \Sigma(t, \dot{u}(t)),\ t \in [0,T], \tag{4.17}$$

$$u(0) = u_0. \tag{4.18}$$

We note that (4.17) is a quasivariational inequality since the set Σ depends on the solution.

Our second result shows that problems P_V and P_D are equivalent.

Theorem 4.2. *Assume that* (4.1)–(4.5) *hold and let* $u \in C^1(0,T;V)$ *and* $\sigma \in C(0,T;\mathcal{H}_1)$. *Then* $\{u,\sigma\}$ *is the solution of Problem* P_V *if and only if* $\{u,\sigma\}$ *is a solution of Problem* P_D.

The proof will be given in Section 6.

We conclude from Theorems 4.1 and 4.2 that, under assumptions (4.1)–(4.5), there exists $\alpha > 0$ such that if $L_\nu + L_\tau < \alpha$, then Problem P_D has a unique solution $\{u,\sigma\}$ and $u \in C^1(0,T;V)$, $\sigma \in C(0,T;\mathcal{H}_1)$. Moreover, the solution $\{u,\sigma\}$ of problem P_D is the weak solution of the mechanical problem P obtained in Theorem 4.1.

5. Proof of Theorem 4.1

The proof is based on fixed point arguments, similar to those used in [12, 13], but in a different setting and with a different choice of the operators. For this reason, we omit some of the details below. The proof is carried out in several steps. To simplify the notation, we do not indicate explicitly the dependence on t, and the equalities and inequalities below hold for all $t \in [0,T]$.

Let $\eta \in C(0,T;\mathcal{H})$ and $g \in C(0,T;V)$. In the first step we consider the following auxiliary problem.

Problem $P_{\eta g}$. Find a displacement field $v_{\eta g} : [0,T] \longrightarrow V$ and a stress field $\sigma_{\eta g} : [0,T] \longrightarrow \mathcal{H}_1$ such that

$$\sigma_{\eta g} = \mathcal{A}(\varepsilon(v_{\eta g})) + \eta, \tag{5.1}$$

$$\langle \sigma_{\eta g}, \varepsilon(w) - \varepsilon(v_{\eta g})\rangle_{\mathcal{H}} + j(g,w) - j(g,v_{\eta g}) \geq \langle f, w - v_{\eta g}\rangle_V \quad \forall w \in V. \tag{5.2}$$

for all $t \in [0,T]$.

Clearly, we solve the problem for the velocity and the stress fields when the elastic part of the stress η and the contact velocity g are given. We have the following result.

Proposition 5.1. *Problem* $P_{\eta g}$ *has a unique solution* $\{v_{\eta g}, \sigma_{\eta g}\}$ *such that*

$$v_{\eta g} \in C(0,T;V), \qquad \sigma_{\eta g} \in C(0,T;\mathcal{H}_1).$$

Proof. Proposition 5.1 follows from classical results for elliptic variational inequalities, and for more details we refer the reader to [12].

We now define the operator $\Lambda_\eta : C(0,T;V) \longrightarrow C(0,T;V)$ by

$$\Lambda_\eta g = v_{\eta g} \qquad g \in C(0,T;V), \tag{5.3}$$

where $v_{\eta g}$ denotes the velocity field given in Proposition 5.1. We have,

Proposition 5.2. *There exists* $\alpha > 0$ *which depends only on* Ω, Γ_1 *and* $\mathcal{A}$, *such that if* $L_\nu + L_\tau < \alpha$, *then the operator* Λ_η *has a unique fixed point* $g_\eta \in C(0,T;V)$.

Proof. Let $g_1,\ g_2 \in C(0,T;V)$ and $\eta \in C(0,T;\mathcal{H})$. Let $\{v_i, \sigma_i\}$, $(i = 1,2)$ denote the solutions of Problem $P_{\eta g_i}$, i.e., $v_i = v_{\eta g_i}$ and $\sigma_i = \sigma_{\eta g_i}$. Using (5.1), (5.2), (4.1) and (2.2) we obtain

$$|v_1 - v_2|_V^2 \le C\Big(j(g_1, v_2) - j(g_1, v_1) + j(g_2, v_1) - j(g_2, v_2)\Big). \tag{5.4}$$

Moreover, from (4.8) and (4.3) it follows that

$$\begin{aligned} j(g_1, v_2) - j(g_1, v_1) \ &+\ j(g_2, v_1) - j(g_2, v_2) \\ &\le C(L_\nu + L_\tau)|g_1 - g_2|_V |v_1 - v_2|_V. \end{aligned} \tag{5.5}$$

Using now (5.4) and (5.5) we obtain

$$|v_1 - v_2|_V \le C(L_\nu + L_\tau)|g_1 - g_2|_V. \tag{5.6}$$

Proposition 5.2 results now from (5.6) and the Banach fixed point theorem.

We assume in the sequel that $L_\nu + L_\tau < \alpha$ and, for each $\eta \in C(0,T;\mathcal{H})$, we denote by g_η the fixed point given in Proposition 5.2. Let $v_\eta \in C(0,T;V)$ and $\sigma_\eta \in C(0,T;\mathcal{H}_1)$ be the functions given by

$$v_\eta = v_{\eta g_\eta}, \qquad \sigma_\eta = \sigma_{\eta g_\eta}. \tag{5.7}$$

Moreover, using (4.5), let $u_\eta \in C^1(0,T;V)$ be the function defined by

$$u_\eta(t) = u_0 + \int_0^t v_\eta(s)\,ds \qquad t \in [0,T]. \tag{5.8}$$

We define the operator $\Lambda : C(0,T;\mathcal{H}) \longrightarrow C(0,T;\mathcal{H})$ by

$$\Lambda\eta = G(\varepsilon(u_\eta)) \qquad \eta \in C(0,T;\mathcal{H}). \tag{5.9}$$

We have the following result.

Proposition 5.3. *The operator Λ has a unique fixed point $\eta^* \in C(0,T,\mathcal{H})$.*

Proof. Let η_1, $\eta_2 \in C(0,T;\mathcal{H})$ and let $v_i = v_{\eta_i}$, $\sigma_i = \sigma_{\eta_i}$, $u_i = u_{\eta_i}$, $g_i = g_{\eta_i}$, for $i = 1,2$. Using Proposition 5.2 we have $g_i = v_i$ and from (5.1) and (5.2) we obtain

$$\sigma_i = \mathcal{A}\varepsilon(v_i) + \eta_i, \tag{5.10}$$

$$\langle \sigma_i, \varepsilon(w) - \varepsilon(v_i)\rangle_{\mathcal{H}} + j(v_i, w) - j(v_i, v_i) \ge \langle f, w - v_i\rangle_V, \quad \forall w \in V, \tag{5.11}$$

where $i = 1,2$. Using (5.11) we deduce that

$$\langle \sigma_1 - \sigma_2, \varepsilon(v_1) - \varepsilon(v_2)\rangle_{\mathcal{H}} \ge 0,$$

and, by (5.10) and (4.1) we find

$$|v_1 - v_2|_V \le C|\eta_1 - \eta_2|_{\mathcal{H}}. \tag{5.12}$$

It follows now from (5.9), (4.2), (5.8) and (5.12) that

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \leq CL\int_0^t |\eta_1(s) - \eta_2(s)|_{\mathcal{H}} ds \qquad \forall t \in [0,T]. \tag{5.13}$$

Proposition 5.3 results from (5.13) and the Banach fixed point theorem.
Proof of Theorem 4.1. Let $L_\nu + L_\tau < \alpha$.
Existence. Let $\eta^* \in C(0,T;\mathcal{H})$ be the fixed point of Λ and let v_{η^*}, σ_{η^*} and u_{η^*} be the functions given by (5.7) and (5.8) for $\eta = \eta^*$. Choosing $\eta = \eta^*$ and $g = g_{\eta^*}$ in (5.1) and (5.2) and using (5.7) we obtain that

$$\sigma_{\eta^*} = \mathcal{A}\varepsilon(v_{\eta^*}) + \eta^*, \tag{5.14}$$

$$\langle \sigma_{\eta^*}, \varepsilon(w) - \varepsilon(v_{\eta^*})\rangle_{\mathcal{H}} + j(g_{\eta^*}, w) - j(g_{\eta^*}, v_{\eta^*}) \geq \langle f, w - v_{\eta^*}\rangle_V \quad \forall w \in V. \tag{5.15}$$

Now, equality (4.10) follows from (5.8), (5.9) and (5.14), since $v_{\eta^*} = \dot{u}_{\eta^*}$, $\eta^* = \Lambda\eta^* = G(\varepsilon(u_{\eta^*}))$. Inequality (4.11) follows from (5.15) since $g_{\eta^*} = \Lambda_{\eta^*} g_{\eta^*} = v_{\eta^*} = \dot{u}_{\eta^*}$. The equality (4.12) results from (5.8), and the fact that $u_{\eta^*} \in C^1(0,T;V)$ and $\sigma_{\eta^*} \in C(0,T;\mathcal{H}_1)$ is a consequence of Proposition 5.1, (4.5) and (5.8).

Uniqueness. It follows from the uniqueness of the fixed point η^* of the operator Λ.

6. Proof of Theorem 4.2

In this section we prove the equivalence between Problems P_V and P_D. Let $\{u, \sigma\}$ be two functions such that $u \in C^1(0,T;V)$ and $\sigma \in C(0,T;\mathcal{H}_1)$. We establish the equivalence of the variational inequalities (4.11) and (4.17).

(i) (4.11) $\Longrightarrow$ (4.17). We choose $w = 2\dot{u}$ and $w = 0$ in (4.11) and deduce that

$$\langle \sigma, \varepsilon(\dot{u})\rangle_{\mathcal{H}} + j(\dot{u}, \dot{u}) = \langle f, \dot{u}\rangle_V. \tag{6.1}$$

Using (4.11) and (6.1) we obtain

$$\langle \sigma, \varepsilon(w)\rangle_{\mathcal{H}} + j(\dot{u}, w) \geq \langle f, w\rangle_V \qquad \forall w \in V,$$

which implies that $\sigma(t) \in \Sigma(t, \dot{u}(t))$ for all $t \in [0,T]$. On the other hand, we obtain from (4.13) and (6.1) that

$$\langle \tau - \sigma(t), \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} \geq 0 \qquad \forall \tau \in \Sigma(t, \dot{u}(t)),\ t \in [0,T]. \tag{6.2}$$

The inequality (4.17) follows now from (6.2).

(4.17) $\Longrightarrow$ (4.11). The subdifferentiability of the function $j(\dot{u}(t), \cdot)$, for $\dot{u}(t) \in V$, implies that there exists a function $\tilde{f} : [0,T] \longrightarrow V$ such that

$$j(\dot{u}, w) - j(\dot{u}, \dot{u}) \geq \langle \tilde{f}, w - \dot{u}\rangle_V \qquad \forall w \in V,$$

thus,

$$\langle f - \tilde{f}, w - \dot{u}\rangle_V + j(\dot{u}, w) - j(\dot{u}, \dot{u}) \geq \langle f, w - \dot{u}\rangle_V \qquad \forall w \in V. \tag{6.3}$$

Choosing $w = 2\dot{u}$ and $w = 0$ we obtain

$$\langle f - \tilde{f}, \dot{u}\rangle_V + j(\dot{u}, \dot{u}) = \langle f, \dot{u}\rangle_V. \tag{6.4}$$

It then follows from (6.3), (6.4) and (2.2) that $\varepsilon(f(t) - \tilde{f}(t)) \in \Sigma(t, \dot{u}(t))$ for all $t \in [0, T]$. Therefore, by using $\tau = \varepsilon(f - \tilde{f})$ as a test function in (4.17) and (2.2) we deduce

$$\langle f - \tilde{f}, \dot{u}\rangle_V \geq \langle \sigma, \varepsilon(\dot{u})\rangle_{\mathcal{H}}.$$

Adding $j(\dot{u}, \dot{u})$ to this inequality and taking (6.4) into account we find

$$\langle f, \dot{u}\rangle_V \geq \langle \sigma, \varepsilon(\dot{u})\rangle_{\mathcal{H}} + j(\dot{u}, \dot{u}). \tag{6.5}$$

On the other hand, since $\sigma(t) \in \Sigma(t, \dot{u}(t))$ and $u(t) \in V$, we obtain

$$\langle \sigma, \varepsilon(\dot{u})\rangle_{\mathcal{H}} + j(\dot{u}, \dot{u}) \geq \langle f, \dot{u}\rangle_V, \tag{6.6}$$

and, also,

$$\langle \sigma, \varepsilon(w)\rangle_{\mathcal{H}} + j(\dot{u}, w) \geq \langle f, w\rangle_V \qquad \forall w \in V. \tag{6.7}$$

Inequality (4.11) is now a consequence of (6.5)–(6.7).

7. The problem with damage

We consider the contact problem when the damage of the material caused by tension or compression is taken into account. We assume that the mechanical strain, when above a given threshold, creates microcracks which may grow and cause the decrease in the load bearing capacity of the material. General models for damage were derived recently in [6, 7] from the virtual power principle. Analysis of one-dimensional problems can be found in [4, 5]. Here we use a variant of one of their models, and we note that a number of other models for damage can be found in the engineering literature.

We assume, following [6, 7], that the effective elastic moduli of the material depend on the damage function $\beta = \beta(x, t)$. There it was assumed that $\beta = E_{\text{eff}}/E$, where E_{eff} is the effective modulus of elasticity and E is the Young modulus in the absence of damage. Therefore, β has values between zero and one; when $\beta = 1$ there is no damage; when $\beta = 0$ the material is completely damaged at the point; when $0 < \beta < 1$ there is partial damage there. We retain the assumption that $0 \leq \beta \leq 1$ but use a more general dependence of the elasticity function on β, given in (1.2). Furthermore, we assume that the source of damage can be represented by $\phi = \phi(\epsilon(u), \beta)$, since the damage depends on the strain. In [6, 7] the damage source was chosen as

$$\phi_{\text{Fr}}(\epsilon(u), \beta) = m\left(\frac{1-\beta}{\beta}\right) - \frac{1}{2}\left(\epsilon(u)\right)^2 + w,$$

where m and w are two positive material parameters. We note that the source becomes unbounded when $\beta \to 0$. However, our assumptions on ϕ do not allow for complete damage at any point. Therefore, we may consider the global solutions of our problem as local solutions of a problem with ϕ_{Fr} as damage source, valid as long as $0 < \beta_* \le \beta$. We assume that the material may recover from damage and cracks may close, thus, we do not impose the restriction $\partial\beta/\partial t \le 0$ which was used in [4, 6, 7, 5].

The physical setting is the same as in Section 3, and in the model we replace the constitutive law (3.1) by (1.2) and (1.3). For the sake of simplicity we assume a homogeneous Neumann boundary condition for the damage field. Next, let K denote the set of admissible damage functions

$$K = \Big\{ \xi \in H^1(\Omega) : \ 0 \le \xi \le 1 \ \text{ a.e. in } \ \Omega \Big\}.$$

Let ψ_K denote the indicator function of K and $\partial\psi_K$ be its subdifferential.

Under these assumptions, the classical formulation of the physical problem is:

Problem DP: Find a displacement field $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{R}^{d\times d}_s$ and a damage field $\beta : \Omega \times [0,T] \longrightarrow \mathbb{R}$ such that

$$\sigma = \mathcal{A}(\varepsilon(\dot{u})) + G(\varepsilon(u), \beta) \quad \text{in} \quad \Omega \times (0,T), \tag{7.1}$$

$$\dot{\beta} - k\Delta\beta + \partial\psi_K(\beta) \ni \phi(\varepsilon(u), \beta) \quad \text{in} \quad \Omega \times (0,T), \tag{7.2}$$

$$\operatorname{Div} \sigma + \mathrm{f}_0 = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{7.3}$$

$$\frac{\partial\beta}{\partial\nu} = 0 \quad \text{on} \quad \Gamma \times (0,T), \tag{7.4}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0,T), \tag{7.5}$$

$$\sigma\nu = \mathrm{f}_2 \quad \text{on} \quad \Gamma_2 \times (0,T), \tag{7.6}$$

$$-\sigma_\nu = p_\nu(\dot{u}_\nu) \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{7.7}$$

$$\begin{aligned} &|\sigma_\tau| \le p_\tau(\dot{u}_\nu) \quad \text{on} \quad \Gamma_3 \times (0,T), \\ &|\sigma_\tau| < p_\tau(\dot{u}_\nu) \quad \Longrightarrow \dot{u}_\tau = 0, \\ &|\sigma_\tau| = p_\tau(\dot{u}_\nu) \quad \Longrightarrow \sigma_\tau = -\lambda \dot{u}_\tau, \ \lambda \ge 0, \end{aligned} \tag{7.8}$$

$$u(0) = u_0, \quad \beta(0) = \beta_0 \quad \text{in} \quad \Omega. \tag{7.9}$$

Here, $\frac{\partial\beta}{\partial\nu}$ represents the normal derivative of β on the boundary and β_0 in (7.9) is a prescribed initial damage.

Next, we derive a weak formulation for the problem DP. To this end we introduce the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ given by

$$a(\xi, \eta) = k \int_\Omega \nabla\xi \cdot \nabla\eta \, dx. \tag{7.10}$$

Using arguments similar to those in Section 3 we obtain the following variational formulation of the viscoelastic problem with friction and damage.

Problem DP_V. Find a displacement field $u : [0,T] \longrightarrow V$, a stress field $\sigma : [0,T] \longrightarrow \mathcal{H}$

and a damage field $\beta : [0,T] \longrightarrow \mathbb{H}^1(\Omega)$ such that for all $t \in [0,T]$

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + G(\varepsilon(u(t)), \beta(t)), \tag{7.11}$$

$$\langle \sigma(t), \varepsilon(w) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(\dot{u}(t), w) - j(\dot{u}(t), \dot{u}(t)) \geq \langle f(t), w - \dot{u}(t)\rangle_V \qquad \forall w \in V, \tag{7.12}$$

$$\beta(t) \in K, \quad \langle \dot{\beta}(t), \xi - \beta(t)\rangle_{L^2(\Omega)} + a(\beta(t), \xi - \beta(t)) \geq \langle \phi(\varepsilon(u(t)), \beta(t)), \xi - \beta(t)\rangle_{L^2(\Omega)} \quad \forall \xi \in K, \text{ a.e. } t \in [0,T], \tag{7.13}$$

$$u(0) = u_0, \quad \beta(0) = \beta_0. \tag{7.14}$$

To study problem DP_V we make the following additional assumptions on the data.

$$G : \Omega \times \mathbb{R}_s^{d\times d} \times \mathbb{R} \longrightarrow \mathbb{R}_s^{d\times d}$$

satisfies

$$\left\{\begin{array}{ll} (a) & \text{there exists } L_G > 0 \text{ such that} \\ & |G(\cdot,\varepsilon_1,\beta_1) - G(\cdot,\varepsilon_2,\beta_2)| \leq L_G\,(|\varepsilon_1 - \varepsilon_2| + |\beta_1 - \beta_2|) \\ & \forall \varepsilon_1, \varepsilon_2 \in \mathbb{R}_s^{d\times d},\ \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. in } \Omega, \\ (b) & x \longmapsto G(x,\varepsilon,\beta) \text{ is Lebesgue measurable on } \Omega\ \forall \varepsilon \in \mathbb{R}_s^{d\times d},\ \beta \in \mathbb{R}, \\ (c) & x \longmapsto G(x,0,0) \in \mathcal{H}. \end{array}\right. \tag{7.15}$$

The *damage source* function

$$\phi : \Omega \times \mathbb{R}_s^{d\times d} \times \mathbb{R} \longrightarrow \mathbb{R}$$

satisfies

$$\left\{\begin{array}{ll} (a) & \text{there exists } L_\phi > 0 \text{ such that} \\ & |\phi(\cdot,\varepsilon_1,\beta_1) - \phi(\cdot,\varepsilon_2,\beta_2)| \leq L_\phi\,(|\varepsilon_1 - \varepsilon_2| + |\beta_1 - \beta_2|) \\ & \forall \varepsilon_1, \varepsilon_2 \in \mathbb{R}_s^{d\times d},\ \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. in } \Omega, \\ (b) & x \longmapsto \phi(x,\varepsilon,\beta) \text{ is Lebesgue measurable on } \Omega\ \ \forall \varepsilon \in \mathbb{R}_s^{d\times d},\ \beta \in \mathbb{R}, \\ (c) & x \longmapsto \phi(x,0,0) \in L^2(\Omega). \end{array}\right. \tag{7.16}$$

$$\beta_0 \in K. \tag{7.17}$$

We have the following existence and uniqueness result for the problem.

Theorem 7.1. *Assume that* (4.1), (4.3)–(4.5) *and* (7.15)–(7.17) *hold. Then there exists* $\alpha > 0$, *which depends only on* Ω, Γ_1 *and* $\mathcal{A}$, *such that if* $L_\nu + L_\tau < \alpha$, *then Problem* DP_V *has a unique solution* $\{u, \sigma, \beta\}$, *and*

$$u \in C^1(0,T;V), \quad \sigma \in C(0,T;\mathcal{H}_1), \quad \beta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

We conclude that when the Lipschitz constants of the contact functions p_ν and p_τ are sufficiently small, Problem DP has a unique weak solution $\{u, \sigma, \beta\}$. Moreover, α does not depend on the damage data. Furthermore, it follows that the problem with an unbounded damage source function of the type ϕ_{Fr} has a local weak solution, as long as $0 < \beta$.

The proof of Theorem 7.1 is similar to that of Theorem 4.1 and is obtained in several steps. Since the modifications are straightforward, we omit most of the details.

Let $\eta \in C(0,T;\mathcal{H})$ and $\theta \in C(0,T;L^2(\Omega))$. In the first step we consider the following uncoupled auxiliary problems.

Problem DP_η. Find a displacement field $u_\eta : [0,T] \longrightarrow V$ and a stress field $\sigma_\eta : [0,T] \longrightarrow \mathcal{H}_1$ such that

$$\sigma_\eta = \mathcal{A}(\varepsilon(\dot{u}_\eta)) + \eta, \tag{7.18}$$

$$\langle \sigma_\eta, \varepsilon(w) - \varepsilon(\dot{u}_\eta)\rangle_{\mathcal{H}} + j(\dot{u}_\eta, w) - j(\dot{u}_\eta, \dot{u}_\eta) \geq \langle f, w - \dot{u}_\eta\rangle_V \quad \forall w \in V \tag{7.19}$$

for all $t \in [0,T]$, and

$$u_\eta(0) = u_0. \tag{7.20}$$

Problem DP_θ. Find a damage field $\beta_\theta : [0,T] \longrightarrow H^1(\Omega)$ such that $\beta_\theta(t) \in K$ for all $t \in [0,T]$ and

$$\langle \dot{\beta}_\theta, \xi - \beta_\theta\rangle_{L^2(\Omega)} + a(\beta_\theta, \xi - \beta_\theta) \geq \langle \theta, \xi - \beta_\theta\rangle_{L^2(\Omega)} \qquad \forall \xi \in K \tag{7.21}$$

a.e. on $[0,T]$, and

$$\beta_\theta(0) = \beta_0. \tag{7.22}$$

We have the following results for these problems.

Proposition 7.2. *There exists $\alpha > 0$, which depends only on Ω, Γ_1 and $\mathcal{A}$, such that if $L_\nu + L_\tau < \alpha$, then Problem DP_η has a unique solution $\{u_\eta, \sigma_\eta\}$, and*

$$u_\eta \in C^1(0,T;V), \qquad \sigma_\eta \in C(0,T;\mathcal{H}_1).$$

Proof. Let $\{u_\eta, \sigma_\eta\}$ be the functions defined by (5.7) and (5.8). Clearly $u_\eta \in C^1(0,T;V)$, $\sigma_\eta \in C(0,T;\mathcal{H}_1)$ and u_η satisfies (7.20). Choosing $g = g_\eta$ in (5.1), (5.2) and keeping in mind (5.3), (5.7), (5.8) we obtain (7.18) and (7.19). The uniqueness follows from (7.18) and (7.19), by using (4.1) and the assumption $L_\nu + L_\tau < \alpha$.

Proposition 7.3. *Problem DP_θ has a unique solution β_θ such that*

$$\beta_\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

Proof. Proposition 7.3 follows from the coercivity of the form a defined by (7.10) and classical results for parabolic variational inequalities, see, e.g., [2](p. 124).

As a consequence of Propositions 7.2, 7.3, (7.15), and (7.16) we may define the operator $\mathcal{L} : C(0,T;\mathcal{H} \times L^2(\Omega)) \longrightarrow C(0,T;\mathcal{H} \times L^2(\Omega))$ by

$$\mathcal{L}(\eta,\theta) = \Big(G(\varepsilon(u_\eta), \beta_\theta),\ \phi(\varepsilon(u_\eta), \beta_\theta)\Big), \tag{7.23}$$

for all $(\eta,\theta) \in C(0,T;\mathcal{H} \times L^2(\Omega))$. We have

Proposition 7.4. *The operator $\mathcal{L}$ has a unique fixed point $(\eta^*, \theta^*) \in C(0,T;\mathcal{H} \times L^2(\Omega))$.*

Proof. Let (η_1,θ_1), $(\eta_2,\theta_2) \in C(0,T;\mathcal{H}\times L^2(\Omega))$ and let $t\in[0,T]$. Using (7.23), (7.15) and (7.16) we deduce

$$|\mathcal{L}(\eta_1,\theta_1)(t)-\mathcal{L}(\eta_2,\theta_2)(t)|_{\mathcal{H}\times L^2(\Omega)} \le C\left(|u_{\eta_1}(t)-u_{\eta_2}(t)|_V + |\beta_{\theta_1}(t)-\beta_{\theta_2}(t)|_{L^2(\Omega)}\right). \tag{7.24}$$

Moreover, it follows from (5.8), (5.12) and (7.18)–(7.20) that

$$|u_{\eta_1}(t)-u_{\eta_2}(t)|_V \le C\int_0^t |\eta_1(s)-\eta_2(s)|_{\mathcal{H}}\,ds. \tag{7.25}$$

On the other hand, we obtain from (7.21)–(7.22) that

$$|\beta_{\theta_1}(t)-\beta_{\theta_2}(t)|_{L^2(\Omega)} \le C\int_0^t |\theta_1(s)-\theta_2(s)|_{L^2(\Omega)}ds. \tag{7.26}$$

Using now (7.24)–(7.26) we find

$$|\mathcal{L}(\eta_1,\theta_1)(t)-\mathcal{L}(\eta_2,\theta_2)(t)|_{\mathcal{H}\times L^2(\Omega)} \le C\int_0^t |(\eta_1,\theta_1)(s)-(\eta_2,\theta_2)(s)|_{\mathcal{H}\times L^2(\Omega)}\,ds. \tag{7.27}$$

Proposition 7.4 follows now from (7.27) and the Banach fixed point theorem.

Proof of Theorem 7.1. Assume that $L_\nu + L_\tau < \alpha$. Let $\{u_{\eta^*},\sigma_{\eta^*}\}$ be the solution of (7.18)–(7.20) for $\eta=\eta^*$ and let β_{θ^*} be the solution of (7.21)–(7.22) for $\theta=\theta^*$. It is straightforward to show that $\{u_{\eta^*},\sigma_{\eta^*},\beta_{\theta^*}\}$ is a solution of problem (7.11)–(7.14) such that $u_{\eta^*}\in C^1(0,T;V)$, $\sigma_{\eta^*}\in C(0,T;\mathcal{H}_1)$ and $\beta_{\theta^*}\in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$. The uniqueness of this solution follows from the uniqueness of the fixed point of the operator $\mathcal{L}$.

We turn now to the dual variational formulation of Problem DP_V which can be obtained as the one in Section 4.

Problem DP_D. Find a displacement field $u:[0,T]\longrightarrow V$, a stress field $\sigma:[0,T]\longrightarrow\mathcal{H}$ and a damage field $\beta:[0,T]\longrightarrow H^1(\Omega)$ such that for all $t\in[0,T]$

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot u(t))) + G(\varepsilon(u(t)),\beta(t)), \tag{7.28}$$

$$\sigma(t)\in\Sigma(t,\dot u(t)), \qquad \langle\tau-\sigma(t),\varepsilon(\dot u(t)\rangle_{\mathcal{H}} \quad \forall\tau\in\Sigma(t,\dot u(t)), \tag{7.29}$$

$$\beta(t)\in K \text{ and for all } \xi\in K, \text{ a.e. } t\in[0,T],$$

$$\langle\dot\beta(t),\xi-\beta(t)\rangle_{L^2(\Omega)} + a(\beta(t),\xi-\beta(t)) \ge \langle\phi(\varepsilon(u(t)),\beta(t)),\xi-\beta(t)\rangle_{L^2(\Omega)}, \tag{7.30}$$

$$u(0)=u_0, \quad \beta(0)=\beta_0. \tag{7.31}$$

We note, again, that problem (7.28)–(7.31) is a quasivariational inequality since Σ depends on the solution. We have have the following equivalence result.

Theorem 7.5. *Assume that* (4.1), (4.3)–(4.5) *and* (7.15)–(7.17) *hold. Then* $\{u,\sigma,\beta\}$ *is the solution of Problem DP_V if and only if* $\{u,\sigma,\beta\}$ *is a solution of Problem DP_D.*

The proof of Theorem 7.5 follows from the same arguments as those used in in Section 6 and it is based on the equivalence of the time dependent inequalities (7.12) and (7.29).

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Existence and Uniqueness for a Quasistatic Frictional Bilateral Contact Problem in Thermoviscoelasticity

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Descriptif

Ce travail porte sur l'étude d'un modèle pour les problèmes quasistatiques de contact bilatéral avec frottement entre un matériau déformable et une fondation rigide en mouvement. La Loi de comportement du matériau est du type viscoélastique de Kelvin-Voigt. Le frottement est modélisé par la version modifiée la loi de Coulomb introduite par N. STRÖMBERG, L. JOHANSSON et A. KLARBRING. On tient compte aussi au niveau des conditions aux limites du dégagement de chaleur due aux frottements.

On considère un milieu continu viscoélastique occupant un domaine Ω de $\mathbb{R}^m$ ($m = 2, 3$), et dont la frontière Γ, supposée suffisamment régulière, est divisée en trois parties disjointes Γ_D, Γ_N et Γ_C. On suppose que, pendant l'intervalle de temps $[0, T]$, des forces volumiques f_A agissent dans Ω, que la partie Γ_D est encastrée dans une structure fixe, que des forces surfaciques f_N s'appliquent sur Γ_N. On suppose en outre que la partie $\Gamma_D \cup \Gamma_N$ est maintenue une température donnée θ_b alors que la fondation rigide, en mouvement tangentiel avec une vitesse $\boldsymbol{\phi}$, est maintenue à une température θ_R. Ce problème mécanique peut se formuler mathématiquement de la façon suivante :

Problème P : Trouver le champ des déplacements $u : \Omega \times [0, T] \longrightarrow \mathbb{R}^m$, le champ des contraintes $\sigma : \Omega \times [0, T] \longrightarrow \mathbb{R}_s^{m\times m}$ et la température $\theta : \Omega \times [0, T] \longrightarrow \mathbb{R}$ tels que

$$\begin{aligned}
\sigma_{ij} = a_{ijkl}u_{k,l} + b_{ijkl}u'_{k,l} - c_{ij}\theta &\quad \text{dans} \quad \Omega \times (0, T),\\
\sigma_{ij,j} + f_{Ai} = 0 &\quad \text{dans} \quad \Omega \times (0, T),\\
\theta' - (k_{ij}\theta_{,j})_{,i} = -c_{ij}u'_{i,j} + q &\quad \text{dans} \quad \Omega \times (0, T),\\
\boldsymbol{u} = 0 &\quad \text{sur} \quad \Gamma_D \times (0, T),\\
\sigma\boldsymbol{n} = \boldsymbol{f}_N &\quad \text{sur} \quad \Gamma_N \times (0, T),\\
\theta = \theta_b &\quad \text{sur} \quad (\Gamma_D \cup \Gamma_N) \times (0, T),\\
u_n = 0 &\quad \text{sur} \quad \Gamma_C \times (0, T),
\end{aligned}$$

$$
\begin{aligned}
|\sigma_\tau| \le \mu |R\sigma_n|\big(1-\delta|R\sigma_n|\big)_+ \qquad & \text{sur} \qquad \Gamma_C \times (0,T),\\
|\sigma_\tau| < \mu |R\sigma_n|\big(1-\delta|R\sigma_n|\big)_+ \qquad & \Longrightarrow \boldsymbol{u}'_\tau = \boldsymbol{\phi},\\
|\sigma_\tau| = \mu |R\sigma_n|\big(1-\delta|R\sigma_n|\big)_+ \qquad & \Longrightarrow \boldsymbol{u}'_\tau = \boldsymbol{\phi} - \lambda\sigma_\tau,\ \lambda \ge 0,\\
k_{ij}\theta_{,i}n_j = \mu |R\sigma_n|\big(1-\delta|R\sigma_n|\big)_+ s_c(\cdot, |\boldsymbol{u}'_\tau - \boldsymbol{\phi}|) \qquad & \\
-k_e(\theta - \theta_R) \qquad & \text{sur} \qquad \Gamma_C \times (0,T),\\
\boldsymbol{u}(\cdot\,,0) = \boldsymbol{u}_0, \qquad \theta(\cdot\,,0) = \theta_0 \qquad & \text{dans} \qquad \Omega.
\end{aligned}
$$

On note par $\mathbb{R}_s^{m\times m}$ l'espace des tenseurs symétriques du second ordre sur $\mathbb{R}^m$. Le "prime" au dessus d'une quantité représente sa dérivée temporelle, $\boldsymbol{n}$ est la normale unitaire sortante à Ω et $\sigma\boldsymbol{n}$ est le vecteur des contraintes de Cauchy. Les fonctions u_n, u'_τ, σ_n et σ_τ représentent respectivement le déplacement normal, la vitesse tangentielle et les contraintes normales et tangentielles. Le réel μ désigne le *coefficient de frottement* et δ est un coefficient positif (assez petit) lié à l'usure et la dureté du matériau.

Le travail commence par une interprétation mécanique des équations et des termes cités dans le problème. L'accent est mis particulièrement sur les conditions aux limites considérées sur la partie de contact potentielle Γ_C. Sous certaines hypothèses, on établit une formulation faible du problème P. Elle se présente comme un système hyperbolique-parabolique d'équations aux dérivées partielles. Le système est formulé ensuite en terme d'opérateurs. Un résultat d'existence est alors établi en utilisant une méthode de régularisation suivie d'estimations *a priori* puis de passages à la limite pour terminer avec une technique de point fixe.

Existence and Uniqueness for a Quasistatic Frictional Bilateral Contact Problem in Thermoviscoelasticity *

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Abstract

We prove the existence and uniqueness of the weak solution for a quasistatic thermoviscoelastic problem which describes bilateral frictional contact between a deformable body and a moving rigid foundation. The model consists of the heat equation for the temperature, the elliptic viscoelasticity system for the displacements, the SJK-Coulomb law of friction and frictional heat generation condition. The proof is accomplished in two steps. First, the existence of solutions for a regularized problem is established and a priori estimates obtained. Then the limit function, which is the weak solution of the original problem, is shown to be the unique fixed point of the solution operator when the friction coefficient is small.

Keywords. Quasistatic frictional contact, viscoelastic, bilateral contact, frictional heat generation, fixed point, SJK-Coulomb law of friction

1. Introduction

This work deals with a model for the quasistatic process of bilateral frictional contact between a viscoelastic body and a rigid moving foundation. The model consists of the heat equation for the temperature and the elliptic viscoelasticity system for the displacements, together with friction and frictional heat generation conditions on the contact surface. We establish the existence and uniqueness of the weak solution for the problem when the coefficient of friction is sufficiently small. The proof is based on the study of a sequence of auxiliary problems, passage to the limit when the regularization parameter vanishes, and an application of a fixed point argument.

The model describes the evolution of the thermomechanical state of a part or component that is in frictional contact with a harder object. The assumption that the acting forces, tractions and possible heat sources change gradually in time allows us to neglect the inertial terms in the equations of motion and use the quasistatic approximation. In particular, we

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neglect any viscoelastic waves in the body, which is the case in a system, such as a pump or a low rpm motor, where slow periodic forces act on the contacting elements. Often in such systems, the contacting elements are partially lubricated and, therefore, the friction coefficient is small.

There exists a large body of research on frictional contact in the engineering literature. Recent results from a number of different points of view can be found in [24] and references therein. General dynamic thermoelastic models, which were derived from thermodynamical principles, can be found in [14, 26]. The mathematical literature on dynamic or quasistatic frictional contact models which include thermal effects is very recent, although in applications frictional contact is very often accompanied by considerable heat generation. Indeed, when car brakes are applied the frictional heat generation can easily exceed 100 HP. Dynamic thermoviscoelastic frictional contact problems can be found in [4, 5, 6, 11, 21]. In [5] the wear of the contacting surface was included and in [6] a model for the grinding process was developed. A one-dimensional thermoviscoelastic problem for a beam was investigated in [16] where the existence of the weak solution was established and numerical simulations of the solutions conducted. A quasistatic thermoviscoelastic problem for a beam can be found in [13, 12] where the wear of the contacting end is included. A frictionless one-dimensional thermoelastic contact problem for a rod was thoroughly analyzed in [3].

Recent existence and uniqueness results for quasistatic contact problems can be found in [2, 8, 19, 22, 23, 25]. However, this paper is the first to investigate the general bilateral quasistatic contact problem in three dimensions in which the thermal effects of friction are taken into account.

The behavior of the bulk material is assumed, for the sake of simplicity, to be linear. The nonlinear effects on which we focus occur on the part of the boundary which is in contact with the rigid foundation. We employ the Kelvin-Voigt viscoelastic law with thermal effects included. We model friction by the SJK-version of Coulomb's law, [26], and assume that the contact is maintained throughout the process, which is the case in many engineering applications. The frictional heat generation enters as a boundary condition for the temperature.

In Section 2 we present the physical setting and formulate the model as a coupled system of parabolic-elliptic partial differential equations, together with initial and boundary conditions. Because of the friction condition there is a regularity ceiling for the solutions and, in general, these problems do not admit classical solutions. Therefore, we introduce the weak formulation in the form of a variational inequality and then rewrite it in an abstract operator form. Then we list the assumptions imposed on the data and state our main existence and uniqueness result in Theorem 2.2. It guarantees that when the coefficient of friction is small the problem has a unique weak solution. Estimating the bound on the friction coefficient remains an important unsolved question, however, partially lubricated surfaces have small friction coefficients. Finally, we present an auxiliary problem in which the contact stress is assumed to be known. The existence of the unique solution of the regularized version of the auxiliary problem is established in Section 3 where we use an abstract existence result for degenerate evolution equations due to [15]. We note in passing that a considerable generalization of this theorem can be found in [17]. Then we obtain the necessary a priori estimates on the solutions. We pass to the regularization limit in Section 4 and, thus, prove the existence of the unique solution to the auxiliary problem. Finally, in Section 5 we use a fixed point argument and establish Theorem 2.2.

There remain many of open questions, in addition to estimating the size of the friction coefficient. Does the solution converge to the steady solution when the forces, tractions and heat sources converge to time independent quantities? If so, what is the rate of convergence? Moreover, the wear of the contacting surfaces needs to be taken into account. Finally, the auxiliary problem and the fixed point argument may be a basis for a convergent numerical algorithm, but will it be sufficiently effective in practice?

2. The model, weak formulation and results

The physical setting consists of a viscoelastic body which, over a part of its surface, is in frictional contact with a rigid moving foundation. The body (in its reference configuration) is represented by Ω, a region in $\mathbb{R}^m$ ($m = 2, 3$), whose boundary $\partial\Omega = \Gamma$, which is assumed to be Lipschitz continuous, is divided into three disjoint parts. On the first part, denoted by Γ_D, the body is clamped; known tractions act on the second part Γ_N; the body is in frictional contact with a rigid obstacle on the third part, Γ_C. The reference configuration is assumed to be stress free and isothermal; its temperature conveniently set as zero, which also serves as the reference temperature. The rigid foundation moves with tangential velocity $\boldsymbol{\phi}$ and this motion is accompanied by frictional heat generation–which is considerable in many applications–on the part of the contact surface where relative slip takes place.

We assume that contact is maintained between the body and the foundation. This is the case in many engineering systems where there is no loss of contact, such as between the piston rings and the engine block in a car. An example from everyday life is the frictional contact of the wheels with the rail when a train is braking. From the mathematical point of view the approach below does not allow us to consider the case when separation occurs between the body and the foundation. We assume that the applied forces, tractions and the heat source vary slowly with time and consequently neglect the accelerations in the system. So we employ the quasistatic approximation for the process. The unknowns in the model are the displacements vector $\boldsymbol{u} = (u_1(x,t), \ldots, u_m(x,t))$, the temperature $\theta = \theta(x,t)$ and the stress tensor $\sigma = (\sigma_{ij}(x,t)$, at location x and time t.

The model for the process of *quasistatic thermoviscoelastic bilateral contact with friction* is as follows:

Find $\{\boldsymbol{u}, \sigma, \theta\}$ such that

$$\sigma_{ij} = a_{ijkl}u_{k,l} + b_{ijkl}u'_{k,l} - c_{ij}\theta \quad \text{in} \quad \Omega_T, \tag{2.1}$$

$$\sigma_{ij,j} + f_{Ai} = 0 \quad \text{in} \quad \Omega_T, \tag{2.2}$$

$$\theta' - (k_{ij}\theta_{,j})_{,i} = -c_{ij}u'_{i,j} + q \quad \text{in} \quad \Omega_T, \tag{2.3}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_D \times (0,T), \tag{2.4}$$

$$\sigma\boldsymbol{n} = \boldsymbol{f}_N \quad \text{on} \quad \Gamma_N \times (0,T), \tag{2.5}$$

$$\theta = \theta_b \quad \text{on} \quad (\Gamma_D \cup \Gamma_N) \times (0,T), \tag{2.6}$$

$$u_n = 0 \quad \text{on} \quad \Gamma_C \times (0,T), \tag{2.7}$$

$$\begin{gathered}|\sigma_\tau| \leq \mu|R\sigma_n|\big(1-\delta|R\sigma_n|\big)_+ \quad \text{on} \quad \Gamma_C \times (0,T), \\ |\sigma_\tau| < \mu|R\sigma_n|\big(1-\delta|R\sigma_n|\big)_+ \Longrightarrow \boldsymbol{u}'_\tau = \boldsymbol{\phi}, \\ |\sigma_\tau| = \mu|R\sigma_n|\big(1-\delta|R\sigma_n|\big)_+ \Longrightarrow \boldsymbol{u}'_\tau = \boldsymbol{\phi} - \lambda\sigma_\tau, \ \lambda \geq 0,\end{gathered} \tag{2.8}$$

$$\begin{aligned}k_{ij}\theta_{,i}n_j = \mu|R\sigma_n|\big(1-\delta|R\sigma_n|\big)_+ s_c(\cdot, |\boldsymbol{u}'_\tau - \boldsymbol{\phi}|) \\ -k_e(\theta - \theta_R) \quad \text{on} \quad \Gamma_C \times (0,T),\end{aligned} \tag{2.9}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \qquad \theta(\cdot,0) = \theta_0 \quad \text{in} \quad \Omega. \tag{2.10}$$

Here, $\Omega_T = \Omega \times (0,T)$, $i,j = 1,\ldots,m$, the repeated index convention is employed, the prime represents time derivative, the portion of a subscript prior to a comma indicates a component and the portion after the comma refers to partial derivatives. All the variables are scaled and form dimensionless quantities.

The thermoviscoelastic constitutive relation is given in (2.1) where $a = (a_{ijkl})$ and $b = (b_{ijkl})$ are the tensors of elastic and viscosity coefficients, respectively. Since we consider the quasistatic approximation for the process, (2.2) represents the equilibrium equations, and $\boldsymbol{f}_A = (f_{A1}(x,t),\ldots,f_{Am}(x,t))$ denotes the density of applied body forces acting in Ω. For the sake of simplicity, the material density is assumed constant, set equal to one. (2.3) is the energy equation where c_{ij} and k_{ij} are the components of the thermal expansion and thermal conductivity tensors, respectively, and q is the density of volume heat sources.

To describe the boundary conditions we introduced the unit outward normal $\boldsymbol{n} = (n_1,\ldots,n_m)$ to Γ, and since Γ is assumed Lipschitz, $\boldsymbol{n}$ exists at almost every point there. We denote by $\sigma_n = \sigma_{ij}n_in_j$ and $u_n = u_in_i$ the normal components of σ and $\boldsymbol{u}$ on Γ, respectively, and let

$$\sigma_\tau = \sigma\boldsymbol{n} - \sigma_n\boldsymbol{n}, \qquad \boldsymbol{u}_\tau = \boldsymbol{u} - u_n\boldsymbol{n}$$

be the tangential parts (see, e.g., [18] or [10]).

In (2.5) and (2.6), $\boldsymbol{f}_N = (f_{N1}(x,t),\ldots,f_{Nm}(x,t))$ denotes the tractions applied on Γ_N, and θ_b is the known and scaled temperature of the part of the boundary $\Gamma_D \cup \Gamma_N$. Condition (2.7) means that there is no loss of contact.

We turn to the conditions on the contact surface. In (2.8) we employ the SJK-generalization of Coulomb's law of dry friction ([26]). μ is the friction coefficient, δ is a small positive coefficient related to the wear and hardness of the surface, $\boldsymbol{u}'_\tau$ is the tangential velocity of the body, $(\,\cdot\,)_+ = \max(\cdot\,, 0)$ and R represents a *normal regularization operator*, a linear and continuous operator $R : H^{-\frac{1}{2}}(\Gamma) \longrightarrow L^2(\Gamma)$ (see, e.g., [9]). We use it in (2.8) and (2.9) to regularize the trace of the stress tensor on Γ. Condition (2.8) may be interpreted physically in the following way. The boundary sticks to the foundation and moves with it when the applied tangential stress is less than the limiting value. The part of Γ_C where this takes place is called the *stick* zone. The part where the tangential stress reaches its limiting value and does not move in tandem with the foundation is called the *slip* zone. λ is an unknown multiplier which indicates the relative direction of the slip.

The modification to Coulomb's law of friction, which was derived in [26] from thermodynamical considerations, consists of the factor $(1 - \delta|\cdot|)_+$. It represents the fact that when the contact stress is large there is a decrease in frictional resistance because of surface wear. This modified condition agrees with Coulomb's law for $|\sigma_n|$ not too large since δ is very small in applications. On the other hand, when the contact tractions are very large it is very likely that the surface will be damaged, and Coulomb's condition, actually the whole model needs to be redone. Thus, from the point of view of applications the modified condition seems more natural. From the mathematical point of view the fact that the tangential stress is bounded is essential to obtaining the necessary estimates below.

Next, the introduction of the regularization operator R is necessary since the contact stress is not sufficiently regular to make the condition of friction meaningful. Indeed, σ on Γ_C lies in $H^{-\frac{1}{2}}(\Gamma)$, which means that it is only a distribution, too irregular to use below. An explanation of the mathematical difficulties and a justification for using R can be found in [9]. The operator R may be chosen to have a small support, in which case it will average the contact stress over the surface asperities in a small region on Γ_C. We do not know how it affects the friction bound since we can not obtain comparable results without it.

We now describe the temperature boundary condition (2.9) on Γ_C. The power generated by the frictional contact forces is proportional to $|\sigma_\tau|$ and $|\boldsymbol{u}'_\tau - \boldsymbol{\phi}|$, so, for the sake of generality, we used the function $s_c(\,\cdot\,, r)$, which is prescribed and generalizes $|\cdot|$, and it is given by $\mu|R\sigma_n|\big(1 - \delta|R\sigma_n|\big)_+ s_c(\cdot, |\boldsymbol{u}'_\tau - \boldsymbol{\phi}|)$. θ_R is the foundation's temperature and k_e is the coefficient of heat exchange between it and the body. Finally, (2.10) represent the initial conditions.

It is well known that, in general, there are no classical solutions to the problem because of the regularity ceiling related to the possible stick-slip motion. Therefore, we consider a weak or variational formulation for the problem. To this end we introduce the following classical Hilbert spaces:

$$
\begin{gathered}
H = L^2(\Omega), \qquad H^m = L^2(\Omega)^m, \\
E = \Big\{ w \in H^1(\Omega)^m : w = 0 \text{ on } \Gamma_D \Big\}, \\
V = \Big\{ \eta \in H^1(\Omega) : \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N \Big\}, \\
W = \Big\{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ \tau_{ij,j} \in L^2(\Omega) \Big\}, \\
\mathbb{H} = L^2(0,T;H), \quad \mathbb{E} = L^2(0,T;E), \quad \mathbb{V} = L^2(0,T;V), \quad \mathbb{W} = L^2(0,T;W).
\end{gathered}
$$

Below, we use $\|\cdot\|_E$, $\|\cdot\|_V$ and $\|\cdot\|_W$ to denote the norms of E, V and W, respectively. Similarly, we use $|\cdot|_H$ and $|\cdot|_{H^m}$ to denote the norms of H and H^m, respectively, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E' and E or V' and V, as the meaning is evident from the context. For standard notation we refer the reader to [1, 20, 18].

We now describe the assumptions on the data. The coefficients of elasticity, viscosity, thermal expansion and thermal conductivity satisfy:

$$
\begin{array}{l}
a_{ijkl} \in L^\infty(\Omega), \quad b_{ijkl} \in L^\infty(\Omega), \quad c_{ij} \in L^\infty(\Omega), \quad k_{ij} \in L^\infty(\Omega); \\
a_{ijkl} = a_{jikl}, \qquad a_{ijkl} = a_{klij}, \qquad a_{ijkl} = a_{ijlk}, \\
a_{ijkl}\chi_{kl}\chi_{ij} \geq \alpha_1 \chi_{ij}\chi_{ij} \quad \text{for all symmetric tensors } \chi = (\chi_{ij}); \\
b_{ijkl} = b_{jikl}, \qquad b_{ijkl} = b_{klij}, \qquad b_{ijkl} = b_{ijlk}, \\
b_{ijkl}\chi_{kl}\chi_{ij} \geq \alpha_2 \chi_{ij}\chi_{ij} \quad \text{for all symmetric tensors } \chi = (\chi_{ij}); \\
c_{ij} = c_{ji}; \\
k_{ij} = k_{ji}, \qquad k_{ij} z_j z_i \geq \alpha_3 z_i z_i \quad \text{for all vectors } z = (z_i).
\end{array}
\tag{2.11}
$$

Here, α_1, α_2 and α_3 are positive constants.
The body forces and the volume heat sources satisfy

$$
\boldsymbol{f}_A \in L^2(0,T;H^m), \qquad q \in \mathbb{V}'. \tag{2.12}
$$

The friction coefficient and the velocity of the foundation satisfy

$$
\begin{array}{l}
\mu \in L^\infty(\Gamma_C), \quad \mu \geq 0, \ \text{ a.e. on } \ \Gamma_C, \\
\boldsymbol{\phi} = \boldsymbol{\phi}(t) \in C([0,T]; \mathbb{R}^{m-1}).
\end{array}
\tag{2.13}
$$

The function s_c satisfies

$$
\begin{array}{l}
s_c : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}_+ \ \text{ is Borel measurable}, \\
s_c(\cdot, r) \leq \alpha_4 \quad \text{for all } \ r \in \mathbb{R}, \ \text{ a.e. on } \ \Gamma_C, \\
|s_c(\cdot, r_1) - s_c(\cdot, r_2)| \leq \alpha_5 |r_1 - r_2| \quad \text{for all } \ r_1, r_2 \in \mathbb{R}, \quad \text{a.e. on } \quad \Gamma_C,
\end{array}
\tag{2.14}
$$

where α_4 and α_5 are positive constants.

The assumptions on the boundary and initial data are

$$
\begin{array}{l}
\boldsymbol{f}_N \in L^2(0,T; L^2(\Gamma_N)^m); \\
\text{there exists } \ \Theta \in H^1(0,T; H^1(\Omega)) \text{ such that } \Theta = \theta_b \text{ on } \Gamma_D \cup \Gamma_N; \\
\theta_R \in L^2(0,T; L^2(\Gamma_C)); \\
\boldsymbol{u}_0 \in E, \qquad \theta_0 \in H.
\end{array}
\tag{2.15}
$$

For technical reasons, it is convenient to shift the temperature function so that it is zero on $\Gamma_D \cup \Gamma_N$. To that end, we introduce $\xi = \theta - \Theta$ and $\xi_0 = \theta_0 - \Theta(\cdot\,, 0)$. To simplify the notation, we will not indicate explicitly the dependence on t.

We can now present the weak formulation of the problem.

Definition 2.1. A triplet $\{\boldsymbol{u}, \sigma, \theta\}$ is said to be a weak solution to (2.1)–(2.10) provided it satisfies

$$\boldsymbol{u} \in \mathbb{E}, \quad \boldsymbol{u}' \in \mathbb{E}, \quad \boldsymbol{u}(\cdot\,, 0) = \boldsymbol{u}_0, \tag{2.16}$$

$$\xi \in \mathbb{V}, \quad \xi' \in \mathbb{V}', \quad \boldsymbol{\xi}(\cdot\,, 0) = \boldsymbol{\xi}_0, \tag{2.17}$$

$$\sigma \in \mathbb{W}, \quad \sigma_{ij} = a_{ijkl}u_{k,l} + b_{ijkl}u'_{k,l} - c_{ij}(\xi + \Theta), \tag{2.18}$$

and for all $w \in \mathbb{E}$

$$\begin{aligned}
&\int_\Omega a_{ijkl}u_{k,l}(w_{i,j} - u'_{i,j})dx + \int_\Omega b_{ijkl}u'_{k,l}(w_{i,j} - u'_{i,j})dx - \int_\Omega c_{ij}\xi(w_{i,j} - u'_{i,j})dx \\
&+ \int_{\Gamma_C} \mu|R\sigma_n|(1 - \delta|R\sigma_n|)_+ (|w_\tau - \phi| - |u'_\tau - \phi|)d\Gamma \\
&\quad \geq \int_\Omega f_{Ai}(w_i - u'_i)dx + \int_\Omega c_{ij}\Theta(w_{i,j} - u'_{i,j})dx + \int_{\Gamma_N} f_{Ni}(w_i - u'_i)d\Gamma,
\end{aligned} \tag{2.19}$$

and, for all $\eta \in \mathbb{V}$

$$\begin{aligned}
&\langle \xi', \eta \rangle + \int_\Omega k_{ij}\xi_{,i}\eta_{,j}dx + \int_\Omega c_{ij}u'_{i,j}\eta dx + \int_{\Gamma_C} k_e \xi \eta dx \\
&- \int_{\Gamma_C} \mu|R\sigma_n|(1 - \delta|R\sigma_n|)_+ s_c(x, |u'_\tau - \phi|)\eta d\Gamma \\
&\quad = \langle q, \eta \rangle - \int_\Omega \Theta' \eta \, dx - \int_{\Gamma_C} k_e(\Theta - \theta_R)\eta \, d\Gamma - \int_\Omega k_{ij}\Theta_{,i}\eta_{,j}dx.
\end{aligned} \tag{2.20}$$

We write the weak formulation in an abstract form, and to that end we define the following operators:

$$\begin{aligned}
A,\ B : E &\longrightarrow E', \\
C_1 : E &\longrightarrow V', \\
C_2 : V &\longrightarrow E', \\
K_1,\ K_2 : V &\longrightarrow V', \\
S : W \times E &\longrightarrow V',
\end{aligned}$$

by

$$\langle Au, w\rangle = \int_\Omega a_{ijkl}u_{k,l}w_{i,j}dx, \tag{2.21}$$

$$\langle Bv, w\rangle = \int_\Omega b_{ijkl}v_{k,l}w_{i,j}dx, \tag{2.22}$$

$$\langle C_1 v, \eta\rangle = \int_\Omega c_{ij}v_{i,j}\eta dx, \tag{2.23}$$

$$\langle C_2 \xi, w\rangle = -\int_\Omega c_{ij}\xi w_{i,j}dx, \tag{2.24}$$

$$\langle K_1 \xi, \eta\rangle = \int_{\Gamma_C} k_e \xi\eta d\Gamma, \tag{2.25}$$

$$\langle K_2 \xi, \eta\rangle = \int_\Omega k_{ij}\xi_{,i}\eta_{,j}dx, \tag{2.26}$$

$$\langle S(\sigma, v), \eta\rangle = -\int_{\Gamma_C} \mu|R\sigma_n|\big(1-\delta|R\sigma_n|\big)_+ s_c(x, |v_\tau - \phi|)\eta d\Gamma. \tag{2.27}$$

We note that each of these operators extends, in a natural way, to an operator defined on the corresponding space of square-integrable vector-valued functions on $(0,T)$. For example, A extends to an operator from $\mathbb{E}$ to $\mathbb{E}'$ by setting $(Au)(t) = A(u(t))$. With a slight abuse of notation, we will use below the same symbol to denote both the original operator and its extension, since the meaning will be clear from the context. We can now formulate Problem (2.16)–(2.20) abstractly as follows.

Problem P: Find $\{\boldsymbol{u}, \sigma, \xi\}$ satisfying (2.16)–(2.18) and such that

$$\xi' + K_1\xi + K_2\xi + C_1\boldsymbol{u}' + S(\sigma, \boldsymbol{u}') = Q \quad \text{in} \quad \mathbf{V}', \tag{2.28}$$

$$B\boldsymbol{u}' + A\boldsymbol{u} + C_2\xi + \partial_2 j(\sigma, \boldsymbol{u}') \ni f \quad \text{in} \quad \mathbb{E}'. \tag{2.29}$$

Here $f \in \mathbb{E}'$ and $Q \in \mathbf{V}'$ are given by

$$\langle f, w\rangle = \int_0^T\int_\Omega f_{Ai}w_i dxdt + \int_0^T\int_\Omega c_{ij}\Theta w_{i,j}dxdt + \int_0^T\int_{\Gamma_N} f_{Ni}w_i d\Gamma dt,$$

$$\langle Q, \eta\rangle = \int_0^T \langle q, \eta\rangle\, dt - \int_0^T\int_\Omega \Theta'\eta\, dxdt - \int_0^T\int_{\Gamma_C} k_e(\Theta - \theta_R)\eta d\Gamma dt - \int_0^T\int_\Omega k_{ij}\Theta_{,i}\eta_{,j}\, dxdt,$$

respectively, and $\partial_2 j(\sigma, v)$ denotes the partial subdifferential with respect to v of

$$j(\sigma, v) = \int_0^T\int_{\Gamma_C} \mu|R\sigma_n|\big(1-\delta|R\sigma_n|\big)_+|v_\tau - \phi| d\Gamma dt.$$

Our main existence and uniqueness result is the following.

Theorem 2.2. *Assume that* (2.11)—(2.15) *hold. Then Problem P has a unique solution when* $|\mu|_{L^\infty(\Gamma_C)}$ *is sufficiently small.*

We conclude that problem (2.1)–(2.10) has a unique weak solution when $|\mu|_{L^\infty(\Gamma_C)}$ is sufficiently small. Estimating the allowed size of the friction coefficient is an open and very interesting problem.

The proof of Theorem 2.2 is accomplished in several steps. The first consists of studying a problem of the form (2.16)–(2.18), (2.28) and (2.29) when the stress σ on the contact boundary is assumed to be known. The existence of a solution for this intermediate problem is obtained as a limit of solutions for a sequence of approximate problems that are presented in the next Section.

In the last step of the proof of the theorem we use a fixed point argument and obtain a solution of *Problem P*.

In the first step,, for each $g \in \mathbb{W}$ we consider the following abstract problem:

Problem P_g : Find $\{\boldsymbol{u}_g, \sigma_g, \xi_g\}$ such that

$$\boldsymbol{u}_g \in \mathbb{E}, \quad \boldsymbol{u}_g' \in \mathbb{E}, \quad \boldsymbol{u}_g(\cdot\,,0) = \boldsymbol{u}_0, \tag{2.30}$$

$$\xi_g \in \mathbb{V}, \quad \xi_g' \in \mathbb{V}', \quad \xi_g(\cdot\,,0) = \xi_0, \tag{2.31}$$

$$\sigma_g \in \mathbb{W}, \quad \{\sigma_g\}_{ij} = a_{ijkl}\{\boldsymbol{u}_g\}_{k,l} + b_{ijkl}\{\boldsymbol{u}_g'\}_{k,l} - c_{ij}(\xi_g - \Theta), \tag{2.32}$$

$$\xi_g' + K_1\xi_g + K_2\xi_g + C_1\boldsymbol{u}_g' + S(g, \boldsymbol{u}_g') = Q \quad \text{in } \mathbb{V}', \tag{2.33}$$

$$B\boldsymbol{u}_g' + A\boldsymbol{u}_g + C_2\xi_g + \partial_2 j(g, \boldsymbol{u}_g') \ni f \quad \text{in } \mathbb{E}'. \tag{2.34}$$

We have the following result.

Theorem 2.3. *Assume that* (2.11)–(2.15) *hold. Then Problem* P_g *has a unique solution.*

The solution of Problem P_g will be obtained in the next Section.

3. Approximate problems

In this section we consider a sequence of regularized approximations to Problem P_g. The solutions of these problems are obtained by using a version of an abstract existence theorem for degenerate first order evolution equations obtained by Kuttler in [15] (see also [17] for a generalization of the theorem) which we now recall.

Let F and G be reflexive Banach spaces such that $F \subseteq G$, $\|\cdot\|_F \geq \|\cdot\|_G$ and F is dense in G; thus, we may write $F \subseteq G \equiv G' \subseteq F'$. Suppose that $\mathcal{B}$ is a linear, bounded, positive and symmetric operator from G to G'. Let $\mathbb{F} = L^2(a,b;F)$, $\mathbb{G} = L^2(a,b;G)$ and define $\mathbb{X} = \{w \in \mathbb{F} : (\mathcal{B}w)' \in \mathbb{F}'\}$ with the graph norm $\|w\|_{\mathbb{X}} = \|w\|_{\mathbb{F}} + \|(\mathcal{B}w)'\|_{\mathbb{F}'}$. Here, the differentiation is taken in the sense of F' valued distributions. It is easy to see that $\mathbb{X}$

is a reflexive Banach space. Let $\mathcal{A}(t,\cdot)$ be an operator from F to F'. We also denote by $\mathcal{A}$ its natural extension from $\mathbb{F}$ to $\mathbb{F}'$ given by $\mathcal{A}w(t) = \mathcal{A}(t, w(t))$. Assume that

$$\mathcal{A} : \mathbb{X} \longrightarrow \mathbb{X}' \text{ is pseudomonotone,} \tag{3.1}$$

(see, e.g., [7]),

$$\mathcal{A} : \mathbb{F} \longrightarrow \mathbb{F}' \text{ is bounded,} \tag{3.2}$$

and, for some $\lambda \in \mathbb{R}$,

$$\lim_{\|w\|_{\mathbb{F}} \to \infty} \frac{\lambda\langle \mathcal{B}w, w\rangle_{\mathbb{G}'\times\mathbb{G}} + \langle \mathcal{A}w, w\rangle_{\mathbb{F}'\times\mathbb{F}}}{\|w\|_{\mathbb{F}}} = \infty, \tag{3.3}$$

then, the following existence theorem is a special case of the result in [15].

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be as described above. Then, for each $w_0 \in G$ and $l \in \mathbb{F}'$ there exists a $w \in \mathbb{X}$ satisfying*

$$(\mathcal{B}w)' + \mathcal{A}w = l \quad \textit{in } \mathbb{F}',$$

$$\mathcal{B}w(0) = \mathcal{B}w_0 \quad \textit{in } G'.$$

We turn to describe the sequence of regularized problems whose solutions will be given by Theorem 3.1. To replace the inclusion in (2.29) with an equality we regularize the norm function on $\mathbb{R}^m$. Let $(\psi^h)_{h>0}$ be a family of smooth approximations to $|\cdot|$ such that, for each $h > 0$, $\psi^h \in C^1(\mathbb{R}^m)$ is positive, convex and

$$|\nabla\psi^h(s)| \le 2, \quad 0 \le \langle \nabla\psi^h(s), s\rangle \quad \text{and} \quad |\psi^h(s) - |s|| \le h,$$

for all $s \in \mathbb{R}^m$. We define the operator $J^h : E \longrightarrow E'$ by

$$\langle J^h v, w\rangle = \int_{\Gamma_C} \mu|Rg_n|\big(1 - \delta|Rg_n|\big)_+ \nabla\psi^h(v_\tau - \phi)\cdot w_\tau d\Gamma. \tag{3.4}$$

We let $R_e : E \longrightarrow E'$ denote the Riesz map, and for the sake of simplicity, we will omit the subscript g below.

We now consider, for each $h > 0$, the following regularized problem.

Problem P_h : Find a triplet $\{u_h, v_h, \xi_h\}$ satisfying

$$\xi_h \in \mathbb{V}, \quad \xi_h' \in \mathbb{V}', \quad u_h \in \mathbb{E}, \quad v_h \in \mathbb{E}, \tag{3.5}$$

$$\xi_h' + K_1\xi_h + K_2\xi_h + C_1 v_h + S(g, v_h) = Q \quad \text{in } \mathbb{V}', \tag{3.6}$$

$$Bv_h + Au_h + C_2\xi_h + J^h v_h = f \quad \text{in } \mathbb{E}', \tag{3.7}$$

$$(R_e u_h)' - R_e v_h = 0 \quad \text{in } \mathbb{E}', \tag{3.8}$$

with the initial conditions

$$\boldsymbol{u}_h(0) = \boldsymbol{u}_0, \quad \xi_h(0) = \xi_0. \tag{3.9}$$

We have the following existence result for this problem.

Theorem 3.2. *Let $h > 0$. Then there exists a solution for Problem P_h.*

Proof. We fit Problem P_h into the framework of Theorem 3.1 by choosing $F = V \times E \times E$ and $G = H \times H^m \times E$, $\boldsymbol{w} \in G$, $\boldsymbol{w}_0 \in G$ and $l \in \mathbb{F}'$ given by

$$\boldsymbol{w} = \begin{pmatrix} \xi \\ \boldsymbol{v} \\ \boldsymbol{u} \end{pmatrix}, \qquad \boldsymbol{w}_0 = \begin{pmatrix} \xi_0 \\ \boldsymbol{v}_0 \\ \boldsymbol{u}_0 \end{pmatrix}, \qquad l = \begin{pmatrix} Q \\ f \\ 0 \end{pmatrix}.$$

The operators $\mathcal{B} : G \longrightarrow G'$ and $\mathcal{A}(t, \cdot) : F \longrightarrow F'$ are chosen as

$$\mathcal{B}\boldsymbol{w} = \begin{pmatrix} \xi \\ 0 \\ R_e \boldsymbol{u} \end{pmatrix}, \qquad \mathcal{A}(t, \boldsymbol{w}) = \begin{pmatrix} K_1 \xi + K_2 \xi + C_1 \boldsymbol{v} + S(g, \boldsymbol{v}) \\ B\boldsymbol{v} + A\boldsymbol{u} + C_2 \xi + J^h \boldsymbol{v} \\ -R_e \boldsymbol{v} \end{pmatrix}.$$

Now, the verification of the assumptions of Theorem 3.1 is routine, so we establish only the pseudomonotonicity (3.1) and the coercivity (3.3) conditions. In checking the pseudomonotonicity of the the operator $\mathcal{A}$, we use the fact that we may write $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1(t, \boldsymbol{w}) = \begin{pmatrix} K_1 \xi + K_2 \xi + C_1 \boldsymbol{v} \\ B\boldsymbol{v} + A\boldsymbol{u} + C_2 \xi \\ -R_e \boldsymbol{v} \end{pmatrix}$$

gives rise to a bounded linear operator from $\mathbb{X}$ to $\mathbb{X}'$ (hence weakly continuous and thus pseudomonotone [7]), and

$$\mathcal{A}_2(t, \boldsymbol{w}) = \begin{pmatrix} S(g, \boldsymbol{v}) \\ J^h \boldsymbol{v} \\ 0 \end{pmatrix}$$

is a weak to norm continuous operator from $\mathbb{X}$ to $\mathbb{X}'$. To establish the latter property we recall that $\mathbb{X} = \{w \in \mathbb{F} : (\mathcal{B}w)' \in \mathbb{F}'\}$, and employ the following lemma.

Lemma 3.3. *Let $\mathbb{Y} = \{u \in \mathbb{E} : (R_e u)' \in \mathbb{E}'\}$ with the graph norm $\|u\|_{\mathbb{Y}} = \|u\|_{\mathbb{E}} + \|(R_e u)'\|_{\mathbb{E}'}$, and let $S(g, \cdot) : \mathbb{Y} \longrightarrow \mathbb{V}'$ and $J^h : \mathbb{Y} \longrightarrow \mathbb{Y}'$ be given by (2.27) and (3.4), respectively. Then $S(g, \cdot)$ and J^h are weak to norm continuous.*

Proof. We establish the result only for the operator $S(g, \cdot)$ since the proof for J^h is similar. It is enough to show that every subsequence of a weakly convergent sequence $\{v_k\}$, which converges to v in $\mathbb{Y}$, has a further subsequence whose images under $S(g, \cdot)$ converge to $S(g, v)$. So let $\{v_{k_j}\}$ be a subsequence of $\{v_k\}$. Since $\{v_{k_j}\}$ is bounded in $\mathbb{Y}$ and since the injection $H^1(\Omega)^m \longrightarrow H^{1-\epsilon}(\Omega)^m$ is compact for any $\varepsilon > 0$ ([7]), we have by a theorem of Lions [20], that $\{v_{k_j}\}$ is relatively compact in $L^2(0,T;H^{1-\epsilon}(\Omega)^m)$. By the continuity of the trace map $L^2(0,T;H^{1-\epsilon}(\Omega)^m) \longrightarrow L^2(0,T;L^2(\Gamma)^m) = \big(L^2(\Gamma \times (0,T))\big)^m$ we may, upon passing to a subsequence denoted also by $\{v_{k_j}\}$, suppose that $v_{k_j}(x,t) \longrightarrow v(x,t)$ for almost all $(x,t) \in \Gamma_C \times (0,T)$. Since $s_c(\cdot\,,r) \leq \alpha_4$ and $|Rg_n|\big(1-\delta|Rg_n|\big)_+ \leq \frac{1}{\delta}$ we obtain that $\big\{|Rg_n|\big(1-\delta|Rg_n|\big)_+ s_c(\cdot\,,|(v_{k_j})_\tau - \phi|)\big\}$ is a bounded sequence in $L^2(\Gamma_C \times (0,T))$. Consequently, it converges to $|Rg_n|\big(1-\delta|Rg_n|\big)_+ s_c\big(\cdot\,,|v_\tau - \phi|\big)$ in $L^2(\Gamma_C \times (0,T))$. Finally, using Cauchy's inequality, we deduce that $S(g,v_{k_j}) \longrightarrow S(g,v)$ in $\mathbb{Y}'$ when $j \to \infty$. This completes the proof.

In checking the coercivity condition (3.3), we estimate various terms using Cauchy's inequality, standard trace theorems and the following result. Here and below c represents a positive constant whose value may change from line to line, but in all cases depends only on the data and coefficients of the approximate problem.

Lemma 3.4. *Assume that* (2.11)-(2.15) *hold. Then there exists a positive constant* α *such that*

$$\langle Au,u\rangle \geq \alpha\|u\|_E^2, \qquad \langle Bv,v\rangle \geq \alpha\|v\|_E^2, \qquad \langle K_2\xi,\xi\rangle \geq \alpha\|\xi\|_V^2, \tag{3.10}$$

and also

$$\langle J^h v, v\rangle \geq -c. \tag{3.11}$$

Proof. The first three inequalities are consequences of assumptions (2.11) and Korn's inequality. The last estimate is a result of the following decomposition

$$\begin{aligned}\langle J^h v, v\rangle = &\int_{\Gamma_C} \mu|Rg_n|\big(1-\delta|Rg_n|\big)_+ \nabla\psi^h(v_\tau - \phi)\cdot(v_\tau-\phi)d\Gamma \\ &+ \int_{\Gamma_C} \mu|Rg_n|\big(1-\delta|Rg_n|\big)_+ \nabla\psi^h(v_\tau-\phi)\cdot\phi d\Gamma,\end{aligned}$$

in which the first integral is nonnegative and the second one is bounded. This concludes the proof of the lemma and of Theorem 3.2.

The next step in the proof of Theorem 2.3 deals with an a priori estimate on the solutions of Problem P_h.

Theorem 3.5. *Let* $\{\boldsymbol{u}_h, \boldsymbol{v}_h, \xi_h\}$ *be a solution of Problem* P_h *corresponding to the parameter* h. *Then there exists a positive constant* c, *independent of* h, *such that, for all* $t \in [0,T]$,

$$\|\boldsymbol{u}_h(t)\|_E^2 + \int_0^t \|\boldsymbol{v}_h(s)\|_E^2\,ds + |\xi_h(t)|_H^2 + \int_0^t \|\xi_h(s)\|_V^2\,ds \leq c. \tag{3.12}$$

Proof. For the sake of simplicity, we will omit the subscript h below. We begin by letting (3.7) act on $\boldsymbol{v}$. Thus,

$$\int_0^t \langle B\boldsymbol{v},\boldsymbol{v}\rangle ds + \int_0^t \langle A\boldsymbol{u},\boldsymbol{v}\rangle ds + \int_0^t \langle J^h\boldsymbol{v},\boldsymbol{v}\rangle ds = -\int_0^t \langle C_2\xi,\boldsymbol{v}\rangle ds + \int_0^t \langle f,\boldsymbol{v}\rangle ds. \tag{3.13}$$

We will now estimate each of the terms in (3.13). Applying Lemma 3.4 yields

$$\int_0^t \langle B\boldsymbol{v},\boldsymbol{v}\rangle ds \geq \alpha \int_0^t \|\boldsymbol{v}(s)\|_E^2 ds, \tag{3.14}$$

$$\begin{aligned}\int_0^t \langle A\boldsymbol{u},\boldsymbol{v}\rangle ds &= \frac{1}{2}\int_0^t \left(\tfrac{d}{ds}\langle A\boldsymbol{u},\boldsymbol{u}\rangle\right) ds = \frac{1}{2}\langle A\boldsymbol{u}(t),\boldsymbol{u}(t)\rangle - \frac{1}{2}\langle A\boldsymbol{u}_0,\boldsymbol{u}_0\rangle \\ &\geq \frac{\alpha}{2}\|\boldsymbol{u}(t)\|_E^2 - c\|\boldsymbol{u}_0\|_E^2,\end{aligned} \tag{3.15}$$

and

$$\int_0^t \langle J^h\boldsymbol{v},\boldsymbol{v}\rangle ds \geq -c. \tag{3.16}$$

The integrals on the right in (3.13) are estimated by using Cauchy's inequality with ε,

$$\left|\int_0^t \langle C_2\xi,\boldsymbol{v}\rangle\, ds\right| \leq c\int_0^t |\xi(s)|_H^2\, ds + \frac{\alpha}{4}\int_0^t \|\boldsymbol{v}(s)\|_E^2\, ds, \tag{3.17}$$

$$\left|\int_0^t \langle f,\boldsymbol{v}\rangle\, ds\right| \leq c\int_0^t \|f(s)\|_{E'}^2\, ds + \frac{\alpha}{4}\int_0^t \|\boldsymbol{v}(s)\|_E^2\, ds. \tag{3.18}$$

Combining the estimates (3.14)–(3.18) in (3.13), we find

$$\|\boldsymbol{u}(t)\|_E^2 + \int_0^t \|\boldsymbol{v}(s)\|_E^2\, ds \leq c\left(1 + \int_0^t |\xi(s)|_H^2\, ds\right). \tag{3.19}$$

We turn now to the energy equation (3.6) and by letting it act on ξ we obtain

$$\begin{aligned}&\int_0^t \langle \xi',\xi\rangle ds + \int_0^t \langle K_1\xi,\xi\rangle ds + \int_0^t \langle K_2\xi,\xi\rangle ds \\ &\quad = -\int_0^t \langle C_1\boldsymbol{v},\xi\rangle ds - \int_0^t \langle S(g,\boldsymbol{v}),\xi\rangle ds + \int_0^t \langle Q,\xi\rangle ds.\end{aligned} \tag{3.20}$$

From Theorem 1(2) in [15] and our choice of the spaces F and G we find

$$\int_0^t \langle \xi',\xi\rangle\, ds = \frac{1}{2}\int_0^t \left(\tfrac{d}{ds}|\xi(s)|_H^2\right) ds = \frac{1}{2}|\xi(t)|_H^2 - \frac{1}{2}|\xi_0|_H^2. \tag{3.21}$$

Moreover, using arguments similar to those above (see (3.14), (3.17) and (3.18)) we get

$$\int_0^t \langle K_1\xi, \xi\rangle ds = \int_0^t\int_{\Gamma_C} k_e|\xi|^2\, ds \geq 0, \tag{3.22}$$

$$\int_0^t \langle K_2\xi, \xi\rangle ds \geq \alpha \int_0^t \|\xi\|_V^2\, ds, \tag{3.23}$$

$$\left|\int_0^t \langle C_1 \boldsymbol{v}, \xi\rangle ds\right| \leq c \int_0^t |\boldsymbol{v}(s)|_{H^m}^2\, ds + \frac{\alpha}{4}\int_0^t \|\xi(s)\|_V^2\, ds, \tag{3.24}$$

$$\left|\int_0^t \langle Q, \xi\rangle ds\right| \leq c \int_0^t \|Q(s)\|_{V'}^2\, ds + \frac{\alpha}{4}\int_0^t \|\xi(s)\|_V^2\, ds. \tag{3.25}$$

To estimate the term involving $S(g,\cdot)$, we use the inequality $|Rg_n|\big(1-\delta|Rg_n|\big)_+ \leq \frac{1}{\delta}$, assumption (2.14) and Cauchy's inequality with ε, and deduce that

$$\left|\int_0^t \langle S(g,\boldsymbol{v}), \xi\rangle\, ds\right| \leq c + \frac{\alpha}{4}\int_0^t \|\xi(s)\|_V^2\, ds. \tag{3.26}$$

Combining the estimates (3.21)–(3.26) in (3.20) we obtain

$$|\xi(t)|_H^2 + \int_0^t \|\xi(s)\|_V^2\, ds \leq c, \tag{3.27}$$

and thus,

$$\int_0^t |\xi(s)|_H^2\, ds \leq c.$$

Inserting this estimate in (3.19) we deduce

$$\|\boldsymbol{u}(t)\|_E^2 + \int_0^t \|\boldsymbol{v}(s)\|_E^2 ds \leq c. \tag{3.28}$$

The estimate (3.12) is now a consequence of (3.27) and (3.28).

4. Proof of Theorem 2.3

We prove the existence of the unique solution of Problem P_g as a limit of a subsequence of solutions for Problem P_h obtained above. It follows from Theorem 3.5 that for a given set of initial conditions the family of solutions $\{\boldsymbol{u}_h, \boldsymbol{v}_h, \xi_h\}$ is bounded in $\mathbb{E}\times\mathbb{E}\times\mathbb{V}$. From this and (3.6)–(3.8) it follows that $\{R_e\boldsymbol{u}_h', \xi_h'\}$ is bounded in $\mathbb{E}'\times\mathbb{V}'$. Consequently, there exists a weak limit point $(\boldsymbol{u}, \boldsymbol{v}, \xi) \in \mathbb{E}\times\mathbb{E}\times\mathbb{V}$ and a subsequence of parameters $\{h_l\}$ such that $h_l \to 0$ when $l\to\infty$ and such that the following limit processes take place when $l\to\infty$:

$$\begin{array}{rcll} \boldsymbol{u}_l & \longrightarrow & \boldsymbol{u} & \text{weakly in } \mathbb{E}, \\ \boldsymbol{v}_l & \longrightarrow & \boldsymbol{v} & \text{weakly in } \mathbb{E}, \\ \xi_l & \longrightarrow & \xi & \text{weakly in } \mathbb{V}, \\ \xi_l' & \longrightarrow & \xi' & \text{weakly in } \mathbb{V}', \\ R_e\boldsymbol{u}_l' & \longrightarrow & R_e\boldsymbol{u}' & \text{weakly in } \mathbb{E}'. \end{array}$$

We may pass now to the limit in (3.6) and (3.7) in all terms except the one involving J^{h_l}. We may suppose, by passing to a subsequence if necessary, that $J^{h_l}(\boldsymbol{v}_l) \longrightarrow \gamma$ weakly in $\mathbb{E}'$, for some $\gamma \in \mathbb{E}'$. It only remains to show that $\gamma \in \partial_2 j(g, \boldsymbol{v})$. Toward that end, we note that when $l \to \infty$ we have

$$\psi^{h_l}(w_\tau - \boldsymbol{v}_\tau + (\boldsymbol{v}_l)_\tau - \boldsymbol{\phi}) \longrightarrow |w_\tau - \boldsymbol{\phi}| \quad \text{in } L^2(\Gamma_C \times (0,T)),$$

$$\psi^{h_l}((\boldsymbol{v}_l)_\tau - \boldsymbol{\phi}) \longrightarrow |\boldsymbol{v}_\tau - \boldsymbol{\phi}| \quad \text{in } L^2(\Gamma_C \times (0,T)),$$

for $w \in \mathbb{E}$. Then, it follows from the convexity of ψ^{h_l} and Hölder's inequality that

$$\begin{aligned}
\langle \gamma, w - \boldsymbol{v} \rangle &= \lim_{l\to\infty} \langle J^{h_l} \boldsymbol{v}_l, w - \boldsymbol{v} \rangle \\
&= \lim_{l\to\infty} \int_0^T \int_{\Gamma_C} \mu |Rg_n| (1 - \delta |Rg_n|)_+ \nabla \psi^{h_l}((\boldsymbol{v}_l)_\tau - \boldsymbol{\phi}) \cdot (w_\tau - \boldsymbol{v}_\tau) d\Gamma dt \\
&\leq \lim_{l\to\infty} \int_0^T \int_{\Gamma_C} \mu |Rg_n| (1 - \delta |Rg_n|)_+ \\
&\qquad \times \left[\psi^{h_l}(w_\tau - \boldsymbol{v}_\tau + (\boldsymbol{v}_l)_\tau - \boldsymbol{\phi}) - \psi^{h_l}((\boldsymbol{v}_l)_\tau - \boldsymbol{\phi}) \right] d\Gamma dt \\
&\leq \int_0^T \int_{\Gamma_C} \mu |Rg_n| (1 - \delta |Rg_n|)_+ \left[|w_\tau - \boldsymbol{\phi}| - |\boldsymbol{v}_\tau - \boldsymbol{\phi}| \right] d\Gamma dt.
\end{aligned}$$

This shows that $\gamma \in \partial_2 j(g, \boldsymbol{v})$. Consequently, the weak limit $\{\boldsymbol{u}, \xi\}$ of the subsequence $\{\boldsymbol{u}_l, \xi_l\}$ is a solution to (2.30), (2.31), (2.33), (2.34). Let now σ given by $\sigma_{ij} = a_{ijkl} u_{k,l} + b_{ijkl} u'_{k,l} - c_{ij}(\xi + \Theta)$. To show that $\sigma \in \mathbb{W}$ we let (2.34) act on $\varphi \in \mathcal{D}(\Omega)^m$ followed by Green's formula which implies that $\sigma_{ij,j} + f_{Ai} = 0$. This completes the proof of the existence part in Theorem 2.3.

We will prove now that the solution of Problem P_g is unique. Indeed, let $\{\boldsymbol{u}_1, \sigma_1, \xi_1\}$ and $\{\boldsymbol{u}_2, \sigma_2, \xi_2\}$ be two solutions corresponding to the same data. Taking (2.32) into account, it suffices to show that $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $\xi_1 = \xi_2$. To this end, we substitute into (2.34) $\boldsymbol{u}_1$ for $\boldsymbol{u}$ and let the resulting expression act on $\boldsymbol{u}'_1 - \boldsymbol{u}'_2$ and then substitute into (2.34) $\boldsymbol{u}_2$ for $\boldsymbol{u}$ and let the resulting expression act on $\boldsymbol{u}'_2 - \boldsymbol{u}'_1$. Adding the two inequalities and using the notation $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\boldsymbol{v} = \boldsymbol{u}'_1 - \boldsymbol{u}'_2$ and $\xi = \xi_1 - \xi_2$, yields

$$\int_0^t \langle B\boldsymbol{v}, \boldsymbol{v} \rangle ds + \int_0^t \langle A\boldsymbol{u}, \boldsymbol{v} \rangle ds \leq - \int_0^t \langle C_2 \xi, \, \boldsymbol{v} \rangle ds. \tag{4.1}$$

We estimate the terms in (4.1) using the same arguments as those used to estimate the corresponding terms in (3.13) (see (3.14), (3.15) and (3.17)) and combining the resulting estimates in (4.1), we find

$$\|\boldsymbol{u}(t)\|_E^2 + \int_0^t \|\boldsymbol{v}(s)\|_E^2 ds \leq c \int_0^t |\xi(s)|_H^2 ds. \tag{4.2}$$

Next, we substitute ξ_1 and ξ_2 into (2.33) in turn, subtract the resulting equations and let the result act on ξ, we obtain

$$\int_0^t \langle \xi', \xi \rangle ds + \int_0^t \langle K_1\xi, \xi \rangle ds + \int_0^t \langle K_2\xi, \xi \rangle ds$$
$$= - \int_0^t \langle C_1 \boldsymbol{v}, \xi \rangle ds - \int_0^t \langle S(g, \boldsymbol{v}_1) - S(g, \boldsymbol{v}_2), \xi \rangle ds. \tag{4.3}$$

All the terms in this equality, except the last one, can be estimated in the same way as (3.21)–(3.24). To estimate the term involving $S(g, \cdot)$ we use (2.27) and get

$$\left| \int_0^t \langle S(g, \boldsymbol{v}_1) - S(g, \boldsymbol{v}_2), \xi \rangle ds \right|$$

$$\leq \int_0^t \int_{\Gamma_C} \mu |Rg_n|(1 - \delta|Rg_n|)_+ \big| s_c(x, |(\boldsymbol{v}_1)_\tau - \boldsymbol{\phi}|) - s_c(x, |(\boldsymbol{v}_2)_\tau - \boldsymbol{\phi}|) \big| |\xi| d\Gamma \, ds.$$

Since $|Rg_n|(1 - \delta|Rg_n|)_+ \leq \frac{1}{\delta}$, by using (2.14) followed by Cauchy's inequality with ε, we obtain

$$\left| \int_0^t \langle S(g, \boldsymbol{v}_1) - S(g, \boldsymbol{v}_2), \, \xi \rangle ds \right| \leq c \int_0^t \|\boldsymbol{v}(s)\|_E^2 ds + \frac{\alpha}{4} \int_0^t \|\xi(s)\|_V^2 ds.$$

Here we used in an essential way the assumption of SJK. Combining these estimates in (4.3) we get

$$|\xi(t)|_H^2 + \int_0^t \|\xi(s)\|_V^2 \, ds \leq c \int_0^t \|\boldsymbol{v}(s)\|_E^2 ds. \tag{4.4}$$

We deduce from (4.2) that

$$\int_0^t \|\boldsymbol{v}(s)\|_E^2 \, ds \leq c \int_0^t |\xi(s)|_H^2 \, ds,$$

which, when used in (4.4), implies that

$$|\xi(t)|_H^2 \leq c \int_0^t |\xi(s)|_H^2 ds. \tag{4.5}$$

Now, it follows from Gronwall's inequality that $\xi(t) = 0$, and then it (4.2) implies that $\boldsymbol{u}(t) = 0$. This completes the proof of Theorem 2.3.

Remark 4.1. The arguments of this section may be used to show that the solution of each of the approximate problems P_h is unique, too.

5. Proof of Theorem 2.2

We use a fixed point argument to establish the existence of a unique solution for Problem P when $|\mu|_{L^\infty(\Gamma_C)}$ is sufficiently small. Theorem 2.3 asserts that for each $g \in \mathbb{W}$, Problem P_g has a unique solution $\{\boldsymbol{u}_g, \sigma_g, \xi_g\}$. Consequently, we consider the operator $\Lambda : \mathbb{W} \longrightarrow \mathbb{W}$ defined by

$$\Lambda g = \sigma_g, \qquad g \in \mathbb{W}. \tag{5.1}$$

We have

Proposition 5.1. *There exists $\mu_0 > 0$, sufficiently small, such that if $|\mu|_{L^\infty(\Gamma_C)} < \mu_0$ then the operator Λ has a unique fixed point $g^* \in \mathbb{W}$.*

Proof. Let g^1, $g^2 \in \mathbb{W}$ and set $g = g^1 - g^2$, $\boldsymbol{u}^i = \boldsymbol{u}_{g^i}$, $\boldsymbol{v}^i = \boldsymbol{u}'_{g^i}$, $\sigma^i = \sigma_{g^i}$ and $\xi^i = \xi_{g^i}$, where $i = 1$ or 2. We substitute into (2.34) $\boldsymbol{u}^1$ for $\boldsymbol{u}$ and let the resulting expression act on $\boldsymbol{v}^1 - \boldsymbol{v}^2$ and then substitute into (2.34) $\boldsymbol{u}^2$ for $\boldsymbol{u}$ and let the resulting expression act on $\boldsymbol{v}^2 - \boldsymbol{v}^1$. We add the two inequalities denote by $\boldsymbol{u} = \boldsymbol{u}^1 - \boldsymbol{u}^2$, $\boldsymbol{v} = \boldsymbol{v}^1 - \boldsymbol{v}^2$ and $\xi = \xi^1 - \xi^2$, and obtain

$$\begin{aligned} &\int_0^t \langle B\boldsymbol{v}, \boldsymbol{v}\rangle ds + \int_0^t \langle A\boldsymbol{u}, \boldsymbol{v}\rangle ds + \int_0^t \langle C_2\xi, \boldsymbol{v}\rangle ds \\ &\quad \le \int_0^t \int_{\Gamma_C} \mu\Big(|Rg_n^1|(1-\delta|Rg_n^1|)_+ - |Rg_n^2|(1-\delta|Rg_n^2|)_+\Big) \\ &\qquad \times \Big(|(\boldsymbol{v}_1)_\tau - \boldsymbol{\phi}| - |(\boldsymbol{v}_2)_\tau - \boldsymbol{\phi}|\Big) d\Gamma ds. \end{aligned} \tag{5.2}$$

We estimate the terms on the left-hand side in (5.2) by the same arguments as those used to estimate the corresponding terms in (3.13) (see (3.14), (3.15) and (3.17)). The integral on the right-hand side is estimated by using the inequality

$$|Rg_n^1|(1-\delta|Rg_n^1|)_+ - |Rg_n^2|(1-\delta|Rg_n^2|)_+ \le c|Rg_n^1 - Rg_n^2|, \tag{5.3}$$

followed by Cauchy's inequality with ε, to obtain

$$\begin{aligned} &\int_0^t \int_{\Gamma_C} \mu\Big(|Rg_n^1|(1-\delta|Rg_n^1|)_+ - |Rg_n^2|(1-\delta|Rg_n^2|)_+\Big)\Big(|(\boldsymbol{v}_1)_\tau - \boldsymbol{\phi}| - |(\boldsymbol{v}_2)_\tau - \boldsymbol{\phi}|\Big) d\Gamma ds \\ &\quad \le c|\mu|^2_{L^\infty(\Gamma_C)} \int_0^t |Rg_n^1(s) - Rg_n^2(s)|^2_{L^2(\Gamma)}\, ds + \frac{\alpha}{4}\int_0^t \|\boldsymbol{v}(s)\|_E^2\, ds. \end{aligned}$$

Combining these estimates in (5.2), we get

$$\|\boldsymbol{u}(t)\|_E^2 + \int_0^t \|\boldsymbol{v}(s)\|_E^2\, ds \le c\int_0^t |\xi(s)|_H^2\, ds + c|\mu|^2_{L^\infty(\Gamma_C)} \int_0^t |Rg_n^1(s) - Rg_n^2(s)|^2_{L^2(\Gamma)}\, ds,$$

which implies

$$\|\boldsymbol{u}(t)\|_E^2 + \int_0^t \|\boldsymbol{v}(s)\|_E^2 ds \le c\int_0^t |\xi(s)|_H^2\, ds + c|\mu|^2_{L^\infty(\Gamma_C)} \int_0^t \|g(s)\|_W^2\, ds. \tag{5.4}$$

Next, we substitute ξ_1 and ξ_2 into (2.33) in turn, subtract the resulting equations and let the result act on the difference ξ and obtain

$$\int_0^t \langle \xi', \xi\rangle ds + \int_0^t \langle K_1\xi,\, \xi\rangle ds + \int_0^t \langle K_2\xi,\, \xi\rangle ds$$
$$= -\int_0^t \langle C_1\boldsymbol{v},\, \xi\rangle ds - \int_0^t \langle S(g^1, \boldsymbol{v}^1) - S(g^2, \boldsymbol{v}^2),\, \xi\rangle ds. \tag{5.5}$$

All the terms in this equality, except the last one, can be estimated in the same way as (3.21)–(3.24). To estimate the term involving S we write

$$\int_0^t \langle S(g^2, \boldsymbol{v}^2) - S(g^1, \boldsymbol{v}^1), \xi\rangle ds$$

$$\le \int_0^t \langle S(g^2, \boldsymbol{v}^1) - S(g^1, \boldsymbol{v}^1), \xi\rangle ds + \int_0^t \langle S(g^2, \boldsymbol{v}^2) - S(g^2, \boldsymbol{v}^1), \xi\rangle ds.$$

Using (2.27), (5.3) and (2.14), the first integral on the right-hand side is estimated by Cauchy's inequality with ε thus,

$$\int_0^t \langle S(g^2, \boldsymbol{v}^1) - S(g^1, \boldsymbol{v}^1),\, \xi\rangle ds \le c|\mu|^2_{L^\infty(\Gamma_C)} \int_0^t \|g(s)\|_W^2\, ds + \frac{\alpha}{4}\int_0^t \|\xi(s)\|_V^2\, ds.$$

Using (2.27), (2.14) and the inequality $|Rg_n|(1 - \delta|Rg_n|)_+ \le \frac{1}{\delta}$, we find that the second integral on the right-hand satisfies

$$\left|\int_0^t \langle S(g^2, \boldsymbol{v}^2) - S(g^2, \boldsymbol{v}^1),\, \xi\rangle ds\right| \le c\int_0^t \|\boldsymbol{v}(s)\|_E^2\, ds + \frac{\alpha}{4}\int_0^t \|\xi(s)\|_V^2\, ds.$$

Combining these estimates in (5.5) we obtain

$$|\xi(t)|_H^2 + \int_0^t \|\xi(s)\|_V^2\, ds \le c\int_0^t \|\boldsymbol{v}(s)\|_E^2\, ds + c|\mu|^2_{L^\infty(\Gamma_C)} \int_0^t \|g(s)\|_W^2\, ds. \tag{5.6}$$

We deduce from (4.4) that

$$\int_0^t \|\boldsymbol{v}(s)\|_E^2\, ds \le c\int_0^t |\xi(s)|_H^2\, ds + c|\mu|^2_{L^\infty(\Gamma_C)} \int_0^t \|g(s)\|_W^2\, ds, \tag{5.7}$$

which, when used in (5.6), yields

$$|\xi(t)|_H^2 \le c\int_0^t |\xi(s)|_H^2\, ds + c|\mu|^2_{L^\infty(\Gamma_C)} \int_0^t \|g(s)\|_W^2\, ds.$$

By using Gronwall's inequality we obtain

$$|\xi(t)|_H^2 \leq c|\mu|^2_{L^\infty(\Gamma_C)} \int_0^t \|g(s)\|_W^2\, ds. \tag{5.8}$$

Applying this estimate in (5.4), we find

$$\|\boldsymbol{u}(t)\|_E^2 \leq c|\mu|^2_{L^\infty(\Gamma_C)} \left(\int_0^t \int_0^s \|g(r)\|_W^2\, drds + \int_0^t \|g(s)\|_W^2 ds \right), \tag{5.9}$$

$$\int_0^t \|\boldsymbol{v}(s)\|_E^2\, ds \leq c|\mu|^2_{L^\infty(\Gamma_C)} \left(\int_0^t \int_0^s \|g(r)\|_W^2\, drds + \int_0^t \|g(s)\|_W^2\, ds \right). \tag{5.10}$$

We conclude, using (5.8)–(5.10), that

$$\|\boldsymbol{u}\|_{\mathbb{E}}^2 + \|\boldsymbol{v}\|_{\mathbb{E}}^2 + |\xi|_{\mathbb{H}}^2 \leq c|\mu|^2_{L^\infty(\Gamma_C)} \|g\|_{\mathbf{W}}^2. \tag{5.11}$$

On the other hand, it follows from (2.32) and (2.11) that

$$\|\sigma^1 - \sigma^2\|_{\mathbf{W}}^2 \leq c\|\boldsymbol{u}\|_{\mathbb{E}}^2 + \|\boldsymbol{v}\|_{\mathbb{E}}^2 + |\xi|_{\mathbb{H}}^2. \tag{5.12}$$

Combining the estimates (5.11) and (5.12) we find

$$\|\sigma^1 - \sigma^2\|_{\mathbf{W}} \leq c|\mu|_{L^\infty(\Gamma_C)} \|g\|_{\mathbf{W}},$$

and, using (5.1), we deduce that

$$\|\Lambda g^1 - \Lambda g^2\|_{\mathbf{W}} \leq c|\mu|_{L^\infty(\Gamma_C)} \|g^1 - g^2\|_{\mathbf{W}}. \tag{5.13}$$

We choose $\mu_0 = 1/c$ and Proposition 5.1 is now a consequence of (5.13) and the Banach fixed point theorem.

Proof of Theorem 2.2. Using (5.1) and Proposition 5.1, it straightforward to show that the solution $\{\boldsymbol{u}_{g^*}, \sigma_{g^*}, \xi_{g^*}\}$ of Problem P_{g^*} is also a solution of Problem P when $|\mu|_{L^\infty(\Gamma_C)}$ is sufficiently small. The uniqueness part in Theorem 2.2 can be deduced from the uniqueness of the solution of Problem P_{g^*} and the uniqueness of the fixed point of the operator Λ.

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A Dynamic Thermoviscoelastic Frictional Contact Problem with Damped Response

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A Dynamic Thermoviscoelastic Frictional Contact Problem with Damped Response

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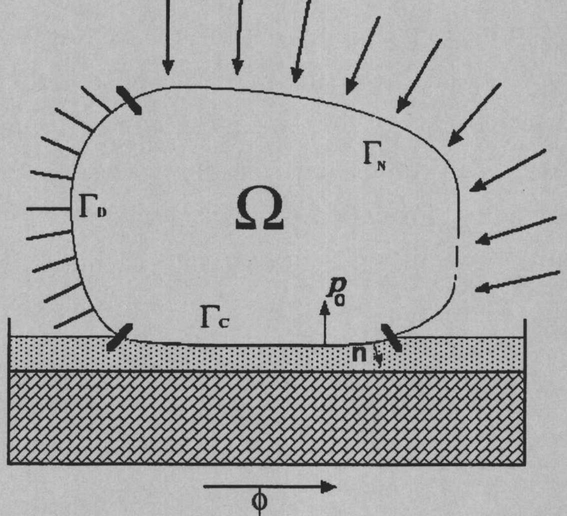
Descriptif

Il s'agit dans ce travail de l'étude d'un modèle général pour les problèmes dynamiques de contact avec frottement entre un matériau déformable et une fondation rigide en mouvement. Le matériau est supposé obéir à la loi de comportement viscoélastique de Kelvin-Voigt en tenant compte des effets thermiques. On suppose la présence d'un lubrifiant entre le matériau et la fondation. Les frottements sont modélisés par la version modifiée la loi de Coulomb introduite par N. STRÖMBERG, L. JOHANSSON et A. KLARBRING. Le dégagement de chaleur due aux frottements est pris en compte dans les conditions aux limites relatives à la température.

On considère un milieu continu viscoélastique occupant un domaine Ω de $\mathbb{R}^m$ ($m = 2, 3$), et dont la frontière Γ, supposée suffisamment régulière, est divisée en trois parties disjointes Γ_D, Γ_N et Γ_C. On suppose que, pendant l'intervalle de temps $[0, T]$, des forces volumiques f_A agissent dans Ω, que la partie Γ_D est encastrée dans une structure fixe, que des forces surfaciques f_N s'appliquent sur Γ_N. On suppose en outre que la partie $\Gamma_D \cup \Gamma_N$ est maintenue à une température donnée θ_b alors que la fondation rigide, en mouvement tangentiel avec une vitesse ϕ, est maintenue à une température θ_R. Ce problème mécanique est illustré par la figure suivante :



Il peut se formuler mathématiquement de la façon suivante :

Problème P : Trouver le champ des déplacements $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^m$, le champ des contraintes $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{R}_s^{m\times m}$ et la température $\theta : \Omega \times [0,T] \longrightarrow \mathbb{R}$ tels que

$$\begin{aligned}
\sigma_{ij} = a_{ijkl}u_{k,l} + b_{ijkl}u'_{k,l} - c_{ij}\theta \quad & \text{dans} \quad \Omega \times (0,T),\\
u''_i - \sigma_{ij,j} = f_{Ai} \quad & \text{dans} \quad \Omega \times (0,T),\\
\theta' - (k_{ij}\theta_{,j})_{,i} = -c_{ij}u'_{i,j} + q \quad & \text{dans} \quad \Omega \times (0,T),\\
u = 0 \quad & \text{sur} \quad \Gamma_D \times (0,T),\\
\sigma\mathbf{n} = f_N \quad & \text{sur} \quad \Gamma_N \times (0,T),\\
\theta = \theta_b \quad & \text{sur} \quad (\Gamma_D \cup \Gamma_N) \times (0,T),\\
-\sigma_n = \beta(u'_n)_+ + p_0 \quad & \text{sur} \quad \Gamma_C \times (0,T),\\
|\sigma_\tau| \leq \mu|\sigma_n|\big(1-\delta|\sigma_n|\big)_+ \quad & \text{sur} \quad \Gamma_C \times (0,T),\\
|\sigma_\tau| < \mu|\sigma_n|\big(1-\delta|\sigma_n|\big)_+ \quad & \Longrightarrow u'_\tau = \phi,\\
|\sigma_\tau| = \mu|\sigma_n|\big(1-\delta|\sigma_n|\big)_+ \quad & \Longrightarrow u'_\tau = \phi - \lambda\sigma_\tau,\ \lambda \geq 0,\\
k_{ij}\theta_{,i}n_j = \mu|\sigma_n|\big(1-\delta|\sigma_n|\big)_+ s_c(\cdot\,, |u'_\tau - \phi|) - k_e(\theta - \theta_R) \quad & \text{sur} \quad \Gamma_C \times (0,T),\\
u(\cdot\,,0) = u_0, \quad u'(\cdot\,,0) = v_0, \quad \theta(\cdot\,,0) = \theta_0 \quad & \text{dans} \quad \Omega.
\end{aligned}$$

On note par $\mathbb{R}_s^{m\times m}$ l'espace des tenseurs symétriques du second ordre sur $\mathbb{R}^m$. Le "prime" au dessus d'une quantité représente sa dérivée temporelle, $\mathbf{n}$ est la normale unitaire sortante à Ω et $\sigma\mathbf{n}$ est le vecteur des contraintes de Cauchy. Les fonctions u_n, u'_τ, σ_n et σ_τ représentent respectivement le déplacement normal, la vitesse tangentielle et les contraintes normales et tangentielles. Les fonctions β et p_0 désignent respectivement le *coefficient de lubrification* et la *pression du lubrifiant* alors que μ est le *coefficient de frottement* et δ est un coefficient positif (assez petit) lié à l'usure et la dureté du matériau.

On commence tout d'abord par donner une interprétation mécanique de chacune des équations et des termes cités dans le problème P. On insiste tout particulièrement sur les conditions aux limites considérées sur la partie de contact potentielle Γ_C pour chacunes des inconnues. On introduit ensuite les hypothèses utilisées suivies d'une formulation faible du problème P. Cette formulation se présente comme un système hyperbolique-parabolique d'équations aux dérivées partielles. Le système est formulé ensuite en terme d'opérateurs. Un résultat d'existence est alors établi en utilisant une méthode de régularisation suivie d'estimations *a priori* puis de passages à la limite. On termine par prouver l'unicité de la solution moyennant quelques hypothèses supplémentaires.

A Dynamic Thermoviscoelastic Frictional Contact Problem with Damped Response

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Abstract

We analyze a problem that describes dynamic contact between a thermoviscoelastic body and a rigid foundation. The contact is modeled with normal damped response and the SJK version of Coulomb's friction law. Frictional heat generation is taken fully into account. The problem is set as a dynamic evolution system. The existence of a weak solution is established by using regularization, the existence theorem for degenerate evolution equations of Kuttler and a priori estimates. We prove the uniqueness of the weak solution when the friction coefficient is sufficiently small.

Keywords: Thermoviscoelastic frictional contact, dynamic contact, degenerate evolution inequalities, SJK-Coulomb friction law, normal damped response, frictional heat generation, weak solutions.

1. Introduction

In this paper we consider a general model for the process of dynamic thermoviscoelastic contact between a deformable body and a rigid obstacle. The material is assumed to behave according to the Kelvin-Voigt constitutive law with added thermal effects. The contact is modeled with the normal damped response and the friction with the recently derived SJK-Coulomb condition. The model is set as a dynamic variational inequality for the displacements coupled with a variational equality for the temperature, and the existence of a weak solution is established. The uniqueness of the weak solution is proved when the friction coefficient is small.

Thermoviscoelastic contact abounds in industry. It can be found in engines, transmissions, brakes and other engineering systems. It is very important, for instance, when a train or a car stops suddenly since a large amount of heat is generated at the contact interface and this may adversely affect the rail wheels or the car brakes and tires. Indeed, the sudden application of car brakes may lead to the dissipation of energy in the form of frictional heating at a rate of 100 HP or more.

General models for dynamic contact with friction have been investigated mathematically in [13, 9, 6, 2, 3]. In [13] the isothermal frictional contact problem with normal compliance was formulated, analyzed and numerical simulations presented. A general existence theorem for dynamic contact with normal compliance, without any restrictions on the power of the normal compliance function, was obtained in [9]. Thermoviscoelastic frictional contact with normal compliance can be found in [6, 2] where the existence of weak solutions was proved, but the effects of the frictional heat generation were included only in the second paper. The dynamic thermoviscoelastic problem, with frictional heat generation, which takes into account the wear of the contacting surfaces was investigated in [3], where the existence of a weak solution was established. Models and numerical simulations for the dynamic thermoelastic processes can be found in [7, 16].

In this paper we model the contact processes with the normal damped response and the SJK version of Coulomb's law of friction ([16]). The surface normal resistance is assumed to be proportional to the normal velocity. This is the case when there is a thin layer of oil, lubricant or a layer of softer material on the contacting surface. A version of the damped normal response condition was considered recently in [15] where it was related to the wear of the contacting surfaces under steady sliding; actually, it was obtained from Archard's law of wear. It was also employed in [14] where the quasistatic contact problem with directional friction was investigated. The SJK version of Coulomb's friction law takes into account the deterioration of the surface under large normal loads, which seems to be a reasonable modification of Coulomb's law for large stresses. For small or medium stresses, the new version reduces to the usual version of the law. This condition, in addition to being more realistic, also turns out to be essential in the existence proof. Moreover, it helps to remove restrictions on the asymptotic power of the frictional heat dissipation function. Apparently, a better description of the physical process leads to removal of certain mathematical restrictions on the asymptotic behavior of the contact functions which had to be imposed in [2, 3] and seem to be of mathematical origin only.

The weak formulation of the problem is in the form of a variational inequality. To establish the existence of weak solutions we regularize the problem, reducing it to a variational equality. The existence of weak solutions to the regularized problem follows from an abstract existence result for implicit evolution equations in [8]. Then the necessary a priori estimates are obtained which allow us to pass to the limit. A similar approach, based on the result of [8], was used in [2, 3]. The estimates turn out to be very delicate, but do not impose unnecessary restrictions on the asymptotic powers of the functions involved in the contact condition, unlike the situation in the above references.

The paper is organized as follows. In Section 2 we describe the classical model for the process. It consists of the equations of motion for a viscoelastic body, in terms of the displacements, together with the energy equation for the temperature. Since the normal velocity of the contact surface may be discontinuous, there exists a regularity ceiling for the problem which precludes, in general, the existence of classical solutions. Therefore, a weak or variational formulation for the problem is presented. It consists of a dynamic variational inequality for the displacements and a variational equality for the temperature. We state the assumptions on the problem data and then cast the problem in an abstract operator setting, as Problem P. The existence result is stated in Theorem 2.2.

The existence proof is given in Section 4 and is based on passing to a limit in a subsequence of solutions to approximate problems, which are investigated in Section 3. These are obtained by regularization of the Euclidean norm, which leads to a variational equality instead of inequality. The regularized problems are set in an abstract operator form and the existence of solutions follows from a result of Kuttler [8] concerning degenerate evolution equations. We recall the setting and the version of his theorem that we employ. Then we obtain the necessary a priori estimates on the approximate solutions. A solution to Problem P is obtained in Section 4 as a limit of a sequence of these solutions. We establish the uniqueness of the solution, in Section 5, under a mild additional assumption on the data, but only when the friction coefficient is sufficiently small.

2. Classical model, weak formulation and results

In this section we present the physical setting and formulate the model as a coupled parabolic-hyperbolic system of partial differential equations. Then we introduce the weak formulation and state the assumptions on the data and our main existence result.

We employ the Kelvin-Voigt viscoelastic law and include thermal effects. We retain the inertial terms in the equations of motion, and thus consider the fully dynamic problem. The behavior of the bulk material is assumed to be linear, the nonlinear effects occur on the part of the boundary which may contact the rigid foundation.

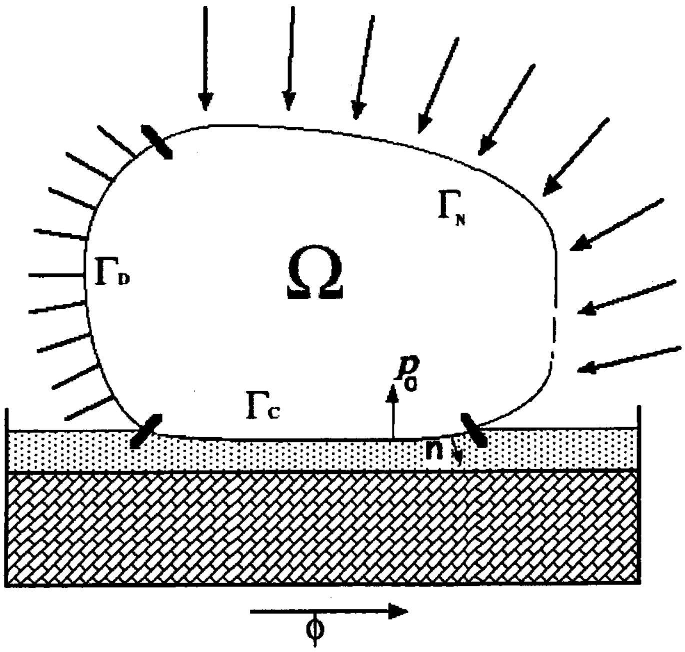

Fig. 1. The physical setting, Γ_c is the contact surface.

The physical setting, depicted in Fig. 1, consists of a viscoelastic body, represented (in its reference configuration) by Ω, a region in $\mathbb{R}^m$ $(m = 2, 3)$, whose boundary $\partial\Omega = \Gamma$ is divided into three disjoint parts. On the first part, denoted by Γ_D, the body is clamped; on the second part, Γ_N, known tractions act; and on the third part, Γ_C, the body may come into frictional contact with a rigid foundation. For the sake of generality, we allow for the motion of the rigid foundation with tangential velocity ϕ. The reference configuration is assumed to be stress free and at a constant temperature, conveniently set as zero, which also serves as the reference temperature. Our interest lies in the evolution process of the system state on the time interval $[0, T]$, $0 < T$. We assume that meas$\Gamma_D > 0$; although, unlike the static or quasistatic cases, in the dynamic problem it is not needed from the mathematical point of view. Indeed, all the results below hold without it with minor modifications. But, without this assumption, the problem becomes invariant under rigid body motions and then the interpretation of the contact surface becomes very cumbersome.

Let $f_A = (f_{A1}(x,t), \dots, f_{Am}(x,t))$ denote the (nondimensional) density of applied body forces acting in Ω and q the density of volume heat sources. For the sake of simplicity, the material density is assumed constant, set equal to one. Let $u = (u_1(x,t), \dots, u_m(x,t))$, $\theta = \theta(x,t)$ and $\sigma = (\sigma_{ij})$ for

$i, j = 1, \ldots, m$, represent the dimensionless displacement vector, temperature and stress tensor, at location x and time t, respectively. The system of viscoelastic and energy equations takes the (nondimensional) form

$$u_i'' - \sigma_{ij,j} = f_{Ai} \qquad \text{in} \quad \Omega \times (0,T), \tag{2.1}$$

$$\theta' - (k_{ij}\theta_{,j})_{,i} = -c_{ij}u'_{i,j} + q \qquad \text{in} \quad \Omega \times (0,T). \tag{2.2}$$

Here and below, $i, j = 1, \ldots, m$; the repeated index convention is employed; the prime represents time derivative; the portion of a subscript prior to a comma indicates a component and the portion after the comma refers to a partial derivative. The constants c_{ij} and k_{ij} are the components of the tensors of thermal expansion and thermal conductivity, respectively. We employ the thermoviscoelastic Kelvin-Voigt stress-strain relation

$$\sigma_{ij} = a_{ijkl}u_{k,l} + b_{ijkl}u'_{k,l} - c_{ij}\theta \qquad \text{in} \quad \Omega \times (0,T). \tag{2.3}$$

Here $a = (a_{ijkl})$ and $b = (b_{ijkl})$ are the tensors of elastic and viscosity coefficients, respectively.

The initial conditions are

$$u(\cdot, 0) = u_0, \qquad u'(\cdot, 0) = v_0, \qquad \theta(\cdot, 0) = \theta_0 \qquad \text{in} \quad \Omega. \tag{2.4}$$

To describe the boundary conditions we introduce the unit normal $\mathbf{n} = (n_1, \ldots, n_m)$ to Γ, and since Γ is assumed Lipschitz, $\mathbf{n}$ exists at almost every point of the boundary. We then let $\sigma_n = \sigma_{ij}n_in_j$ and $u_n = u_in_i$ be the normal components of σ and u on Γ, and let

$$\sigma_\tau = \sigma\mathbf{n} - \sigma_n\mathbf{n}, \qquad u_\tau = u - u_n\mathbf{n}$$

be the tangential components, see, for instance, [10, 5]. We impose the following conditions on the $\Gamma_D \cup \Gamma_N$ portion of the boundary:

$$u = 0 \qquad \text{on} \quad \Gamma_D \times (0,T), \tag{2.5}$$

$$\sigma\mathbf{n} = f_N \qquad \text{on} \quad \Gamma_N \times (0,T), \tag{2.6}$$

$$\theta = \theta_b \qquad \text{on} \quad (\Gamma_D \cup \Gamma_N) \times (0,T). \tag{2.7}$$

Here, $f_N = (f_{N1}(x,t), \ldots, f_{Nm}(x,t))$ denotes the tractions that are applied on Γ_N, and θ_b is the known, scaled temperature of the boundary.

We turn to consider the boundary conditions on the potential contact surface Γ_C, which is where our main interest lies. Mechanically, the contact surface is assumed to be covered with a lubricant that contains solid particles, such as a new smart lubricant or oil with worn metallic particles. The resistance of the foundation, actually of this lubricant layer, is assumed to be proportional to the normal velocity when pressing, but it offers no resistance when receding; thus,

$$-\sigma_n = \beta(u'_n)_+ + p_0 \qquad \text{on} \quad \Gamma_C \times (0,T), \tag{2.8}$$

where $(f)_+ = \max\{0, f\}$, β is the normal damped response function and p_0 is the oil pressure. Alternatively, we may consider the surface of the foundation as covered by a thin soft layer with asperities, which resists the contact. If we replace u'_n with u_n we obtain a version of the *normal compliance condition* which was investigated in [13, 11, 2, 3, 9, 6] (see also the references therein).

We remark that all the results below hold when $(u_n')_+$ is replaced by u_n' in (2.8), but then the interpretation is different. Moreover, a condition similar to (2.8), but withoutp_0, has been derived in [15] in connection with t he wear of the contacting surface when in sliding contact.

We model the friction by the SJK generalization of Coulomb's law of dry friction ([16]) which may be stated as

$$\begin{aligned} &|\sigma_\tau| \leq \mu|\sigma_n|\big(1-\delta|\sigma_n|\big)_+ \qquad \text{on} \quad \Gamma_C \times (0,T), \\ &|\sigma_\tau| < \mu|\sigma_n|\big(1-\delta|\sigma_n|\big)_+ \implies u_\tau' = \phi, \\ &|\sigma_\tau| = \mu|\sigma_n|\big(1-\delta|\sigma_n|\big)_+ \implies u_\tau' = \phi - \lambda\sigma_\tau, \ \lambda \geq 0. \end{aligned} \tag{2.9}$$

Here μ is the friction coefficient, δ is a small positive coefficient related to the wear and hardness of the surface, ϕ is the tangential velocity of the rigid foundation, and u_τ' is the tangential velocity of the body. Condition (2.9) may be interpreted as follows. When the applied tangential stress is less than the limiting value, the boundary sticks to the foundation and moves with it: the part of the boundary where this takes place is called the *stick* zone. When the tangential stress reaches its limiting value, the boundary does not move in tandem with the foundation: this is the so-called *slip* zone. The scalar λ is a multiplier indicating the relative direction of the slip between the body and the foundation. This condition represents a modified version for Coulomb's law of friction and was derived in [16] from thermodynamical considerations. The modification consists of the factor $(1-\delta|\cdot|)_+$ which states that when the contact stress is large there is a decrease in frictional resistance because of surface cracking, and when it is too large, i.e., when it exceeds $1/\delta$, the surface disintegrates and offers no resistance to the motion. Clearly, this modified condition is more realistic, and since δ is very small, it agrees well with Coulomb's law for $|\sigma_n|$ not too large. On the other hand, as was mentioned above, mathematically it leads to considerable simplifications.

Finally, we describe the boundary condition for the temperature on Γ_C. Since the power that is generated by frictional contact forces is proportional to $|\sigma_\tau|$ and $|u_\tau' - \phi|$, we choose:

$$k_{ij}\theta_{,i}n_j = \mu|\sigma_n|\big(1-\delta|\sigma_n|\big)_+ s_c(\cdot\,, |u_\tau' - \phi|) - k_e(\theta - \theta_R) \qquad \text{on} \quad \Gamma_C \times (0,T). \tag{2.10}$$

Here, $s_c(\cdot\,, r)$ is a prescribed function which generalizes the Euclidean norm $|\cdot|$, θ_R is the foundation's temperature, and k_e is the coefficient of heat exchange between it and the surface. The inclusion of the frictional heat generation is essential in many situations, as was mentioned in the Introduction.

The classical formulation of the *thermoviscoelastic frictional contact problem with normal damping* is to find $\{u, \theta\}$ such that (2.1)–(2.10) hold.

It is well known that, in general, there are no classical solutions to the problem because of the regularity ceiling related to the possible jump in the velocity upon impact on the boundary. Therefore, we turn to the weak or variational formulation of the problem. To this end we introduce the following classical Hilbert spaces:

$$E = \Big\{ w \in H^1(\Omega)^m \ : \ w = 0 \quad \text{on} \quad \Gamma_D \Big\},$$

$$V = \Big\{ \eta \in H^1(\Omega) \ : \ \eta = 0 \quad \text{on} \quad \Gamma_D \cup \Gamma_N \Big\},$$

$$H = L^2(\Omega), \qquad H^m = L^2(\Omega)^m, \qquad \mathbb{E} = L^2(0,T;E), \qquad \mathbb{V} = L^2(0,T;V).$$

Below, we use $\|\cdot\|_E, \|\cdot\|_V, |\cdot|_H$ and $|\cdot|_{H^m}$ to denote the norms of E, V, H and H^m, respectively. Also, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E' and E, or V' and V, where the meaning is evident from the context. For standard notation we refer the reader to [1, 12, 10].

We now describe the assumptions on the data. The coefficients of elasticity, viscosity, thermal expansion and thermal conductivity satisfy

$$\begin{aligned}
&a_{ijkl} \in L^\infty(\Omega), \quad b_{ijkl} \in L^\infty(\Omega), \quad c_{ij} \in L^\infty(\Omega), \quad k_{ij} \in L^\infty(\Omega);\\
&a_{ijkl} = a_{jikl}, \quad a_{ijkl} = a_{klij}, \quad a_{ijkl} = a_{ijlk},\\
&a_{ijkl}\chi_{ij}\chi_{kl} \geq \alpha_1\chi_{ij}\chi_{ij} \quad \text{for all symmetric tensors } \chi = (\chi_{ij});\\
&b_{ijkl} = b_{jikl}, \quad b_{ijkl} = b_{klij}, \quad b_{ijkl} = b_{ijlk},\\
&b_{ijkl}\chi_{ij}\chi_{kl} \geq \alpha_2\chi_{ij}\chi_{ij} \quad \text{for all symmetric tensors } \chi = (\chi_{ij});\\
&c_{ij} = c_{ji};\\
&k_{ij} = k_{ji}, \quad k_{ij}z_iz_j \geq \alpha_3 z_i z_i \quad \text{for all vectors } z = (z_i).
\end{aligned} \tag{2.11}$$

Here, α_1, α_2 and α_3 are positive constants.
The body forces and the volume heat sources satisfy

$$f_A \in \mathbf{E}', \qquad q \in \mathbf{V}'. \tag{2.12}$$

The damping and friction coefficients, the oil pressure and the velocity of the foundation satisfy

$$\begin{aligned}
&\beta \in L^\infty(\Gamma_C), \quad \beta \geq 0, \ \text{a.e. on } \Gamma_C,\\
&\mu \in L^\infty(\Gamma_C), \quad \mu \geq 0, \ \text{a.e. on } \Gamma_C,\\
&p_0 \in L^2(0,T; L^\infty(\Gamma_C)), \quad p_0 \geq 0, \ \text{a.e. on } \Gamma_C,\\
&\phi = \phi(t) \in C([0,T]; \mathbb{R}^{m-1}).
\end{aligned} \tag{2.13}$$

The function s_c satisfies

$$\begin{aligned}
&s_c : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}_+ \ \text{is Borel measurable},\\
&s_c(x,\cdot) \ \text{is continuous for each } x \in \Gamma_C,\\
&s_c(\cdot, r) \leq \alpha_4|r|, \quad r \in \mathbb{R}, \ \text{a.e. on } \Gamma_C,
\end{aligned} \tag{2.14}$$

where α_4 is a positive constant. We note that assumption (2.14) is a considerable relaxation of the restrictions which were imposed on s_c in [2, 3], indeed, in [2] the function was assumed to be bounded, and in [3] it was allowed to grow asymptotically as r^m for $0 < m < 1/3$. On the other hand, since s_c is a generalization of $|\cdot|$ clearly the conditions (2.14) are appropriate.
The regularity assumptions on the boundary and initial data are

$$\begin{aligned}
&f_N \in L^2(0,T; L^2(\Gamma_N)^m);\\
&\text{there exists } \Theta \in H^1(0,T; H^1(\Omega)) \text{ such that } \Theta = \theta_b \quad \text{on } \Gamma_D \cup \Gamma_N;\\
&\theta_R \in L^2(0,T; L^2(\Gamma_C));\\
&u_0 \in E, \quad v_0 \in H^m, \quad \theta_0 \in H.
\end{aligned} \tag{2.15}$$

For technical reasons, it is convenient to shift the temperature function so that it is zero on $\Gamma_D \cup \Gamma_N$. For this purpose, we introduce the quantities $\xi = \theta - \Theta$ and $\xi_0 = \theta_0 - \Theta(\cdot, 0)$. To simplify the notation, we will not indicate explicitly the dependence on t.

We can now present the weak formulation of the problem.

Definition 2.1. A pair of functions $\{u, \theta\}$ is said to be a *weak solution* to (2.1)–(2.10) provided that $\{u, \xi\}$ satisfies

$$\begin{gathered} u, u' \in \mathbb{E}, \quad u'' \in \mathbb{E}', \quad u(\cdot\,,0) = u_0, \quad u'(\cdot\,,0) = v_0, \\ \xi \in \mathbb{V}, \quad \xi' \in \mathbb{V}', \quad \xi(\cdot\,,0) = \xi_0, \end{gathered} \tag{2.16}$$

for all $w \in \mathbb{E}$ and a.e. on $(0,T)$,

$$\begin{aligned} &\langle u_i'', w_i - u_i' \rangle + \int_\Omega a_{ijkl} u_{k,l} (w_{i,j} - u'_{i,j}) dx + \int_\Omega b_{ijkl} u'_{k,l} (w_{i,j} - u'_{i,j}) dx - \int_\Omega c_{ij} \xi (w_{i,j} - u'_{i,j}) dx \\ &+ \int_{\Gamma_C} \beta(u_n')_+ (w_n - u_n') d\Gamma + \int_{\Gamma_C} \mu \left(\beta(u_n')_+ + p_0\right) \left(1 - \delta(\beta(u_n')_+ + p_0)\right)_+ \left(|w_\tau - \phi| - |u_\tau' - \phi|\right) d\Gamma \\ &\geq \langle f_{Ai}, w_i - u_i' \rangle + \int_\Omega c_{ij} \Theta (w_{i,j} - u'_{i,j}) dx + \int_{\Gamma_N} f_{Ni} (w_i - u_i') d\Gamma - \int_{\Gamma_C} p_0 (w_n - u_n') d\Gamma, \end{aligned} \tag{2.17}$$

and, for all $\eta \in \mathbb{V}$ and a.e. on $(0,T)$,

$$\begin{aligned} &\langle \xi', \eta \rangle + \int_\Omega k_{ij} \xi_{,i} \eta_{,j} dx + \int_\Omega c_{ij} u'_{i,j} \eta dx + \int_{\Gamma_C} k_e \xi \eta dx \\ &- \int_{\Gamma_C} \mu(\beta(u_n')_+ + p_0) \left(1 - \delta(\beta(u_n')_+ + p_0)\right)_+ s_c(x, |u_\tau' - \phi|) \eta d\Gamma \\ &\qquad = \langle q, \eta \rangle - \int_\Omega \Theta' \eta \, dx - \int_{\Gamma_C} k_e (\Theta - \theta_R) \eta \, d\Gamma - \int_\Omega k_{ij} \Theta_{,i} \eta_{,j} dx. \end{aligned} \tag{2.18}$$

To write the problem in an abstract form, we define the following operators:

$$\begin{aligned} A,\ B,\ D : E &\longrightarrow E', \\ C_1,\ S : E &\longrightarrow V', \\ K_1,\ K_2 : V &\longrightarrow V', \\ C_2 : V &\longrightarrow E', \end{aligned}$$

by

$$\langle Au, w\rangle = \int_\Omega a_{ijkl}u_{k,l}w_{i,j}dx, \tag{2.19}$$

$$\langle Bv, w\rangle = \int_\Omega b_{ijkl}v_{k,l}w_{i,j}dx, \tag{2.20}$$

$$\langle C_1 v, \eta\rangle = \int_\Omega c_{ij}v_{i,j}\eta dx, \tag{2.21}$$

$$\langle C_2 \xi, w\rangle = -\int_\Omega c_{ij}\xi w_{i,j}dx, \tag{2.22}$$

$$\langle Dv, w\rangle = \int_{\Gamma_C} \beta(v_n)_+ w_n d\Gamma, \tag{2.23}$$

$$\langle K_1 \xi, \eta\rangle = \int_{\Gamma_C} k_e \xi\eta d\Gamma, \tag{2.24}$$

$$\langle K_2 \xi, \eta\rangle = \int_\Omega k_{ij}\xi_{,i}\eta_{,j}dx, \tag{2.25}$$

$$\langle Sv, \eta\rangle = -\int_{\Gamma_C} \mu\left(\beta(v_n)_+ + p_0\right)\left(1-\delta(\beta(v_n)_+ + p_0)\right)_+ s_c(x, |v_\tau - \phi|)\eta d\Gamma. \tag{2.26}$$

We note that each of these operators extends to an operator defined on the corresponding space of square-integrable vector-valued functions on $(0, T)$ in a natural way. For example, A extends to an operator from $\mathbb{E}$ to $\mathbb{E}'$ by setting $(Au)(t) = A(u(t))$. With a slight abuse of notation, we will use below the same letter to denote both the original operator and its extension, since the meaning will be clear from the context. We can now formulate Problem (2.16)–(2.19) abstractly as follows.

Problem P : Find $\{u, \xi\}$ satisfying (2.16) and such that

$$\xi' + K_1\xi + K_2\xi + C_1 u' + Su' = Q \quad \text{in } \mathbf{V}', \tag{2.27}$$

$$u'' + Bu' + Du' + Au + C_2\xi + \partial_2 j(u', u') \ni f \quad \text{in } \mathbb{E}'. \tag{2.28}$$

Here $f \in \mathbb{E}'$ and $Q \in \mathbf{V}'$ are given by

$$\langle f, w\rangle_{\mathbb{E}'\times\mathbb{E}} = \int_0^T \langle f_{Ai}, w_i\rangle dt + \int_0^T\int_\Omega c_{ij}\Theta w_{i,j}dxdt + \int_0^T\int_{\Gamma_N} f_{Ni}w_i d\Gamma dt - \int_0^T\int_{\Gamma_C} p_0 w_n d\Gamma dt,$$

$$\langle Q, \eta\rangle_{\mathbf{V}'\times\mathbf{V}} = \int_0^T \langle q, \eta\rangle\, dt - \int_0^T\int_\Omega \Theta'\eta\, dxdt - \int_0^T\int_{\Gamma_C} k_e(\Theta - \theta_R)\eta d\Gamma dt - \int_0^T\int_\Omega k_{ij}\Theta_{,i}\eta_{,j}\, dxdt,$$

respectively, and $\partial_2 j(v, w)$ denotes the partial subdifferential with respect to w of

$$j(v, w) = \int_0^T\int_{\Gamma_C} \mu\left(\beta(v_n)_+ + p_0\right)\left(1-\delta(\beta(v_n)_+ + p_0)\right)_+ |w_\tau - \phi| d\Gamma dt.$$

Our main existence result is:

Theorem 2.2. *Assume that* (2.11)–(2.15) *hold. Then Problem P has a solution.*

We conclude that Problem (2.1)–(2.10) has a weak solution.

The proof of the theorem will be given in Section 4, where the solution will be obtained as a limit of solutions to a subsequence of approximate problems that will be investigated in the next section.

3. Approximate problems

In this section we consider a sequence of regularized approximations to Problem P. The solutions are obtained by the application of a special version of an abstract existence theorem for first order evolution equations obtained by Kuttler in [8] which we now recall.

Let F and G be reflexive Banach spaces such that $F \subseteq G$, $\|\cdot\|_F \geq \|\cdot\|_G$ and F is dense in G; thus, we may write $F \subseteq G \equiv G' \subseteq F'$. Suppose that $\mathcal{B}$ is a linear, bounded, positive and symmetric operator from G to G'. Let $\mathbb{F} = L^2(a,b;F)$, $\mathbb{G} = L^2(a,b;G)$ and define $\mathbb{X} = \{w \in \mathbb{F} : (\mathcal{B}w)' \in \mathbb{F}'\}$ with the graph norm $\|w\|_{\mathbb{X}} = \|w\|_{\mathbb{F}} + \|(\mathcal{B}w)'\|_{\mathbb{F}'}$. Here the differentiation is taken in the sense of F' valued distributions. It is easy to see that $\mathbb{X}$ is a reflexive Banach space. Let $\mathcal{A}(t,\cdot)$ be an operator from F to F'. We also denote by $\mathcal{A}$ its natural extension from $\mathbb{F}$ to $\mathbb{F}'$ given by $\mathcal{A}w(t) = \mathcal{A}(t, w(t))$. Assume that

$$\mathcal{A}: \mathbb{X} \longrightarrow \mathbb{X}' \text{ is pseudomonotone,} \tag{3.1}$$

(see, e.g., [4]),

$$\mathcal{A}: \mathbb{F} \longrightarrow \mathbb{F}' \text{ is bounded,} \tag{3.2}$$

and, for some $\lambda \in \mathbb{R}$,

$$\lim_{\|w\|_{\mathbb{F}}\to\infty} (\|w\|_{\mathbb{F}})^{-1}\big(\lambda\langle \mathcal{B}w, w\rangle_{\mathbb{G}'\times\mathbb{G}} + \langle \mathcal{A}w, w\rangle_{\mathbb{F}'\times\mathbb{F}}\big) = \infty, \tag{3.3}$$

then, the following existence theorem is a special case of the result in [8].

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be as described above. Then for each $w_0 \in G$ and $l \in \mathbb{F}'$ there exists a $w \in \mathbb{X}$ satisfying*

$$(\mathcal{B}w)' + \mathcal{A}w = l \quad \text{in } \mathbb{F}',$$

$$\mathcal{B}w(0) = \mathcal{B}w_0 \quad \text{in } G'.$$

We turn to describe the sequence of regularized problems whose solutions will be given by Theorem 3.1. In order to replace the inclusion in (2.28) with an equality we regularize the norm function on $\mathbb{R}^m$. Let $(\psi^h)_{h>0}$ be a family of smooth approximations to $|\cdot|$ such that, for each $h > 0$, $\psi^h \in C^1(\mathbb{R}^m)$ is positive, convex and

$$|\nabla\psi^h(s)| \leq 2, \quad 0 \leq \langle\nabla\psi^h(s), s\rangle \quad \text{and} \quad |\psi^h(s) - |s|| \leq h.$$

for all $s \in \mathbb{R}^m$. We define the operator $J^h : E \longrightarrow E'$ by

$$\langle J^h v, w\rangle = \int_{\Gamma_C} \mu\left(\beta(v_n)_+ + p_0\right)\left(1 - \delta(\beta(v_n)_+ + p_0)\right)_+ \nabla\psi^h(v_\tau - \phi)\cdot w_\tau d\Gamma. \tag{3.4}$$

Finally, we let $R : E \longrightarrow E'$ be the Riesz map.

We now consider the following regularized problem, for each $h > 0$.

<u>*Problem P_h*</u> : Find a triplet $\{u_h, v_h, \xi_h\}$ satisfying

$$\xi_h \in \mathbf{V}, \quad \xi_h' \in \mathbf{V}', \quad u_h \in \mathbb{E}, \quad v_h \in \mathbb{E}, \quad v_h' \in \mathbb{E}', \tag{3.5}$$

$$\xi_h' + K_1\xi_h + K_2\xi_h + C_1v_h + Sv_h = Q \quad \text{in } \mathbf{V}', \tag{3.6}$$

$$v_h' + Bv_h + Dv_h + Au_h + C_2\xi_h + J^h v_h = f \quad \text{in } \mathbb{E}', \tag{3.7}$$

$$(Ru_h)' - Rv_h = 0 \quad \text{in } \mathbb{E}', \tag{3.8}$$

with the initial conditions

$$u_h(0) = u_0, \qquad v_h(0) = v_0, \qquad \xi_h(0) = \xi_0. \tag{3.9}$$

We have the following existence result for this problem.

Theorem 3.2. *Let $h > 0$. Then there exists a solution to Problem P_h.*

Proof. We fit Problem P_h into the framework of Theorem 3.1 by taking $F = V \times E \times E$ and $G = H \times H^m \times E$, $w \in G$, $w_0 \in G$ and $l \in \mathbb{F}'$ given by

$$w = \begin{pmatrix} \xi \\ v \\ u \end{pmatrix}, \qquad w_0 = \begin{pmatrix} \xi_0 \\ v_0 \\ u_0 \end{pmatrix}, \qquad l = \begin{pmatrix} Q \\ f \\ 0 \end{pmatrix}.$$

The operators $\mathcal{B} : G \to G'$ and $\mathcal{A}(t, \cdot) : F \to F'$ are chosen as

$$\mathcal{B}w = \begin{pmatrix} \xi \\ v \\ Ru \end{pmatrix}, \qquad \mathcal{A}(t, w) = \begin{pmatrix} K_1\xi + K_2\xi + C_1v + Sv \\ Bv + Dv + Au + C_2\xi + J^h v \\ -Rv \end{pmatrix}.$$

Now, the verification of the assumptions of Theorem 3.1 is routine. So we single out only two items for special mention: the pseudomonotonicity (3.1) and the coercivity (3.3). In checking the pseudomonotonicity of the the operator $\mathcal{A}$, we use the fact that $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1(t, w) = \begin{pmatrix} K_1\xi + K_2\xi + C_1v \\ Bv + Au + C_2\xi \\ -Rv \end{pmatrix}$$

gives rise to a bounded linear operator from $\mathbf{X}$ to $\mathbf{X}'$ (hence weakly continuous and thus pseudomonotone [4]), and

$$\mathcal{A}_2(t, w) = \begin{pmatrix} Sv \\ Dv + J^h v \\ 0 \end{pmatrix}$$

is a weak to norm continuous operator from $\mathbf{X}$ to $\mathbf{X}'$. To establish the latter property we recall that $\mathbf{X} = \{w \in \mathbb{F} : (\mathcal{B}w)' \in \mathbb{F}'\}$, and employ the following lemma.

Lemma 3.3. *Let $\mathbb{Y} = \{u \in \mathbb{E} : (Ru)' \in \mathbb{E}'\}$ with norm given by $\|u\|_{\mathbb{Y}} = \|u\|_{\mathbb{E}} + \|(Ru)'\|_{\mathbb{E}'}$, and let $S : \mathbb{Y} \longrightarrow \mathbf{V}'$, $D : Y \longrightarrow Y'$ and $J^h : \mathbb{Y} \longrightarrow \mathbb{Y}'$ be given by* (2.26), (2.23) *and* (3.4), *respectively. Then S, D and J^h are weak to norm continuous.*

Proof. We establish the result only for the operator S since the proof for D and J^h is similar. It is enough to show that every subsequence of a weakly convergent sequence $\{v_k\}$ in $\mathbb{Y}$, such that $v_k \rightharpoonup v$ weakly, has a further subsequence whose images under S converge to Sv. So let $\{v_{k_j}\}$ be a subsequence of $\{v_k\}$. Since $\{v_{k_j}\}_j$ is bounded in $\mathbb{Y}$ and since the injection $H^1(\Omega)^m \longrightarrow H^{1-\varepsilon}(\Omega)^m$ is compact for any $\varepsilon > 0$ ([4]), we have by a theorem of Lions [12], that $\{v_{k_j}\}_j$ is relatively compact in $L^2(0,T;H^{1-\varepsilon}(\Omega)^m)$. By the continuity of the trace map $L^2(0,T;H^{1-\varepsilon}(\Omega)^m) \longrightarrow L^2(0,T;L^2(\Gamma)^m) = \left(L^2(\Gamma \times (0,T))\right)^m$ we may, upon passing to a subsequence denoted also by $\{v_{k_j}\}_j$, suppose that $v_{k_j}(x,t) \longrightarrow v(x,t)$ for almost all $(x,t) \in \Gamma_C \times (0,T)$. Since $s_c(x,r) \le \alpha_4|r|$ and $\{v_{k_j}\}_j$ is bounded in $\mathbb{Y}$, by the continuity of the trace map we obtain that

$$\left\{ (\beta((v_{k_j})_n)_+ + p_0)\left(1 - \delta(\beta((v_{k_j})_n)_+ + p_0)\right)_+ s_c(\cdot\,, |(v_{k_j})_\tau - \phi|) \right\}_j$$

is a bounded sequence in $L^2(\Gamma_C \times (0,T))$. Consequently, it converges to

$$(\beta(v_n)_+ + p_0)\left(1 - \delta(\beta(v_n)_+ + p_0)\right)_+ s_c(\cdot\,, |v_\tau - \phi|)$$

in $L^2(\Gamma_C \times (0,T))$. Finally, using Cauchy's inequality, we deduce that $Sv_{k_j} \longrightarrow Sv$ in $\mathbb{Y}'$ when $j \to \infty$. This completes the proof.

In checking the coercivity condition (3.3), we estimate various terms using Cauchy's inequality, standard trace theorems and the following result. Here and below c represents a positive constant whose value may change from line to line, but in all cases depends only on the data and coefficients of the approximate problem.

Lemma 3.4. *Assume that* (2.11)–(2.15) *hold. Then there exists a positive constant* α *such that*

$$\langle Au, u\rangle \ge \alpha\|u\|_E^2, \qquad \langle Bv, v\rangle \ge \alpha\|v\|_E^2, \qquad \langle K_2\xi, \xi\rangle \ge \alpha\|\xi\|_V^2, \tag{3.10}$$

and also

$$\langle J^h v, v\rangle \ge -c. \tag{3.11}$$

Proof. The three first inequalities are a consequence of assumptions (2.11) followed by Korn's inequality. The latter estimate is a result of the following decomposition

$$\begin{aligned}\langle J^h v, v\rangle = &\int_{\Gamma_C} \mu(\beta(v_n)_+ + p_0)\left(1 - \delta(\beta(v_n)_+ + p_0)\right)_+ \nabla\psi^h(v_\tau - \phi)\cdot(v_\tau - \phi)d\Gamma \\ &+ \int_{\Gamma_C} \mu(\beta(v_n)_+ + p_0)\left(1 - \delta(\beta(v_n)_+ + p_0)\right)_+ \nabla\psi^h(v_\tau - \phi)\cdot\phi d\Gamma,\end{aligned}$$

in which the first integral is nonnegative and the second one is bounded. This concludes the proof of the lemma and of Theorem 3.2.

The next step in the proof of Theorem 2.2 deals with an a priori estimate on the solutions to Problem P_h.

Theorem 3.5. *Let* $\{u_h, v_h, \xi_h\}$ *be a solution to Problem* P_h *corresponding to the parameter* h. *Then there exists a positive constant* c, *independent of* h, *such that, for all* $t \in [0,T]$,

$$\|u_h(t)\|_E^2 + |v_h(t)|_{H^m}^2 + \int_0^t \|v_h(s)\|_E^2\,ds + |\xi_h(t)|_H^2 + \int_0^t \|\xi_h(s)\|_V^2\,ds \le c. \tag{3.12}$$

Proof. For the sake of simplicity, we will omit the subscript h below. We begin by letting (3.7) act on v. Thus,

$$\int_0^t \langle v', v\rangle ds + \int_0^t \langle Bv, v\rangle ds + \int_0^t \langle Dv, v\rangle ds + \int_0^t \langle Au, v\rangle ds + \int_0^t \langle J^h v, v\rangle ds$$
$$= -\int_0^t \langle C_2\xi, v\rangle ds + \int_0^t \langle f, v\rangle ds. \tag{3.13}$$

We will now estimate each of the terms in (3.13). Using Theorem 1(2) in [8] and our choice of the spaces F and G we obtain that

$$\int_0^t \langle v', v\rangle ds = \frac{1}{2}\int_0^t \left(\tfrac{d}{ds}|v(s)|^2_{H^m}\right) ds = \frac{1}{2}|v(t)|^2_{H^m} - \frac{1}{2}|v_0|^2_{H^m}. \tag{3.14}$$

Applying Lemma 3.4 yields

$$\int_0^t \langle Bv, v\rangle ds \geq \alpha \int_0^t \|v(s)\|^2_E ds, \tag{3.15}$$

$$\begin{aligned} \int_0^t \langle Au, v\rangle ds &= \frac{1}{2}\int_0^t \left(\tfrac{d}{ds}\langle Au, u\rangle\right) ds = \frac{1}{2}\langle Au(t), u(t)\rangle - \frac{1}{2}\langle Au_0, u_0\rangle \\ &\geq \frac{\alpha}{2}\|u(t)\|^2_E - c\|u_0\|^2_E, \end{aligned} \tag{3.16}$$

and

$$\int_0^t \langle J^h v, v\rangle ds \geq -c. \tag{3.17}$$

The integrals on the right in (3.13) are estimated by using Cauchy's inequality with ε:

$$\left|\int_0^t \langle C_2\xi, v\rangle ds\right| \leq c\int_0^t |\xi(s)|^2_H ds + \frac{\alpha}{4}\int_0^t \|v(s)\|^2_E ds, \tag{3.18}$$

$$\left|\int_0^t \langle f, v\rangle ds\right| \leq c\int_0^t \|f(s)\|^2_{E'} ds + \frac{\alpha}{4}\int_0^t \|v(s)\|^2_E ds. \tag{3.19}$$

Combining the estimates (3.14)–(3.19) in (3.13) and using the fact that the integral involving the operator D is nonnegative, results in

$$\|u(t)\|^2_E + |v(t)|^2_{H^m} + \int_0^t \|v(s)\|^2_E ds \leq c\left(1 + \int_0^t |\xi(s)|^2_H ds\right). \tag{3.20}$$

We turn now to the energy equation (3.6) and by letting it act on ξ we obtain

$$\int_0^t \langle \xi', \xi\rangle ds + \int_0^t \langle K_1\xi, \xi\rangle ds + \int_0^t \langle K_2\xi, \xi\rangle ds$$
$$= -\int_0^t \langle C_1 v, \xi\rangle ds - \int_0^t \langle Sv, \xi\rangle ds + \int_0^t \langle Q, \xi\rangle ds. \tag{3.21}$$

Using arguments similar to those above (see (3.14), (3.15), (3.18) and (3.19)) yields

$$\int_0^t \langle \xi', \xi \rangle ds = \frac{1}{2}|\xi(t)|_H^2 - \frac{1}{2}|\xi_0|_H^2, \tag{3.22}$$

$$\int_0^t \langle K_1\xi, \xi \rangle ds = \int_0^t \int_{\Gamma_C} k_e |\xi|^2 \, ds \geq 0, \tag{3.23}$$

$$\int_0^t \langle K_2\xi, \xi \rangle ds \geq \alpha \int_0^t \|\xi\|_V^2 \, ds, \tag{3.24}$$

$$\left| \int_0^t \langle C_1 v, \xi \rangle ds \right| \leq c \int_0^t |v(s)|_{H^m}^2 \, ds + \frac{\alpha}{4} \int_0^t \|\xi(s)\|_V^2 \, ds, \tag{3.25}$$

$$\left| \int_0^t \langle Q, \xi \rangle ds \right| \leq c \int_0^t \|Q(s)\|_{V'}^2 \, ds + \frac{\alpha}{4} \int_0^t \|\xi(s)\|_V^2 \, ds. \tag{3.26}$$

To estimate the term involving the operator S, we use the inequality

$$(\beta(v_n)_+ + p_0) \left(1 - \delta(\beta(v_n)_+ + p_0)\right)_+ \leq \frac{1}{\delta},$$

assumption (2.14) and Cauchy's inequality with ε, and deduce:

$$\left| \int_0^t \langle Sv, \xi \rangle \, ds \right| \leq c \int_0^t \|v(s)\|_E^2 \, ds + \frac{\alpha}{4} \int_0^t \|\xi(s)\|_V^2 \, ds. \tag{3.27}$$

Combining the estimates (3.22)–(3.27) in (3.21) yields

$$|\xi(t)|_H^2 + \int_0^t \|\xi(s)\|_V^2 \, ds \leq c \left(1 + \int_0^t |v(s)|_{H^m}^2 \, ds + \int_0^t \|v(s)\|_E^2 ds \right). \tag{3.28}$$

It follows now from (3.20) that

$$\int_0^t \|v(s)\|_E^2 \, ds \leq c \left(1 + \int_0^t |\xi(s)|_H^2 \, ds \right).$$

Inserting this estimate in (3.28) we deduce

$$|\xi(t)|_H^2 + \int_0^t \|\xi(s)\|_V^2 \, ds \leq c \left(1 + \int_0^t |v(s)|_{H^m}^2 \, ds + \int_0^t |\xi(s)|_H^2 \, ds \right). \tag{3.29}$$

Finally, adding the inequalities (3.20) and (3.29) we obtain

$$\begin{aligned} &\|u(t)\|_E^2 + |v(t)|_{H^m}^2 + \int_0^t \|v(s)\|_E^2 \, ds + |\xi(t)|_H^2 + \int_0^t \|\xi(s)\|_V^2 \, ds \\ &\leq c \left(1 + \int_0^t |v(s)|_{H^m}^2 \, ds + \int_0^t |\xi(s)|_H^2 \, ds \right), \end{aligned} \tag{3.30}$$

which implies that

$$|v(t)|_{H^m}^2 + |\xi(t)|_H^2 \leq c \left(1 + \int_0^t |v(s)|_{H^m}^2 \, ds + \int_0^t |\xi(s)|_H^2 \, ds \right).$$

Using now Gronwall's inequality yields

$$|v(t)|_{H^m}^2 + |\xi(t)|_H^2 \leq c. \tag{3.31}$$

The a priori estimate (3.12) is now a consequence of (3.30) and (3.31).

4. Proof of Theorem 2.2

We prove the existence of a solution to Problem P as a limit of a subsequence of the solutions to the regularized Problem P_h obtained above. It follows from Theorem 3.5 that for a given set of initial conditions the family of solutions $\{u_h, v_h, \xi_h\}_h$ is bounded in $\mathbb{E} \times \mathbb{E} \times \mathbf{V}$. From this and (3.6)–(3.8) it follows that $\{Ru'_h, v'_h, \xi'_h\}_h$ is bounded in $\mathbb{E}' \times \mathbb{E}' \times \mathbf{V}'$. Consequently, there exists a weak limit point $(u, v, \xi) \in \mathbb{E} \times \mathbb{E} \times \mathbf{V}$ and a subsequence of parameters $\{h_l\}$ such that $h_l \to 0$ when $l \to \infty$ and such that the following limit processes take place when $l \to \infty$:

$$\begin{array}{rcll} u_l & \rightharpoonup & u & \text{weakly in } \mathbb{E}, \\ v_l & \rightharpoonup & v & \text{weakly in } \mathbb{E}, \\ \xi_l & \rightharpoonup & \xi & \text{weakly in } \mathbf{V}, \\ \xi'_l & \rightharpoonup & \xi' & \text{weakly in } \mathbf{V}', \\ v'_l & \rightharpoonup & v' & \text{weakly in } \mathbb{E}', \\ Ru'_l & \rightharpoonup & Ru' & \text{weakly in } \mathbb{E}'. \end{array}$$

With these results and Lemma 3.3 we may pass to the limit in (3.6) and (3.7) in all terms except the one involving J^{h_l}. For this term we may suppose, by passing to a subsequence that, for some $\gamma \in \mathbb{E}'$, $J^{h_l}(v_l) \rightharpoonup \gamma$ weakly in $\mathbb{E}'$. It only remains to show that $\gamma \in \partial_2 j(v, v)$. Toward this end, we note that when $l \to \infty$

$$(\beta((v_l)_n)_+ + p_0)\,(1 - \delta(\beta((v_l)_n)_+ + p_0))_+ \longrightarrow (\beta(v_n)_+ + p_0)\,(1 - \delta(\beta(v_n)_+ + p_0))_+,$$

strongly in $L^{\frac{4}{3}}(\Gamma_C \times (0,T))$. On the other hand, for any $w \in \mathbb{E}$ we have

$$\psi^{h_l}(w_\tau - v_\tau + (v_l)_\tau - \phi) \longrightarrow |w_\tau - \phi| \quad \text{in } L^4(\Gamma_C \times (0,T)),$$

$$\psi^{h_l}((v_l)_\tau - \phi) \longrightarrow |v_\tau - \phi| \quad \text{in } L^4(\Gamma_C \times (0,T)).$$

Then it follows from the convexity of ψ^{h_l} and Hölder's inequality that

$$\begin{aligned} \langle \gamma, w-v \rangle &= \lim_{l\to\infty} \langle J^{h_l} v_l, w-v \rangle \\ &= \lim_{l\to\infty} \int_0^T\!\!\int_{\Gamma_C} \mu\,(\beta((v_l)_n)_+ + p_0)\,(1 - \delta(\beta((v_l)_n)_+ + p_0))_+ \nabla\psi^{h_l}((v_l)_\tau - \phi)\cdot(w_\tau - v_\tau)\,d\Gamma dt \\ &\le \lim_{l\to\infty} \int_0^T\!\!\int_{\Gamma_C} \mu\,(\beta((v_l)_n)_+ + p_0)\,(1 - \delta(\beta((v_l)_n)_+ + p_0))_+ \\ &\qquad\qquad \times \left[\psi^{h_l}(w_\tau - v_\tau + (v_l)_\tau - \phi) - \psi^{h_l}((v_l)_\tau - \phi)\right] d\Gamma dt \\ &\le \int_0^T\!\!\int_{\Gamma_C} \mu\,(\beta(v_n)_+ + p_0)\,(1 - \delta(\beta(v_n)_+ + p_0))_+ \left[|w_\tau - \phi| - |v_\tau - \phi|\right] d\Gamma dt. \end{aligned}$$

This shows that $\gamma \in \partial_2 j(v, v)$ which completes the proof of Theorem 2.2.

5. Uniqueness

In this section we introduce additional, reasonably mild, assumptions on the function s_c and prove that the solution to Problem P is unique when $|\mu|_{L^\infty(\Gamma_C)}$ is sufficiently small. Estimating the allowed coefficient μ is left open.

Theorem 5.1. *Assume that assumptions* (2.11)–(2.15) *hold and that s_c satisfies the additional boundedness and Lipschitz conditions*

$$s_c(\cdot\,,r) \le \alpha_5 \quad \textit{for all} \;\; r \in \mathbb{R}, \tag{5.1}$$

$$|s_c(\cdot\,,r_1) - s_c(\cdot\,,r_1)| \le \alpha_5 |r_1 - r_2| \quad \textit{for all} \;\; r_1,\, r_2 \in \mathbb{R}, \tag{5.2}$$

where α_5 is a positive constant. Then, the solution to Problem P is unique when $|\mu|_{L^\infty(\Gamma_C)}$ is sufficiently small.

Proof. Let $\{u_1, \xi_1\}$ and $\{u_2, \xi_2\}$ be two solutions to Problem P corresponding to the same data. We substitute into (2.28) u_1 for u and let the resulting expression act on $u_1' - u_2'$ and then substitute into (2.28) u_2 for u and let the resulting expression act on $u_2' - u_1'$. Adding the two inequalities and using the notation $u = u_1 - u_2$, $v = u_1' - u_2'$, $\xi = \xi_1 - \xi_2$ and $\lambda_i = (\beta((v_i)_n)_+ + p_0)\big(1 - \delta(\beta((v_i)_n)_+ + p_0)\big)_+$ for $i = 1, 2$, yields

$$\int_0^t \langle v', v\rangle ds + \int_0^t \langle Bv, v\rangle ds + \int_0^t \langle Dv, v\rangle ds + \int_0^t \langle Au, v\rangle ds + \int_0^t \langle C_2\xi, v\rangle ds$$
$$\le \int_0^t\!\!\int_{\Gamma_C} \mu(\lambda_1 - \lambda_2)\big(|(v_1)_\tau - \phi| - |(v_2)_\tau - \phi|\big) d\Gamma ds. \tag{5.3}$$

We estimate the terms on the left-hand side in (5.3) using the same arguments as those used to estimate the corresponding terms in (3.13) (see (3.14)–(3.16) and (3.18)). The integral on the right-hand side is estimated by taking into account $\mu|\lambda_1 - \lambda_2| \le c|\mu|_{L^\infty(\Gamma_C)}|v_1 - v_2|$; thus,

$$\int_0^t\!\!\int_{\Gamma_C} \mu(\lambda_1 - \lambda_2)\big(|(v_1)_\tau - \phi| - |(v_2)_\tau - \phi|\big) d\Gamma ds \le c|\mu|_{L^\infty(\Gamma_C)} \int_0^t \|v(s)\|_E^2\, ds.$$

Combining these estimates in (5.3), when $|\mu|_{L^\infty(\Gamma_C)}$ is sufficiently small, yields

$$\|u(t)\|_E^2 + |v(t)|_{H^m}^2 + \int_0^t \|v(s)\|_E^2 ds \le c \int_0^t |\xi(s)|_H^2\, ds. \tag{5.4}$$

Next, we substitute ξ_1 and ξ_2 into (2.27) in turn, subtract the resulting equations and let the result act on the difference $\xi = \xi_1 - \xi_2$ to obtain

$$\int_0^t \langle \xi', \xi\rangle ds + \int_0^t \langle K_1\xi, \xi\rangle ds + \int_0^t \langle K_2\xi, \xi\rangle ds$$
$$= -\int_0^t \langle C_1 v, \xi\rangle ds - \int_0^t \langle Sv_1 - Sv_2, \xi\rangle ds. \tag{5.5}$$

All the terms in this equality, except the last one, are estimated in the same way as (3.22)–(3.25). To estimate the term involving S we use (2.26) and obtain

$$\left|\int_0^t \langle Sv_1 - Sv_2, \xi\rangle ds\right| = \left|\int_0^t\int_{\Gamma_C} \mu(\lambda_1 s_c(x, |(v_1)_\tau - \phi|) - \lambda_2 s_c(x, |(v_2)_\tau - \phi|))\xi d\Gamma ds\right|,$$
$$\leq \int_0^t\int_{\Gamma_C} \mu|\lambda_1 - \lambda_2| s_c(x, |(v_1)_\tau - \phi|)|\xi| d\Gamma ds$$
$$+ \int_0^t\int_{\Gamma_C} \mu\lambda_2 \big| s_c(x, |(v_1)_\tau - \phi|) - s_c(x, |(v_1)_\tau - \phi|)\big| |\xi| d\Gamma ds.$$

Since $\mu|\lambda_1 - \lambda_2| \leq c|v_1 - v_2|$, using assumptions (5.1) and (5.2) followed by Cauchy's inequality with ε in the two integrals on the right-hand side, we obtain

$$\left|\int_0^t \langle Sv_1 - Sv_2, \xi\rangle\, ds\right| \leq \left(c\int_0^t \|v(s)\|_E^2 ds + \frac{\alpha}{4}\int_0^t \|\xi(s)\|_V^2\, ds\right).$$

Combining these estimates in (5.5) yields

$$|\xi(t)|_H^2 + \int_0^t \|\xi(s)\|_V^2\, ds \leq c\left(\int_0^t |v(s)|_{H^m}^2\, ds + \int_0^t \|v(s)\|_E^2\, ds\right). \tag{5.6}$$

We deduce from (5.4) that

$$\int_0^t \|v(s)\|_E^2\, ds \leq c\int_0^t |\xi(s)|_H^2\, ds,$$

which, when applied in (5.6), gives

$$|\xi(t)|_H^2 + \int_0^t \|\xi(s)\|_V^2\, ds \leq c\left(\int_0^t |v(s)|_{H^m}^2\, ds + \int_0^t |\xi(s)|_H^2\, ds\right). \tag{5.7}$$

Finally, we add (5.4) and (5.7) and obtain

$$\|u(t)\|_E^2 + |v(t)|_{H^m}^2 + |\xi(t)|_H^2 + \int_0^t \|v(s)\|_E^2 ds + \int_0^t \|\xi(s)\|_V^2 ds$$
$$\leq c\left(\int_0^t |v(s)|_{H^m}^2 ds + \int_0^t |\xi(s)|_H^2 ds\right),$$

which implies

$$|v(t)|_{H^m}^2 + |\xi(t)|_H^2 \leq c\left(\int_0^t |v(s)|_{H^m}^2 ds + \int_0^t |\xi(s)|_H^2 ds\right).$$

Applying now Gronwall's inequality yields $v(t) = 0$ and $\xi(t) = 0$, and then it follows from (5.4) that $u(t) = 0$. The proof of Theorem 5.1 is complete.

Remark 5.2. Under the additional assumptions (5.1) and (5.2) we can establish, using the arguments of this section, that for $|\mu|_{L^\infty(\Gamma_C)}$ sufficiently small the solution to each of the approximate problems P_h is unique, too.

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Simulations of Beam Vibrations between Stops

Y. Dumont, D. Goeleven, M. Rochdi et M. Shillor

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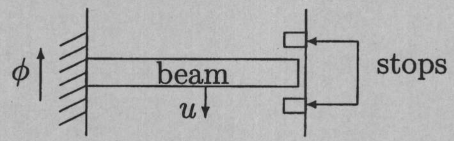
Descriptif

Ce travail est une présentation de simulations numériques concernant les vibrations d'une barre élastique entre deux obstacles.

On considère une barre élastique horizontale fixée par l'une de ses extrémités à un mur oscillant verticalement avec une vitesse ϕ. L'oscillation du mur entraîne l'oscillation de toute la barre. Les mouvements de l'autre extrémité restent bloqués entre deux obstacles fixes. La figure suivante illustre le modèle étudié :



La barre élastique est modélisée par l'intervalle $0 \leq x \leq 1$ alors que g_1 et g_2 ($g_1 < 0 < g_2$) sont les positions verticales des deux obstacles. On note $\Omega_T = (0,1) \times (0,T)$ pour $T > 0$. Soit $u = u(x,t)$ le déplacement vertical de la barre et $\sigma = \sigma(x,t)$ les contraintes dans la direction verticale. Le problème étudié ici se modélise en partie de la façon suivante :
Trouver une fonction u telle que :

$$u_{tt} - \sigma_x = f,$$
$$\sigma(x,t) = -k^2 u_{xxx}(x,t),$$
$$u(0,t) = \phi(t), \quad u_x(0,t) = 0 \text{ for } 0 < t \leq T,$$
$$g_1 \leq u(1,t) \leq g_2,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x).$$

Ici, f représente la densité de forces appliquées sur la barre (par unité de longueur), les indices en x et en t indiquent respectivement les dérivées partielles en espace et en temps et k^2 représente le module d'élasticité.

Pour les conditions aux limites au niveau de l'éxtrémité droite de la barre, on considère deux types de conditions : les conditions de Signorini ou les conditions de compliance normale. Les conditions de Signorini s'écrivent sous la forme

$$-\sigma(1,t) \in \partial\chi(u(1,t)),$$

pour $0 < t \leq T$, où $\partial\chi$ le sous-différentiel de la fonction indicatrice $\chi = \chi_{[g_1,g_2]}$ de l'intervalle $[g_1, g_2]$, i.e.,

$$\partial\chi(r) = \begin{cases} [0, +\infty) & \text{si} \quad r = g_2, \\ 0 & \text{si} \quad g_1 < r < g_2, \\ (-\infty, 0] & \text{si} \quad r = g_1. \end{cases}$$

Les conditions de compliance normale s'écrivent sous la forme

$$\sigma(1,t) = -\kappa\left[(u(1,t) - g_2)_+ - (g_1 - u(1,t))_+\right],$$

où κ est un coefficient d'amortissement et $(f)_+ = \max\{f, 0\}$.

L'étude théorique ayant été faite dans un travail antérieur (cf. références de l'article), le but de ce travail concerne l'étude numérique du problème suivant les conditions de compliance normale. Pour l'algorithme, une méthode des lignes a été considérée. Elle est basée sur la discrétisation du système d'équations aux dérivées partielles obtenu suivant la variable espace x, ce qui le transforme en un système d'équations différentielles ordinaires en temps. La dernière partie de ce travail porte sur des simulations numériques suivant différentes paramètres. La transformée de Fourier rapide (FFT) a été utilsée pour mettre en évidence les bruits générés au niveau du contact de la barre avec les deux obstacles.

Simulations of Beam Vibrations between Stops *

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Abstract. We present numerical simulations of vibrations of an elastic beam between two stops. The beam is clamped at one end to a vibrating device; the motion of the other end is restricted by two stops. The contact is modeled by the normal compliance condition which describes nonlinear flexible stops, and approximates the Signorini condition for rigid stops. We use a finite difference scheme for the numerical approximations of the behavior of the model. The numerical simulations indicate that the stops may cause very complicated oscillations. We also use the FFT to describe the noise characteristics of the system.

AMS(MOS) classification: 73T05; 35L35, 35L85.

Keywords: Dynamic contact, elastic beam, Signorini condition, normal compliance, constrained vibrations.

1. Introduction

This research is a continuation of our investigation, started in (Kuttler and Shillor, 1998), of the vibration characteristics of contacting structures. We study numerically the vibrations of an elastic beam between two rigid stops. The model and the existence of its weak solutions can be found in (Kuttler and Shillor, 1998). Here we describe a finite difference numerical algorithm for the problem, and present numerical simulations of the solutions.

There is considerable interest in industry in the dynamic vibrations of mechanical systems. In particular, it is recognized in the automotive industry that the noise and vibration characteristics of cars are an important factor in the product customer satisfaction. Indeed, appreciable effort has been made recently in designing automotive components to reduce undesired or disturbing noise. For instance, when the mounting of the components on the engine is not perfect, the motion in the clearances leads to dynamic contact, which may generate this undesired noise.

The purpose of this study is to investigate contact noise characteristics by simulating numerically a dynamic model for constrained

* This paper is dedicated to the memory of P. D. Panagiotopoulos

vibrations in a simple setting which avoids mathematical and numerical difficulties associated with two- or three-dimensions. The problem we consider is one-dimensional and describes an elastic beam that is clamped at one end to a vibrating device while the other end oscillates between two rigid or flexible stops. Such a setting was considered by Moon and Shaw (Moon and Shaw, 1983) (see also (Moon, 1992) and references therein). There, the mathematical problem was considerably simplified by reducing it to a nonlinear ordinary differential equation. They showed that even such an approximation can exhibit complicated behavior under periodic forcing. Indeed, it can oscillate periodically, quasiperiodically or chaotically. Since we consider the full problem, we expect it also to exhibit such varied types of behavior.

Related results for elastic rods can be found in (Schatzman and Bercovier, 1989) and a simplified model for vibrations in (Paoli and Schatzman, 1998) and in (Schatzman, 1998). In (Kuttler, Park, Shillor and Zhang, 1999) a model for the transmission of vibrations between two beams, across a mechanical joint with a clearance, was investigated. The model was formulated in a variational form and the existence of weak solutions established. Then, the solutions were numerically simulated using the method of lines. The noise characteristics of the system were simulated too, by using the FFT.

The model is described in Section 2, following (Kuttler and Shillor, 1998). The contact is modeled with either the Signorini condition which represents two perfectly rigid stops (see, e.g., (Duvaut and Lions, 1976; Kikuchi and Oden, 1988)), or with the normal compliance condition (see, e.g., (Martins and Oden, 1983; Klarbring, Mikelic and Shillor, 1988; Andrews, Kuttler and Shillor, 1997; Andrews, Shillor and Wright, 1996; Kuttler, Park, Shillor and Zhang, 1999) and references therein), which describes flexible stops with spring–like nonlinear reaction. Each problem is set in a variational form, and the existence results of (Kuttler and Shillor, 1998) extend to these problems, guaranteeing the existence of weak solutions. We quote these in Section 2, as well. In Section 3 we present the numerical algorithm which is based on the finite differences approximations of the beam equation, and time retarding at the contacting end of the beam. In particular we use the method of lines to convert our partial differential equations problem into a system of ordinary differential equations; then, we integrate the system by using a backward time discretization. Our numerical simulations are presented in Section 4. We show that the vibrations of the beam can exhibit very complicated patterns. Moreover, using the FFT we show that the contact excites frequencies which are higher than the driving frequency.

2. The model

In this section, we present the classical model and its variational formulation, following (Kuttler and Shillor, 1998). Then we quote the existence results obtained there. For the sake of simplicity we consider only the elastic beam and omit the viscosity effects which were included in (Kuttler and Shillor, 1998). The mechanical setting is depicted in Fig.1. A uniform elastic beam is clamped at its left end to an oscillating device. The motion of its right end is constrained by two obstacles – the stops.

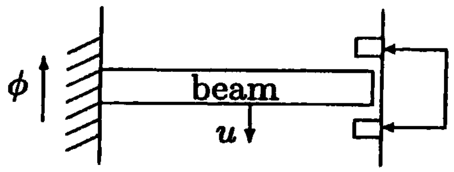


Fig. 1. The physical setting.

The area-center of gravity of the beam in its (stress free) reference configuration coincides with the interval $0 \leq x \leq 1$; g_1 and g_2 ($g_1 < 0 < g_2$) are the positions of the stops. We set $\Omega_T = (0,1) \times (0,T)$, for $T > 0$. Let $u = u(x,t)$ represent the vertical displacement of the beam, and $\sigma(x,t)$ the stress in the vertical direction. Then, the equation of motion, in nondimensional form, is

$$u_{tt} - \sigma_x = f, \tag{1}$$

in Ω_T, where f denotes the density (per unit length) of applied forces, and the subscripts x and t indicate partial derivatives. The material is assumed to be elastic and of constant cross section, thus $\sigma(x,t) = -k^2 u_{xxx}(x,t)$, where k^2 is the scaled elastic modulus. The initial conditions are

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x), \tag{2}$$

for $0 \leq x \leq 1$. The beam is rigidly attached at its left end,

$$u(0,t) = \phi(t) \qquad \text{and} \qquad u_x(0,t) = 0, \text{ for } 0 < t \leq T. \tag{3}$$

Here $\phi = \phi(t)$ represents the motion of the supporting device. In this paper we present simulation results when the device oscillates periodically (see also (Moon, 1992)). At the free end we use either the classical Signorini condition (see, e.g., (Duvaut and Lions, 1976; Kikuchi and Oden, 1988), or the normal compliance condition (see, e.g., (Kikuchi and Oden, 1988; Klarbring, Mikelic and Shillor, 1988; Kuttler and

Shillor, 1998; Kuttler, Park, Shillor and Zhang, 1999)). In the numerical simulations we use the latter as an approximation of the former. The Signorini condition describes the idealized case of completely rigid stops. The displacement $u(1,t)$ of the beam's end is constrained between the stops, thus,

$$g_1 \leq u(1,t) \leq g_2. \tag{4}$$

Then either the end is free and $\sigma(1,t) = 0$; or it is in contact and the stress is opposite to the displacement,

$$\sigma(1,t) \leq 0 \quad \text{if} \quad u(1,t) = g_2; \qquad \sigma(1,t) \geq 0 \quad \text{if} \quad u(1,t) = g_1.$$

Since only one of the cases can take place we require that

$$\sigma(1,t)(g_2 - u(1,t))_+(u(1,t) - g_1)_+ = 0,$$

where $(f)_+ = \max\{f, 0\}$ is the positive part of f. The condition may be restated as follows. Let $\chi = \chi_{[g_1,g_2]}$ be the indicator function of the interval $[g_1, g_2]$, i.e. $\chi(r) = 0$ when $r \in [g_1, g_2]$ and $\chi(r) = +\infty$, otherwise. Then Signorini's condition, in addition to (4), states that

$$-\sigma(1,t) \in \partial\chi(u(1,t)), \tag{5}$$

for $0 < t \leq T$, where $\partial\chi$ is the subdifferential of χ, i.e.,

$$\partial\chi(r) = \begin{cases} [0, +\infty) & \text{if} \quad r = g_2, \\ 0 & \text{if} \quad g_1 < r < g_2, \\ (-\infty, 0] & \text{if} \quad r = g_1. \end{cases}$$

The second condition we use is the *normal compliance* condition, where the stops are assumed to be flexible, with resistance force proportional to the deflection,

$$\sigma(1,t) = -\kappa\left[(u(1,t) - g_2)_+ - (g_1 - u(1,t))_+\right]. \tag{6}$$

The stops behave as one-sided nonlinear springs, with spring constant κ. Conditions (4) and (5) can be obtained from (6) in the limit when $\kappa \to +\infty$, and therefore, we may employ (6) as a regularization of the Signorini condition. Finally, we assume that no moments act on the free end,

$$u_{xx}(1,t) = 0. \tag{7}$$

The classical statement of the problem of *vibrations of a beam between two stops* with Signorini's unilateral condition is: *Find a function u such that* (1)–(5) *and* (7) *hold.*

The classical problem with normal compliance is: *Find a function u such that* (1)–(3), (6) *and* (7) *hold.*

Both condition impose regularity ceilings which, generally, preclude the existence of classical solutions to dynamic problems containing them. Thus, it is natural to consider weak or variational inequality formulations of the problems. To that end we introduce the following spaces and notation. For definitions of any unexplained notation we refer the reader to (Adams, 1975).

Let $H = L^2(0,1)$ and $V = \{w \in H^2(0,1) : w(0) = w'(0) = 0\}$, then $V \subseteq H = H' \subseteq V'$, where V' is the topological dual of V. We denote by $(\cdot,\cdot)$ the inner product in H and by $\langle\cdot,\cdot\rangle$ the duality pairing between V and V'. The associated norm on H is denoted by $|\cdot|_H$ and the one on V by $||\cdot||_V$. To incorporate Signorini's conditions (4) and (5) we set

$$K = \{w \in V : g_1 \leq w(1) \leq g_2\},$$

which is the convex set of all admissible displacements. Next, to have zero boundary condition at $x = 0$ we change the dependent variable to $u(\cdot,t) = u(\cdot,t) - (1-x)\phi(t)$, then the forcing function f changes to $f(\cdot,t) = f(\cdot,t) - (1-x)\phi''(t)$. Here and below " $'$ " denotes the time derivative.

The following existence result for the Signorini problem has been established in (Kuttler and Shillor, 1998):

Theorem 1. Assume that $f \in L^2(0,T;H)$, $\phi \in H^2(0,T)$, $u_0 \in K$ and $v_0 \in H$. Then, for each $T < \infty$, there exists a function u satisfying

$$u \in L^2(0,T;V),\ u' \in L^2(0,T;H),\ u(t,\cdot) \in K,\ u(0) = u_0,$$

and the variational inequality,

$$\int_0^T\int_0^1 u'(w'-u')\,dxdt + k^2\int_0^T\int_0^1 u_{xx}(w_{xx}-u_{xx})\,dxdt$$

$$+\int_0^1 v_0(w(0)-u_0)\,dx \geq -\int_0^T\int_0^1 f(w-u)\,dxdt, \qquad (8)$$

which holds for all $w \in L^2(0,T;V)$ with $w' \in L^2(0,T;H)$, $w(t,\cdot) \in K$ and $w(T,\cdot) = u(T,\cdot)$.

The proof of the theorem was based on a priori estimates for the solutions of a family of problems with the normal compliance condition. We conclude that Problem (1)–(5) and (7) has a weak solution.

For the problem with normal compliance the following result of (Kuttler and Shillor, 1998) holds.

Theorem 2. Assume that the data satisfy the above conditions. Then, the problem (1) – (3), (6) and (7) has a weak solution for each $T < \infty$, such that

$$u \in L^\infty(0,T;V),\ u(0)=u_0,\quad u' \in L^\infty(0,T;H),\ u'' \in L^2(0,T;V'),$$

and for each $w \in L^2(0,T;V)$ with $w' \in L^2(0,T;H)$, $w(t,\cdot) \in K$ and $w(\cdot,T) = u(\cdot,T)$, the following equality holds

$$\int_0^T\int_0^1 u'(w'-u')\,dxdt + k^2\int_0^T\int_0^1 u_{xx}(w_{xx}-u_{xx})\,dxdt$$

$$-\kappa\int_0^T \left[(u(1,t)-g_2)_+ - (g_1-u(1,t))_+\right](w(1,t)-u(1,t))dt+$$

$$+\int_0^1 v_0(w(0)-u_0)\,dx = -\int_0^T\int_0^1 f(w-u)\,dxdt. \tag{9}$$

When the material is considered viscoelastic the results in (Kuttler and Shillor, 1998) show that, in addition, the solution of the problem with normal compliance is unique and better behaved.

3. Numerical algorithm

Assuming enough regularity on the solution, we present a numerical algorithm for the problem with normal compliance. We use Rothe's method (Kacur, 1985), or the so-called method of lines, to discretize the problem. It is based on discretizing the partial differential equations system with respect to the x variable and in this way transforming it into a system of ordinary differential equations in time. We assume, for the sake of simplicity, that there are no external forces, thus, $f = 0$. Next, we make the following change of variable $v(x,t) = u(x,t) - \phi(t)(1-x)$, which makes the displacements at $x = 0$ vanish, but does not affect the conditions at $x = 1$. The problem can be written as

$$\begin{aligned}
\frac{\partial^2 v(x,t)}{\partial t^2} + k^2\frac{\partial^4 v(x,t)}{\partial x^4} &= -\phi''(t)(1-x),\\
v(x,0) &= u_0(x) - \phi(0)(1-x),\\
v_t(x,0) &= v_0(x) - \phi'(0)(1-x),\\
v_x(0,t) &= \phi(t),\\
v_t(0,t) &= 0,\\
k^2 v_{xxx}(1,t) &= \kappa\left((u(1,t)-g_2)_+ - (g_1-u(1,t))_+\right),\\
v_{xx}(1,t) &= 0.
\end{aligned}$$

We denote by V the vector $(v_t, v_{xx}) = (v_t(x,t), v_{xx}(x,t))$. Under sufficient regularity on the solution, the equation in the interior of the beam can be written as

$$\begin{aligned}\frac{\partial v_t}{\partial t} &= -k^2\frac{\partial^2 v_{xx}}{\partial x^2} - \phi''(t)(1-x),\\ \frac{\partial v_{xx}}{\partial t} &= \frac{\partial^2 v_t}{\partial x^2}.\end{aligned}$$

That is

$$\frac{\partial V}{\partial t} = \frac{\partial^2}{\partial x^2}(AV) + F,$$

where

$$A = \begin{pmatrix} 0 & -k^2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} -\phi''(t)(1-x) \\ 0 \end{pmatrix}.$$

Next, we discretize the problem. Let $\Delta t = T/N$ be the time step and let $h = 1/M$ be the space step, then the grid points are given by $(x_i = ih,\ t_n = n\Delta t)$, where, here and below, $i = 1, \cdots, M-1$ and $n = 0, \cdots, N$. We first discretize with respect to the space variable using the classical finite difference scheme. We obtain a system of M ordinary differential equations

$$\frac{\partial V_i}{\partial t} = \frac{1}{h^2}A(V_{i+1} - 2V_i + V_{i-1}) + F_i. \tag{10}$$

To integrate this ODE system we use a second order backward finite difference approximation. Let $V_i^n = \left(v_{t,i}^n,\ v_{xx,i}^n\right)$ denote the approximation of V at the grid point $(ih, n\Delta t)$; we have

$$\frac{3}{2\Delta t}\left(V_i^n - \frac{4}{3}V_i^{n-1} + \frac{1}{3}V_i^{n-2}\right) = \frac{1}{h^2}A\left(V_{i+1}^n - 2V_i^n + V_{i-1}^n\right) + F_i^n, \tag{11}$$

where $F_i^n = F(x_i, t_n)$. We set $\lambda = \frac{\Delta t}{h^2}$ and rewrite (11) in the following way

$$\begin{aligned}&-\frac{2\lambda}{3}AV_{i-1}^n + \left(I + \frac{4\lambda}{3}A\right)V_i^n - \frac{2\lambda}{3}AV_{i+1}^n\\ &\qquad = \frac{4}{3}V_i^{n-1} - \frac{1}{3}V_i^{n-2} + 2\frac{\Delta t}{3}F_i^n.\end{aligned}$$

The discretized initial conditions are $v_{t,i}^0 = v_{0,i} - \phi'(0)(1-ih)$ and $v_{xx,i}^0 = u_{0_{xx,i}}$ The boundary conditions $v_{xx}(1,t) = 0$ and $v_t(0,t) = 0$ can be expressed as $v_{xx,N}^n = 0$ and $v_{t,0}^n = 0$.

Next, we derive a boundary condition for v_{xx} at $x = 0$. Recalling that $v_x(0,t) = \phi(t)$ and using Taylor expansions around 0 at $x = h$ and $x = 2h$, we obtain

$$v_t(h,t) = h\phi'(t) + v_{xxt}(0,t)\frac{h^2}{2} + v_{xxxt}(0,t)\frac{h^3}{3!} + O(h^4),$$

$$v_t(2h,t) = 2h\phi'(t) + v_{xxt}(0,t)\,2h^2 + v_{xxxt}(0,t)\frac{8h^3}{3!} + O(h^4).$$

Then,

$$8v_t(h,t) - v_t(2h,t) = 6h\phi'(t) + 2h^2 v_{xxt}(0,t) + O(h^4),$$

from which we deduce

$$v_{xxt}(0,t) = \frac{1}{2h^2}\left(8v_t(h,t) - v_t(2h,t)\right) - \frac{3}{h}\phi'(t) + O(h^2).$$

Using again the second order backward differentiation approximation, we find

$$v_{xx,0}^{n+1} - \frac{8\Delta t}{3h^2}v_{t,1}^{n+1} + \frac{\Delta t}{3h^2}v_{t,2}^{n+1} = \frac{4}{3}v_{xx,0}^{n} - \frac{1}{3}v_{xx,0}^{n-1} - \frac{2\Delta t}{h}(\phi')^{n+1},$$

where $(\phi')^{n+1} = \phi'(t_{n+1})$. We turn to the discretization of the contact condition. We model the contact with the normal compliance, and as explained above, we may also think of it as an approximation for the Signorini condition. First, we use Taylor expansions around 1 at $x = 1-h$ and $x = 1-2h$ to discretize $v_{xxxx}(1,t)$, thus,

$$v_{xx}(1-h,t) = v_{xx}(1,t) - hv_{xxx}(1,t) + \frac{h^2}{2}v_{xxxx}(1,t)$$
$$-\frac{h^3}{3!}v_{xxxxx}(1,t) + O(h^4),$$

$$v_{xx}(1-2h,t) = v_{xx}(1,t) - 2hv_{xxx}(1,t) + \frac{4h^2}{2}v_{xxxx}(1,t)$$
$$-\frac{8h^3}{3!}v_{xxxxx}(1,t) + O(h^4).$$

Using the assumption $v_{xx}(1,t) = 0$, we have

$$8v_{xx}(1-h,t) - v_{xx}(1-2h,t) = -6hv_{xxx}(1,t) + 2h^2 v_{xxxx}(1,t) + O(h^4).$$

We deduce

$$v_{xxxx}(1,t) = \frac{1}{2h^2}\left(8v_{xx}(1-h,t) - v_{xx}(1-2h,t)\right) + \frac{3}{h}v_{xxx}(1,t) + O(h^2).$$

Using now the equation

$$\frac{\partial v_t(1,t)}{\partial t} = -k^2 v_{xxxx}(1,t),$$

and the discretization with respect to x, we obtain

$$\frac{\partial v_{t,N}}{\partial t} = \frac{-k^2}{2h^2}\left(8v_{xx,N-1} - v_{xx,N-2}\right) - \frac{3k^2}{h} v_{xxx,N}.$$

Recalling that $\sigma(1,t) = -k^2 v_{xxx}(1,t)$, we have

$$\frac{\partial v_{t,N}}{\partial t} = \frac{-k^2}{2h^2}\left(8v_{xx,N-1} - v_{xx,N-2}\right) + \frac{3}{h}\sigma_N,$$

where the contact pressure is given by

$$\sigma_N = -\kappa\left((v_N - g_2)_+ - (g_1 - v_N)_+\right),$$

and v_N is the displacement at the end of the beam. Below, we linearize the system by using the value of this term computed from the previous time step, i.e., we use time retarding of the argument.

Then, using again the same approximation, we obtain

$$v_{t,N}^{n+1} + \frac{8\Delta t k^2}{3h^2} v_{xx,N-1}^{n+1} - \frac{k^2 \Delta t}{3h^2} v_{xx,N-2}^{n+1} = \frac{4}{3} v_{t,N}^n - \frac{1}{3} v_{t,N}^{n-1} + \frac{2\Delta t}{h}\sigma_N^{n+1}.$$

To summarize the algorithm, at each time step we solve the following linearized system:

$$BV^n = \frac{4}{3}\mathrm{I}V^{n-1} - \frac{1}{3}\mathrm{I}V^{n-2} + G^n, \tag{12}$$

where $B \in \mathbb{R}^{2N\times 2N}$ is given below, $\mathbf{I}$ is the $2N \times 2N$ identity matrix and $V^n \in \mathbb{R}^{2N}$ is the column vector

$$V^n = (v_{xx,0}^n \;\; V_1^n \;\; V_2^n \;\; \dots \;\; \dots \;\; \dots \;\; \dots \;\; V_{N-2} \;\; V_{N-1}^n \;\; v_{t,N}^n)^T.$$

We note that $v_{t,0}^n$ and $v_{xx,N}^n$ are omitted, since both vanish. Next, $G^n \in \mathbb{R}^{2N}$ is given by

$$G^n = \Big(\tfrac{-2\Delta t}{h}\phi'(0, n\Delta t) \quad \tfrac{2\Delta t}{3}F_1^n \quad \dots \quad \tfrac{2\Delta t}{3}F_i^n \quad \dots \quad \dots \quad \dots \quad \tfrac{2\Delta t}{3}F_{N-1}^n \quad \tfrac{-2\Delta t\kappa}{h}\left((v_N^n - g_2)_+ - (g_1 - v_N^n)_+\right)\Big)^T.$$

The term G_{2N}^n is computed from the known values v_N^{n-1}, v_N^{n-2}, $v_{t,N}^{n-1}$ and $v_{t,N}^{n-2}$, which is the linearization of the contact condition and therefore of the whole problem.

The matrix B is given by

$$B=\begin{pmatrix}
1 & -\frac{8}{3}\lambda & 0 & \frac{1}{3}\lambda & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{2k^2}{3}\lambda & 1 & -\frac{4k^2}{3}\lambda & 0 & \frac{2k^2}{3}\lambda & 0 & 0 & 0 & 0 & 0\\
0 & \frac{4}{3}\lambda & 1 & -\frac{2}{3}\lambda & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{2k^2}{3}\lambda & 1 & -\frac{4k^2}{3}\lambda & 0 & \frac{2k^2}{3}\lambda & 0 & 0 & 0\\
0 & -\frac{2}{3}\lambda & 0 & \frac{4}{3}\lambda & 1 & -\frac{2}{3}\lambda & 0 & 0 & \cdots & 0\\
\vdots & \ddots & 0 & 0 & \ddots & \ddots & & & & \vdots\\
\vdots & & & & 0 & 0 & \ddots & & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & & \\
\vdots & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{3}\lambda & 1 & -\frac{2}{3}\lambda\\
0 & \cdots & 0 & 0 & 0 & 0 & -\frac{k^2}{3}\lambda & 0 & \frac{8k^2}{3}\lambda & 1
\end{pmatrix}.$$

The convergence of this algorithm, as $\Delta x \to 0$ and $\Delta t \to 0$, remains an open question. The numerical simulations, obtained by solving the linear system (12) are presented in the next section.

4. Numerical simulations

We describe a number of our numerical simulations, showing the types of behavior the solutions exhibit. To start the algorithm we need to compute the values of V^0 and V^1 from the initial conditions u_0 and v_0. We use the Crank-Nicholson scheme to compute V_i^1, for $0 \leq i \leq M$. Once we have the initial values we march in time using the scheme described in the previous Section. The motion of the left end is given by $\phi(t) = E\sin(at)$, where a is the frequency of the driving device. We depict the numerical solutions below, and in each figure the values of the various constants, and the value of the normal compliance coefficient κ, are given in the table above the figure. We employed very large values of κ, in the range $10^6 - 10^{10}$, which are found to approximate the Signorini condition closely, indeed, the displacements of the stops are seen to be very small.

In each case we give, in addition, the fundamental and the second vibration frequencies ω_0 and ω_1, respectively, of a beam which is clamped at the left end and free at the right end. We found in the simulations that when the free end is not in contact it vibrates at higher frequencies, often at the fundamental free frequency, although higher frequencies are also prsent.

The free vibration frequencies are given by

$$\omega_n = k\beta_{n+1}^2, \qquad n = 0, 1, \cdots,$$

where the β_n are ordinate solutions of the transcendental equation

$$\cosh\beta\ \cos\beta = -1.$$

A straightforward calculation shows that

$$\omega_0 = 3.5148k \quad [\text{rads/sec}], \qquad \omega_1 = 22.3480k \quad [\text{rads/sec}].$$

We provide these values in each table.

For each simulation we present the noise characteristics of the system. We depict the (Fast) Fourier Transforms (FFT) of the motion of the beam's end, $u(1,t)$, which show the frequency distribution of the vibrations. It is seen that the high frequencies are, in part, the free vibrations of the end during the transits between contacts. Also, it is seen that higher modes are excited by the contact and the system becomes rather 'noisy' at high driving frequencies. In Fig. 2 we present a very low driving frequency ($a = 1$) solution. There is contact between the beam's end and the stops over considerable periods of time, and there is very little deflection of the stops. The phase portrait of the beam's end, $u_t(1,t)$ vs. $u(1,t)$ is shown in Fig. 3, together with the FFT. The end velocity oscillates rapidly, at the fundamental frequency $\omega_0 = 314.4$, during the free motion between the stops. Two higher modes are excited too, with frequencies $3a$ and $5a$, as can be seen in the FFT.

k	a	h	Δt	g_1	g_2	κ	ω_0	ω_1
$\sqrt{8000}$	1	0.02	4×10^{-4}	-0.1	0.1	10^6	314.4	1971.3

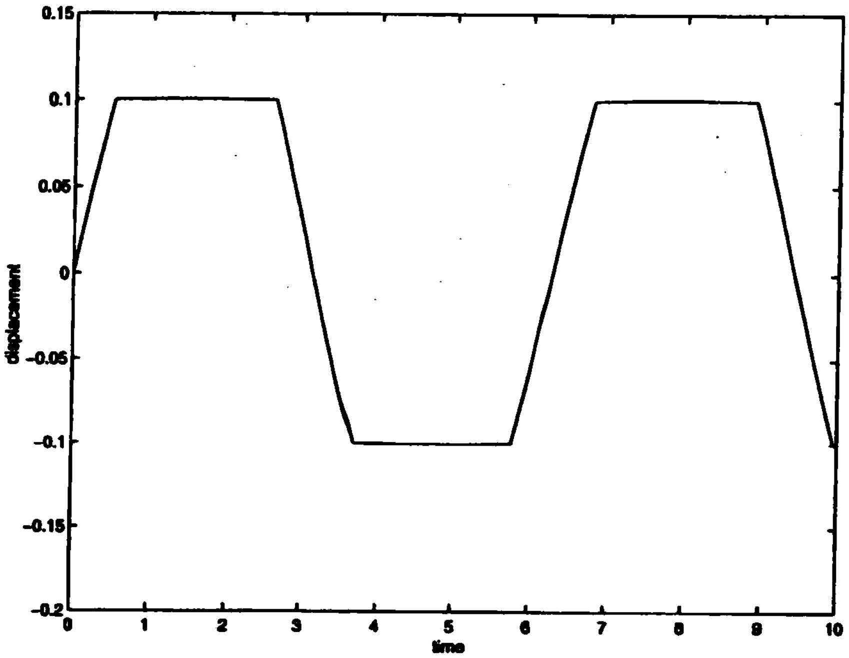


Fig. 2. The displacement $u(1,t)$ vs. t; low frequency

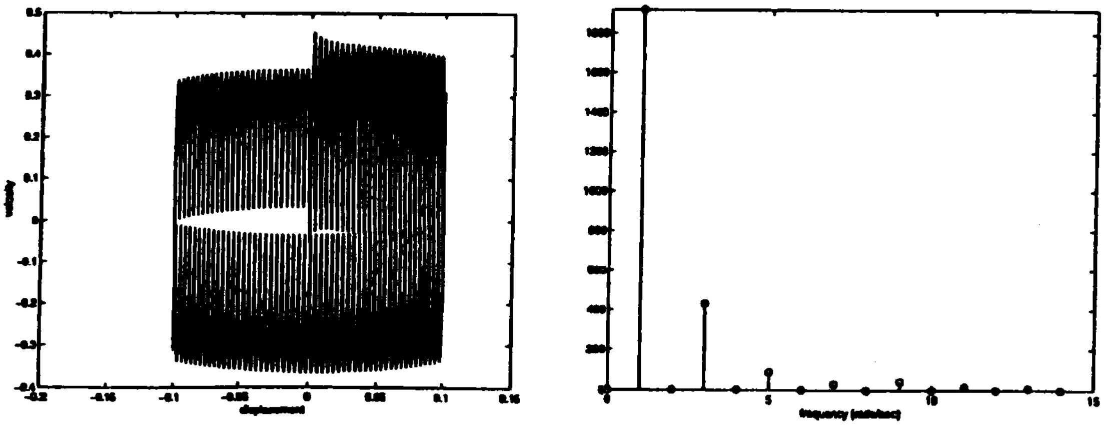


Fig. 3. The phase portrait $u_t(1,t)$ vs. $u(1,t)$, and the FFT, as in Fig. 2

k	a	h	Δt	g_1	g_2	κ	ω_0	ω_1
$\sqrt{8000}$	10	0.02	2×10^{-4}	-0.1	0.1	10^6	314.4	1971.3

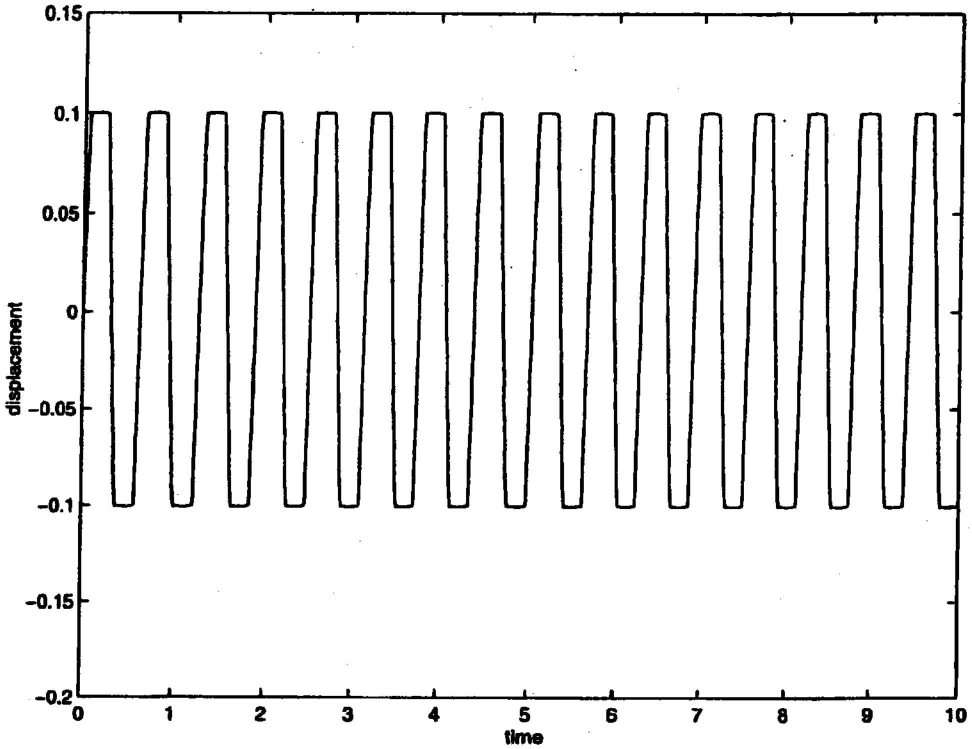


Fig. 4. The displacement $u(1,t)$ vs. t, low frequency

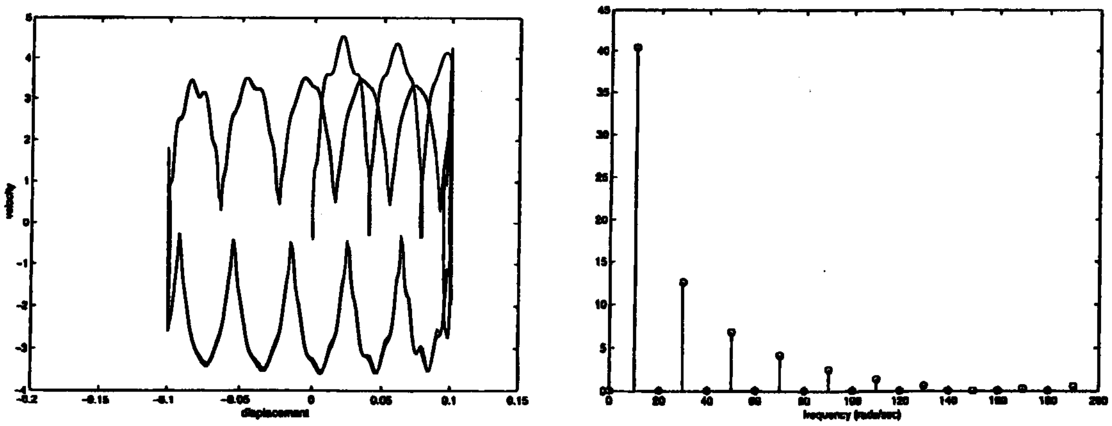


Fig. 5. The phase portrait $u_t(1,t)$ vs. $u(1,t)$, and the FFT as in Fig. 4

In Fig. 4 we depict a low frequency oscillations ($a = 10$), and we see that the contact times are shorter. The phase portrait, Fig. 5, shows that the motion is periodic. Again, the end's velocity oscillates about five times during the transit between the stops, which corresponds to the fundamental free frequency $\omega_0 = 314.4$. The FFT is shown in Fig. 5, and now the excited frequencies are, in addition, $3a$, $5a$, $7a$, $9a$ and $11a$.

k	a	h	Δt	g_1	g_2	κ	ω_0	ω_1
$\sqrt{8000}$	50	0.02	10^{-4}	-0.1	0.1	10^6	314.4	1971.3

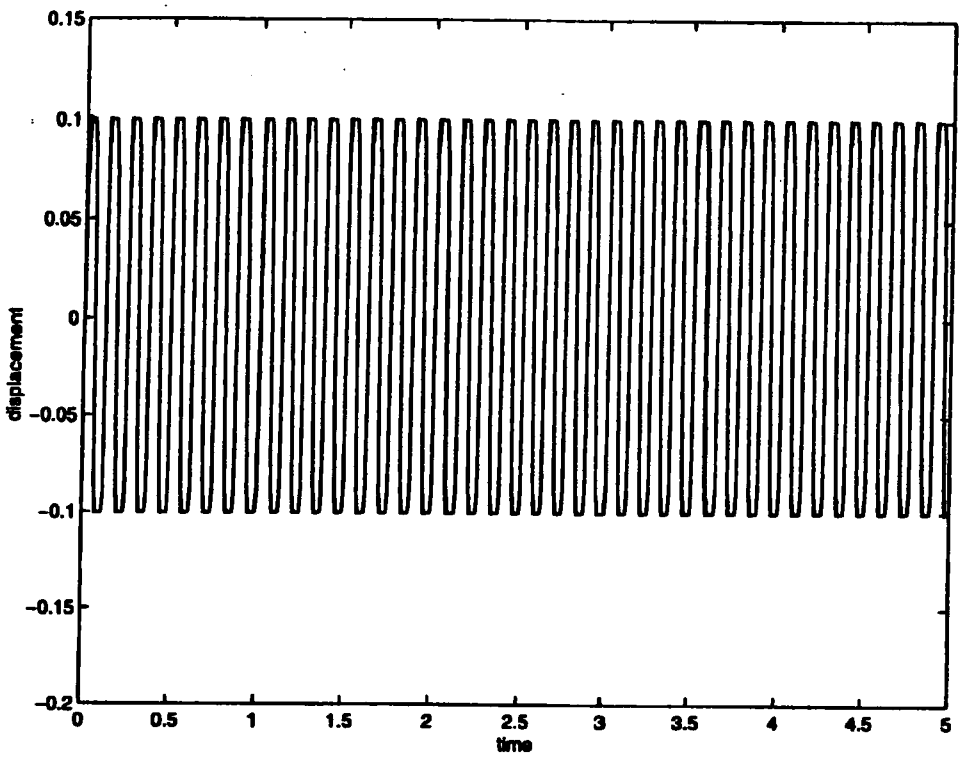


Fig. 6. The displacement $u(1,t)$ vs. t, medium frequency

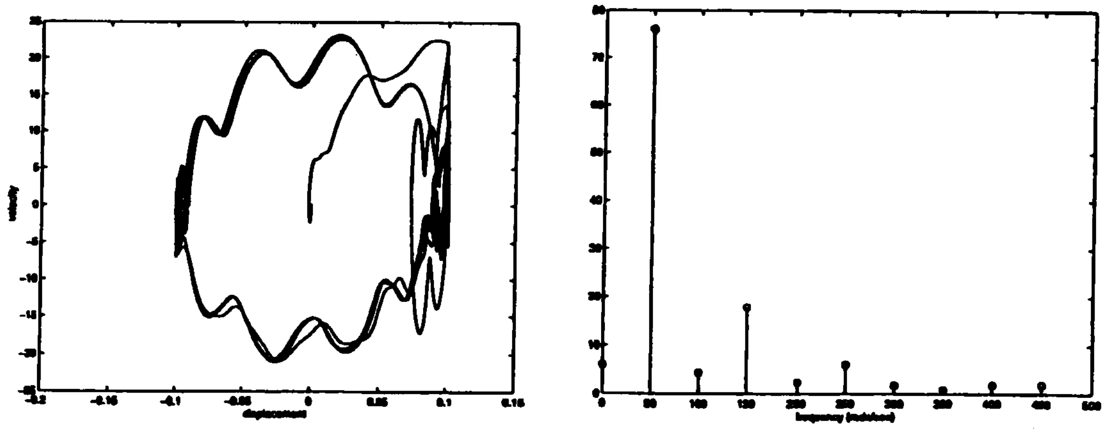


Fig. 7. The phase portrait $u_t(1,t)$ vs. $u(1,t)$, and the FFT, as in Fig. 6

A higher driving frequency was used in Fig. 6, with short contact periods. The driving frequency is very close to the natural frequency. The phase portrait, Fig. 7, shows that near contact the motion is not as regular as away from the stops, and it may be that the end oscillates with frequency ω_1. We note in the FFT, Fig. 7, that multiples of a are excited, and, interestingly, the amplitudes of the odd multiples are much higher than the even ones.

k	a	h	Δt	g_1	g_2	κ	ω_0	ω_1
1000	10	0.02	$2*10^{-5}$	-0.02	0.02	$5*10^8$	3514.81	22348.03

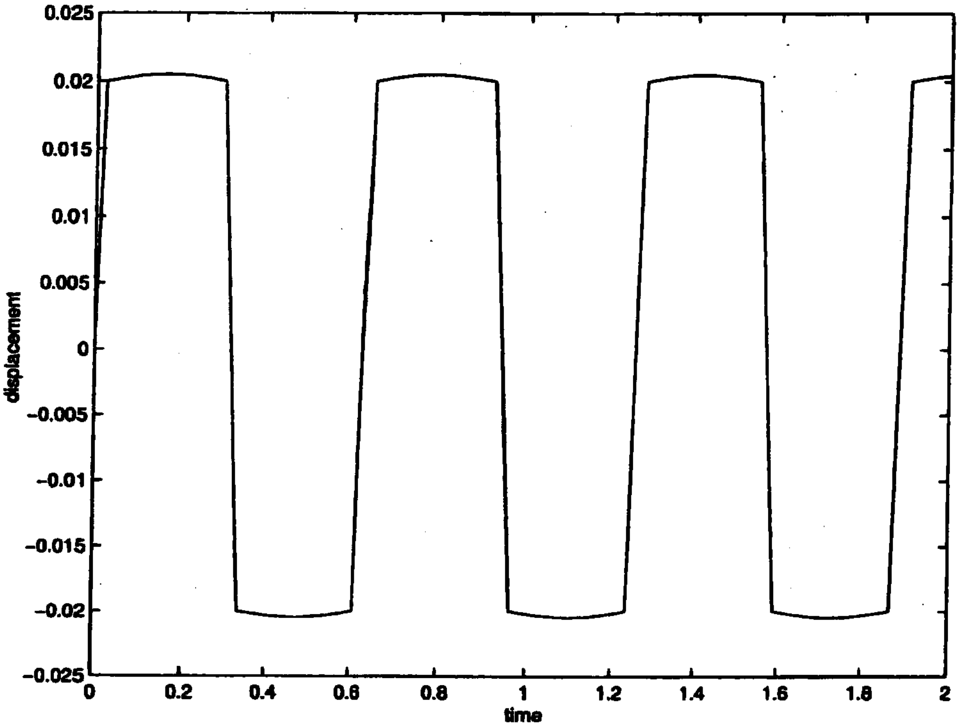


Fig. 8. The displacement $u(1,t)$ vs. t, low frequency; with contact

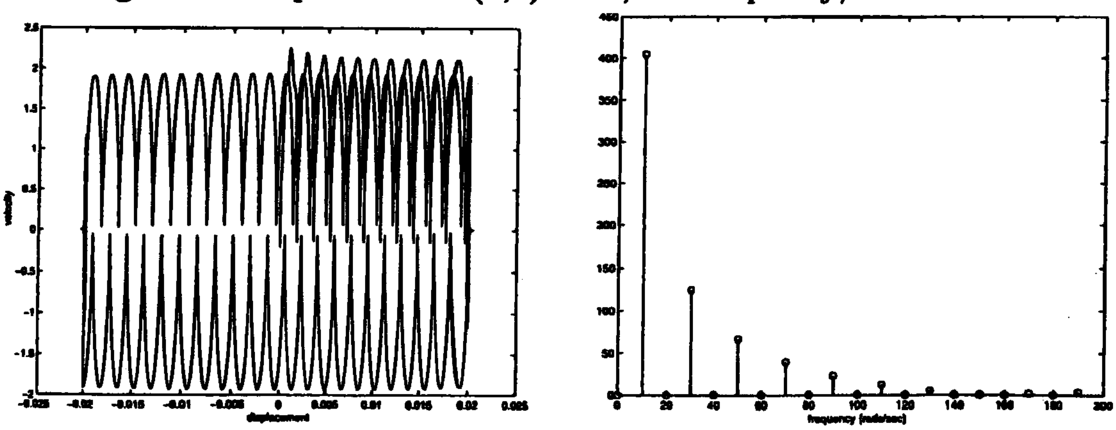


Fig. 9. The phase portrait $u_t(1,t)$ vs. $u(1,t)$ and the FFT, as in Fig. 8

Another low frequency simulation ($a = 10$), with wider clearance and stiffer material, is given in Fig. 8. Small deflection of the stops can be seen during contact. The phase portrait and the FFT are depicted in Fig. 9, and we see that the tip oscillates rapidly between the stops, however, the frequency is lower than ω_0. Only odd multiples of a are excited.

k	a	h	Δt	g_1	g_2	κ	ω_0	ω_1
1000	100	0.02	$2 * 10^{-6}$	-0.02	0.02	10^9	3514.81	22348.03

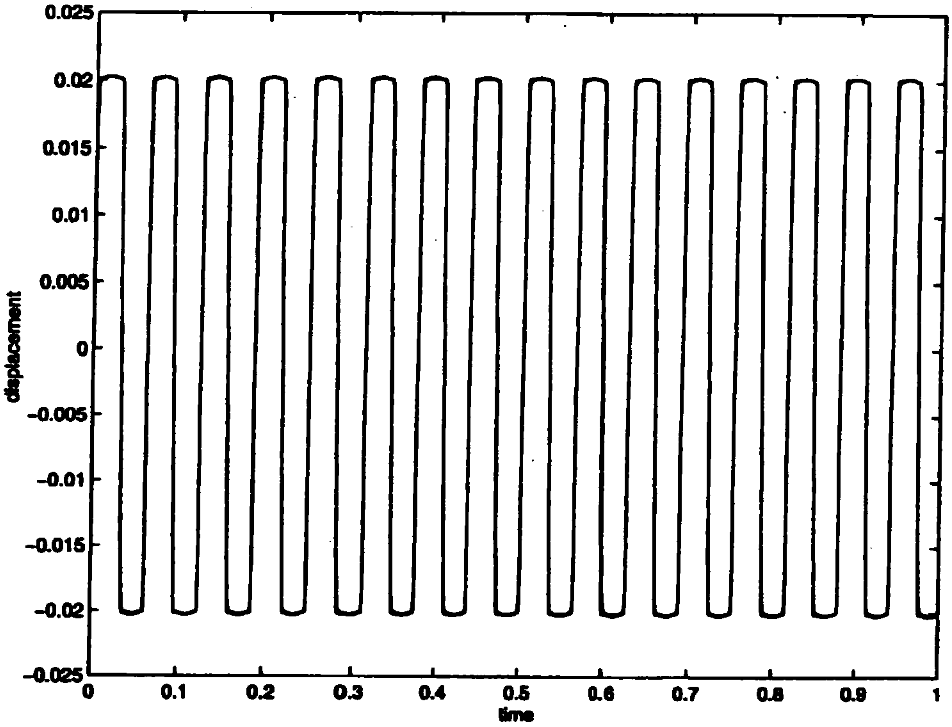


Fig. 10. The displacement $u(1,t)$ vs. t, high frequency

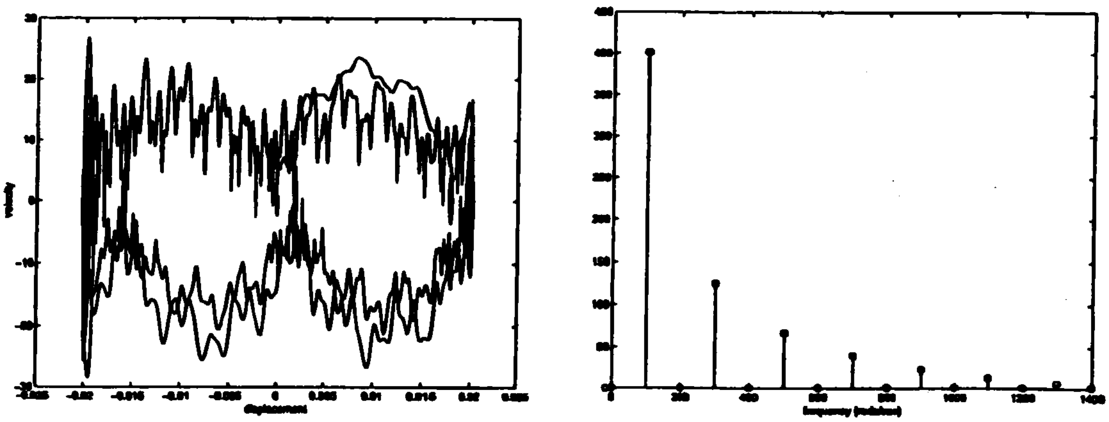


Fig. 11. The phase portrait $u_t(1,t)$ vs. $u(1,t)$ and the FFT, as in Fig. 10

A simulation with a larger clearance and medium driving frequency ($a = 100$) can be seen in Fig. 10. The phase portrait, Fig. 11, shows that the motion of the end is becoming irregular. The FFT is depicted in Fig. 11 too. We do not know if this irregularity is a numerical artifact or the property of the motion itself, and it is possible that we may attribute it to both. The FFT shows that higher frequencies, only odd multiples of a are excited.

k	a	h	Δt	g_1	g_2	κ	ω_0	ω_1
1000	400π	0.02	10^{-6}	-0.09	0.02	10^{10}	3514.81	22348.03

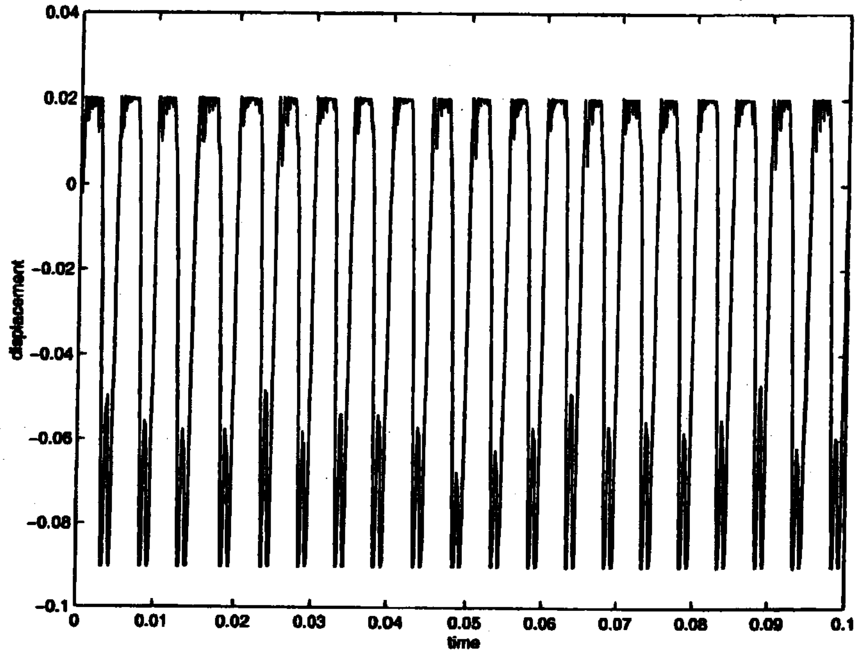


Fig. 12. The displacement $u(1,t)$ vs. t, high frequency

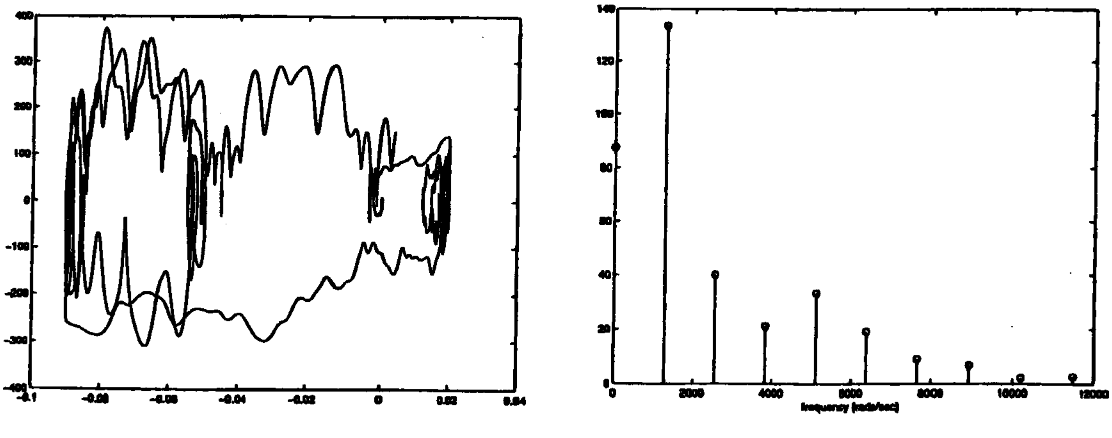


Fig. 13. The phase portrait $u_t(1,t)$ vs. $u(1,t)$ and the FFT, as in Fig. 12

In Fig. 12 we present the simulation of a stiff beam with high driving frequency ($a = 400\pi$) which is higher than the fundamental frequency, and asymmetric stops. We see that the oscillations near the lower stop are much larger than those near the top one. Moreover, it seems as if the frequency near the bottom doubles in an interesting fashion. The phase portrait, Fig. 13, indicates that the motion is becoming irregular.

Again, we conjecture that only part of it is numerical. In the FFT the odd multiples of a have higher amplitudes than the even ones.

k	a	h	Δt	g_1	g_2	κ	ω_0	ω_1
1000	1000	0.02	10^{-6}	-0.02	0.02	$5 * 10^{10}$	3514.81	22348.03

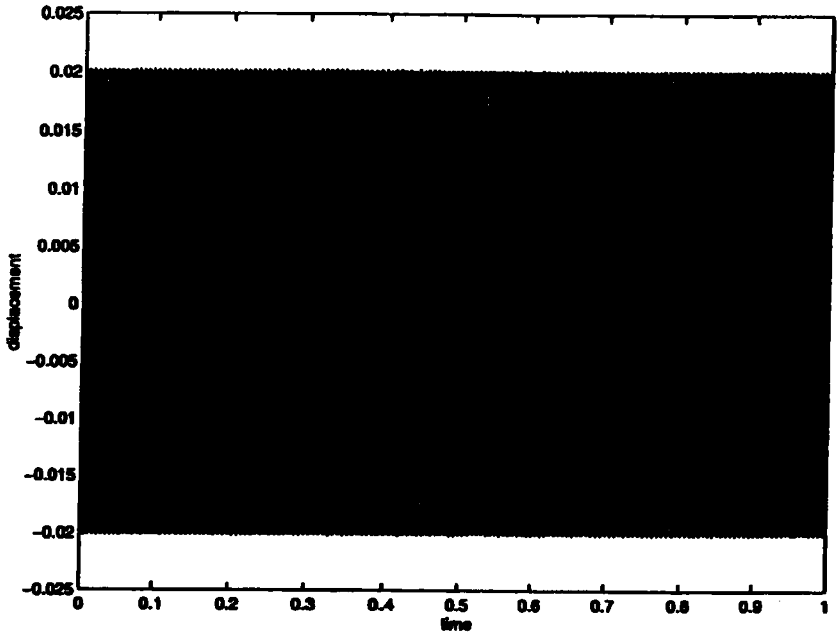
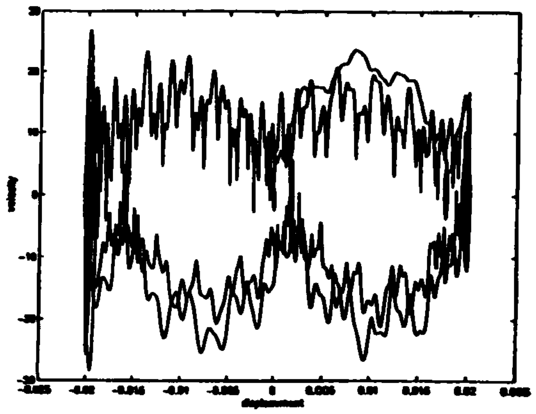


Fig. 14. The displacement $u(1,t)$ vs. t, high frequency



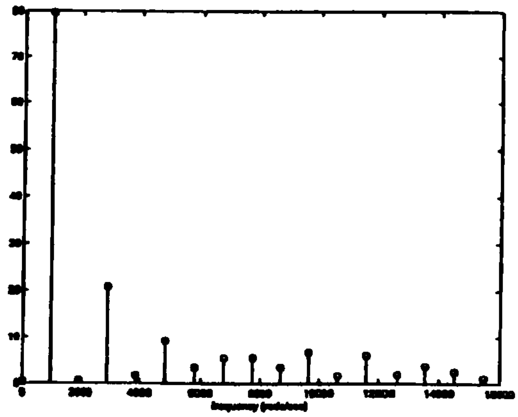


Fig. 15. The phase portrait $u_t(1,t)$ vs. $u(1,t)$ and the FFT, as in Fig. 14

In Fig. 14 we depict a simulation with very high driving frequency ($a = 1000$) which is higher than ω_0. The phase portrait, Fig. 15, shows that the motion seems to be unpredictable, and rather messy. The FFT in Fig. 15 shows a broad range of excited frequencies.

We consider now simulations when the device driving function is

$$\phi(t) = E \sin(at) \cos(bt).$$

We show the simulations for a stiff beam and low driving frequency $b < a$. The motion is depicted in Fig. 16. It seems to be periodic. However, the phase portrait, Fig. 17, has considerable noise, and we conjecture that only part of it is numerical. The FFT indicates that a wide range of frequencies is excited.

k	a	b	h	Δt	g_1	g_2	κ	ω_0	ω_1
1000	10	50	0.02	$2 * 10^{-6}$	−0.02	0.02	10^9	3514.81	22348.03

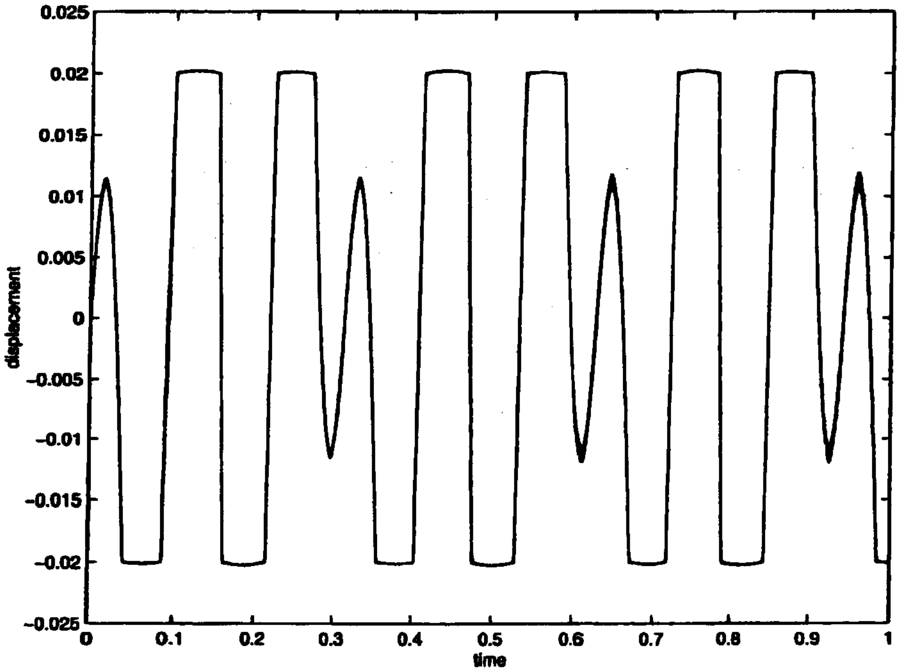


Fig. 16.The displacement $u(1,t)$ vs. t, low frequencies, $a < b$

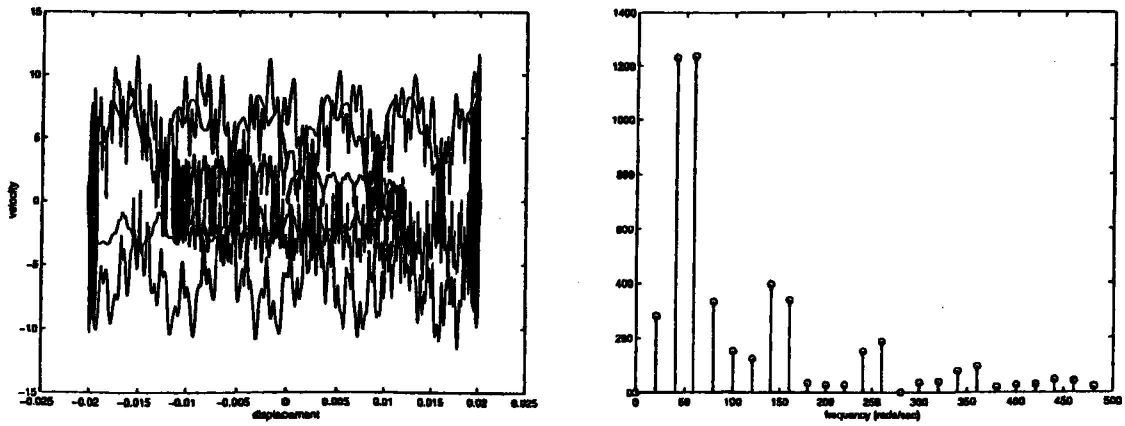


Fig. 17. The phase portrait $u_t(1,t)$ vs. $u(1,t)$ and the FFT, as in Fig. 16

Next, we consider a stiff beam and low driving frequency with $b < a$. The motion is depicted in Fig. 18. It does not seem to be periodic, and the phase portrait, Fig. 19, is very noisy, indicating an irregular motion. The FFT shows a wide band of excited frequencies, and the motion may be chaotic.

k	a	b	h	Δt	g_1	g_2	κ	ω_0	ω_1
1000	50	10π	0.02	$2 * 10^{-6}$	-0.02	0.02	10^9	3514.81	22348.03

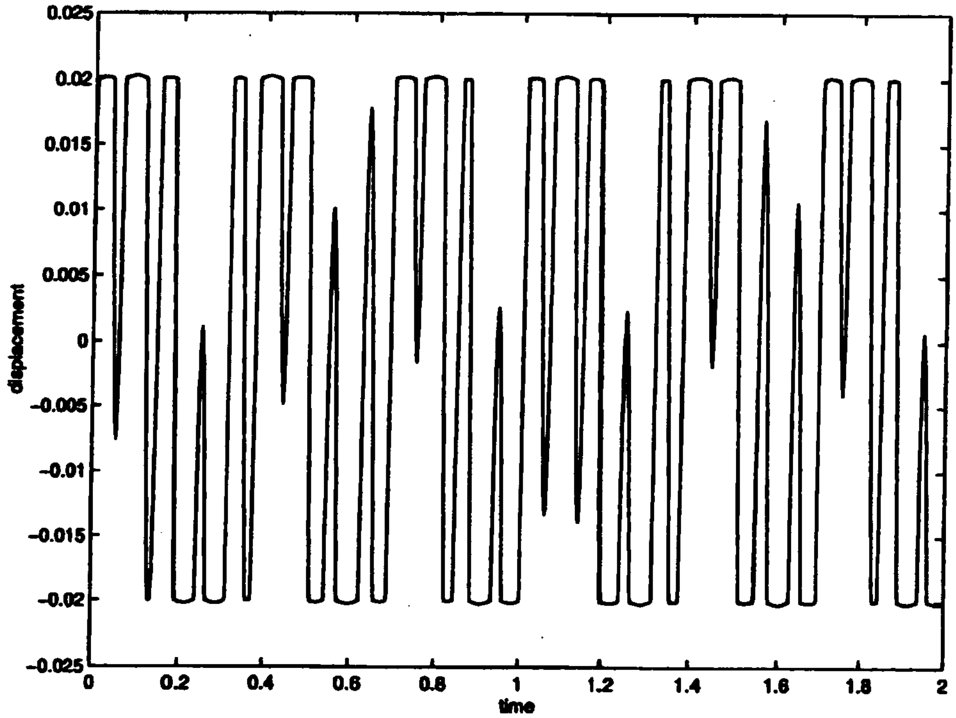


Fig. 18. The displacement $u(1,t)$ vs. t, low frequencies, $a > b$

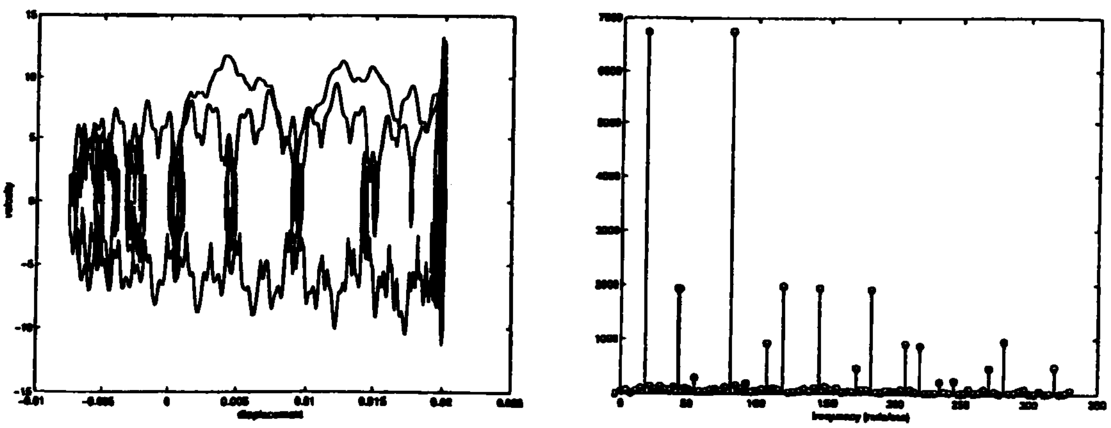


Fig. 19. The phase portrait $u_t(1,t)$ vs. $u(1,t)$ and the FFT, as in Fig. 18

The computations were done on the SUN workstation Ultra 10, using Matlab for computing and graphing. Typical low frequency simulations took 235 seconds of CPU time while the high frequency simulations took 4993 seconds of CPU time. The scheme was found to be reliable and easy to work with.

5. Conclusions

We present a model for the dynamic contact of a vibrating beam with two flexible, nonlinear stops. Since we deal with contact conditions, which lead to velocity discontinuities, we employ a weak formulation of the problem. We construct an algorithm for the numerical approximations of the solutions, based on finite differences. The convergence of the scheme is an open question. Then we present extensive numerical simulations of the solutions. We show that the stops introduce a varied and complicated types of oscillations, some of them seem to be irregular and might be chaotic. For low driving frequencies we see that the end vibrates at the fundamental free beam frequency ω_0 when in transit between the stops. This is not the case for high driving frequencies, where other frequencies can be seen.

The FFT of the solutions shows that multiples of the driving frequency are excited in the process. Sometimes only the odd multiples, in other cases all multiples appear but the odd ones have higher amplitudes. At high driving frequencies it is seen that a wide range of frequencies is exited, indicating a complicated motion.

The numerical scheme was found to be quick and reliable. Only the computer runs with high frequency took relatively long CPU times. We conclude that even such a simple setting exhibits complex and intricate types of behavior.

We plan to continue this investigation, especially of the transition to a possible chaotic behavior. Moreover, we will extend this study to more realistic settings of mechanical joints. There is a need to investigate the numerical noise in the solutions, and to determine how much of their irregular behavior is caused by the numerics. It is very likely that the system is capable of genuine chaotic oscillations, however, to prove it will take considerable effort, since the some of the needed mathematical tools are not available, yet.

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