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Ingleton's axiom and the Axiom of Choice

33rd Summer Conference on Topology and its Applications

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The Hahn-Banach axiom

Given a vector space E over the field \mathbb{R} of real numbers, a *semi-norm* on E is a mapping $N : E \rightarrow \mathbb{R}_+$ such that for every $\lambda \in \mathbb{R}$ and every $x, y \in E$, $N(\lambda \cdot x) = |\lambda|_{\mathbb{R}} N(x)$ and $N(x + y) \leq N(x) + N(y)$, where $|\cdot|_{\mathbb{R}}$ is the usual absolute value $x \mapsto \max(x, -x)$ on \mathbb{R} .

HB: *Given a \mathbb{R} -vector space E , a semi-norm $N : E \rightarrow \mathbb{R}_+$, a vector subspace V of E and a linear form $f : V \rightarrow \mathbb{R}$ such that for every $x \in V$, $|f(x)|_{\mathbb{R}} \leq N(x)$, there exists a linear form $\tilde{f} : E \rightarrow \mathbb{R}$ extending f such that for every $x \in E$, $|\tilde{f}(x)|_{\mathbb{R}} \leq N(x)$.*

Remark. In set-theory without the axiom of choice:

- **AC** \Rightarrow **HB** \Rightarrow “The Hausdorff-Banach-Tarski” paradox.
- None of these two arrows is reversible.

See Jech’s book “The Axiom of Choice” or Howard and Rubin’s book “Consequences of the Axiom of Choice”.

The Hahn-Banach Lemma (set theory without choice)

The usual proof of **HB** can be obtained by transfinitely iterating the following Lemma (for example using Zorn's lemma or a transfinite recursion and the Axiom of Choice).

Lemma (Hahn-Banach, 1932, "one step")

Let E be a \mathbb{R} -vector space, let $N : E \rightarrow \mathbb{R}_+$ be a semi-norm on E , let V be a vector subspace of E and let $f : V \rightarrow \mathbb{R}$ be a linear form such that $|f|_{\mathbb{R}} \leq N|_V$. For every $a \in E \setminus V$, there exists a linear form $\tilde{f} : V + \mathbb{R}.a \rightarrow \mathbb{R}$ extending f such that $|\tilde{f}|_{\mathbb{R}} \leq N|_{V + \mathbb{R}.a}$.

A similar result to Hahn-Banach's Lemma

The following result is choiceless:

Lemma (Ingleton, 1952, “one step”)

Let E be a vector space over a spherically complete ultrametric valued field $(\mathbb{K}, |\cdot|)$, let $N : E \rightarrow \mathbb{R}_+$ be a ultrametric semi-norm, let V be a vector subspace of E and let $f : V \rightarrow \mathbb{K}$ be a linear form such that $|f| \leq N|_V$. If $a \in E \setminus V$, then there exists a linear form $\tilde{f} : V + \mathbb{K}.a \rightarrow \mathbb{K}$ extending f such that $|\tilde{f}| \leq N|_{V+\mathbb{K}.a}$.

Valued fields

An *absolute value* on a (commutative) field \mathbb{K} is a mapping $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}_+$ satisfying the following properties for every $\lambda, \mu \in \mathbb{K}$: $|\lambda| = 0 \Leftrightarrow \lambda = 0$; $|\lambda\mu| = |\lambda||\mu|$ and $|\lambda + \mu| \leq |\lambda| + |\mu|$. Each valued field $(\mathbb{K}, |\cdot|)$ gives rise to a metric $d : \mathbb{K} \times \mathbb{K} \rightarrow \mathbb{R}_+$ defined by $d(x, y) = |x - y|$ for every $x, y \in \mathbb{K}$. An absolute value $|\cdot|$ on \mathbb{K} is said to be *ultrametric* if the associated metric d is ultrametric, equivalently if for every $\lambda, \mu \in \mathbb{K}$, $|\lambda + \mu| \leq \max(|\lambda|, |\mu|)$.

- For each commutative field \mathbb{K} , the mapping $|\cdot|_{triv} : \mathbb{K} \rightarrow \mathbb{R}_+$ associating to each $\lambda \in \mathbb{K}$ the real number 0 if $\lambda = 0$ and 1 otherwise is a ultrametric absolute value, called the *trivial* absolute value on \mathbb{K} . If \mathbb{K} is finite, $|\cdot|_{triv}$ is the only absolute value on \mathbb{K} .
- For each prime number p , the mapping $x \mapsto |x|_p := p^{-v_p(x)}$ is a ultrametric absolute value on the field \mathbb{Q} of rational numbers, where $v_p : \mathbb{Q} \rightarrow \mathbb{Z} \cup \{+\infty\}$ is the p -adic valuation on \mathbb{Q} .
- Every *non trivial* absolute value on \mathbb{Q} is of the form $|\cdot|_{\mathbb{R}}^\tau$ where $0 < \tau < 1$, or of the form $|\cdot|_p^\tau$ for some prime number p and some $\tau > 0$ (Ostrowski's theorem).

Spherically complete ultrametric valued fields

A ultrametric valued field $(\mathbb{K}, |\cdot|)$ is *spherically complete* if every chain of balls with “large inequalities” (i.e. of the form $\{x \in \mathbb{K} : |x - a| \leq r\}$ where $a \in \mathbb{K}$ and $r \in \mathbb{R}_+$) of the metric space (\mathbb{K}, d) has a non-empty intersection.

Examples

-Each commutative field \mathbb{K} endowed with the trivial absolute value is spherically complete.

-For each prime number p , the valued field $(\mathbb{Q}, |\cdot|_p)$ is not spherically complete, however, the Cauchy completion \mathbb{Q}_p of $(\mathbb{Q}, |\cdot|_p)$ is spherically complete (because the unit ball of \mathbb{Q}_p is compact).

Semi-normed vector spaces over a valued field

Given a vector space E over a valued field $(\mathbb{K}, |\cdot|)$, a *semi-norm* on E is a mapping $N : E \rightarrow \mathbb{R}_+$ satisfying for every $x, y \in E$ and $\lambda \in \mathbb{K}$ the properties $N(\lambda \cdot x) = |\lambda|N(x)$ and $N(x + y) \leq N(x) + N(y)$.

For a ultrametric valued field $(\mathbb{K}, |\cdot|)$, the semi-norm N is *ultrametric* if the semi-metric associated to N is ultrametric, equivalently if for every $x, y \in E$, $N(x + y) \leq \max(N(x), N(y))$.

AC implies Ingleton's statement

From Ingleton's Lemma and the Axiom of Choice, it follows for each spherically complete ultrametric valued field $(\mathbb{K}, |\cdot|)$:

Ingleton's statement

$\mathbf{I}_{\mathbb{K},|\cdot|}$: "Let E be a \mathbb{K} -vector space, let $N : E \rightarrow \mathbb{R}_+$ be a ultrametric semi-norm, let V be a vector subspace of E and let $f : V \rightarrow \mathbb{K}$ be a linear form such that $|f| \leq N|_V$. Then there exists a linear form $\tilde{f} : E \rightarrow \mathbb{K}$ extending f such that $|\tilde{f}| \leq N$."

- A.C.M. van Rooij (1992) asked whether the "full Ingleton theorem" (i.e. the conjunction of all statements $\mathbf{I}_{\mathbb{K},|\cdot|}$) implies **AC**.
- We shall show that in set theory **ZFA** (set theory without choice weakened to allow "atoms"), the "full Ingleton theorem" + **HB** does not imply **AC** (unless **ZFA** is inconsistent).

A model of $\mathbf{ZFA} + \neg \mathbf{AC}$ with “multiple choices”

Levy (1962) built a model of \mathbf{ZFA} in which there exists a sequence $(F_n)_{n \in \mathbb{N}}$ of finite sets such that for every $n \in \mathbb{N}$, $\#F_n = n + 1$ and $\prod_{n \in \mathbb{N}} F_n = \emptyset$: such a model does not satisfy \mathbf{AC} .

However, Levy showed that this model satisfies the following consequences of \mathbf{AC} :

- **MC**: (“Multiple Choice”) *“For every family $(A_i)_{i \in I}$ of non-empty sets, there exists a family $(B_i)_{i \in I}$ of non-empty finite sets such that for every $i \in I$, $B_i \subseteq A_i$.”*

For every prime number $p \geq 2$, the following refined statement:

- **MC**(p): *“For every family $(A_i)_{i \in I}$ of nonempty sets, there exists a family $(B_i)_{i \in I}$ of finite sets such that for every $i \in I$, $B_i \subseteq A_i$ and $\#B_i$ is not a multiple of p .”*

Remark. In set-theory \mathbf{ZFA} , **MC** does not imply \mathbf{AC} . In set-theory \mathbf{ZF} (without atoms), **MC** implies \mathbf{AC} .

$\mathbf{MC}_{+\forall}^{Prime} p \mathbf{MC}(p)$ implies $\mathbf{HB} + \text{“Full Ingleton”}$

We shall prove the following Lemma:

Extension Lemma

Let $(\mathbb{K}, |\cdot|)$ be a spherically complete ultrametric valued field or the usual valued field \mathbb{R} . Let E be a \mathbb{K} -vector space endowed with a semi-norm N which is assumed to be ultrametric if $\mathbb{K} \neq \mathbb{R}$. Then $\mathbf{MC}_{+\forall}^{Prime} p \mathbf{MC}(p)$ implies the existence of a mapping associating to each ordered pair (V, f) where V is a proper vector subspace of E and $f : V \rightarrow \mathbb{K}$ is a linear form such that $|f| \leq N|_V$, an ordered pair (V', f') such that V' is a vector subspace of E strictly including V and $f' : V' \rightarrow \mathbb{K}$ is a linear mapping extending f with $|f'| \leq N|_{V'}$.

The “full Ingleton theorem” follows from this Lemma in set theory **ZFA**.

Proof of the Lemma in $\mathbf{ZFA} + \mathbf{MC} + \forall^{Prime} p \mathbf{MC}(p)$

With \mathbf{MC} , let Φ be a mapping associating to each non-empty subset X of $E \cup \mathbb{K}^E$ a finite non-empty subset of X . Given a proper vector subspace V of E and a linear form $f : V \rightarrow \mathbb{K}$ satisfying $|f| \leq N|_V$, let $F := \Phi(E \setminus V)$ and let $V_F := \text{span}(V \cup F)$. Using Hahn-Banach's lemma (for $\mathbb{K} = \mathbb{R}$) or Ingleton's lemma (otherwise), the set \mathcal{G} of linear forms $g : V_F \rightarrow \mathbb{K}$ extending f such that $|g| \leq N|_{V_F}$ is non-empty.

For the first two cases below, we let $G := \Phi(\mathcal{G})$.

- *Case $\mathbb{K} = \mathbb{R}$.* Consider the linear form $\tilde{f} := \frac{1}{\#G} \sum_{g \in G} g$ on V_F : then \tilde{f} extends f and $|\tilde{f}|_{\mathbb{R}} \leq N|_{V_F}$ (whence $\mathbf{MC} \Rightarrow \mathbf{HB}$).

- *Case \mathbb{K} has characteristic zero and the restriction $|\cdot|_{\mathbb{Q}}$ is the trivial absolute value.* Consider the same linear form

$$\tilde{f} := \frac{1}{\#G} \sum_{g \in G} g. \text{ Then } |\#G| = 1 \text{ thus for every } x \in V_F, |\tilde{f}(x)| = \frac{1}{|\#G|} \left| \sum_{g \in G} g(x) \right| = \left| \sum_{g \in G} g(x) \right| \leq \max_{g \in G} |g(x)| \leq N(x)$$

whence $|\tilde{f}| \leq N|_{V_F}$.

Proof of the Lemma in $\mathbf{ZFA} + \mathbf{MC} + \forall^{Prime} p \mathbf{MC}(p)$: cont'd

- *Other cases.*

-*Subcase a)*: The characteristic of the field \mathbb{K} is zero; then \mathbb{K} extends the field \mathbb{Q} of rational numbers; $|\cdot|_{\mathbb{Q}}$ is non-trivial. Using Ostrowski's theorem, the absolute value induced by $|\cdot|$ on \mathbb{Q} is equivalent to the p -adic absolute value for some prime number p .

-*Subcase b)*: The characteristic of \mathbb{K} is not zero. Then, this characteristic is a prime number p .

With $\mathbf{MC}(p)$, let Φ_p be a mapping associating to each non-empty subset X of \mathbb{K}^E a finite subset G of X such that p does not divide $\#G$. Let $G := \Phi_p(\mathcal{G})$: then G is a finite subset of \mathcal{G} such that p does not divide $\#G$. Let $n := \#G$. Then $|n| = 1$: in Subcase a), $|n| = |n|_p = 1$ because p does not divide n ; in Subcase b), $n \in \mathbb{F}_p \setminus \{0\} \subseteq \mathbb{K}$ thus $|n| = 1$.

Now we consider the linear form $\tilde{f} := \frac{1}{n} \sum_{g \in G} g$: this linear form extends f , and for every $x \in V_F$, $|\tilde{f}(x)| = \frac{1}{|n|} \left| \sum_{g \in G} g(x) \right| = \left| \sum_{g \in G} g(x) \right| \leq \max_{g \in G} (|g(x)|) \leq N(x)$, whence $|\tilde{f}| \leq N|_{V_F}$.

Some questions

- Are there links in set-theory without choice between the statements $\mathbf{I}_{\mathbb{K}}$ obtained for various spherically complete ultrametric valued fields \mathbb{K} ?
- Does the conjunction of the statements $\mathbf{I}_{\mathbb{Q}_p}$ for p prime number imply $\mathbf{I}_{\mathbb{Q}, |\cdot|_{triv}}$ or **HB**?
- Given two different prime numbers p and q , are the statements $\mathbf{I}_{\mathbb{Q}_p}$ and $\mathbf{I}_{\mathbb{Q}_q}$ equivalent?

Remark

For each ultrametric spherically complete valued field $(\mathbb{K}, |\cdot|)$, the statement $\mathbf{I}_{(\mathbb{K}, |\cdot|)}$ is equivalent to the following one (see MM-2017):

“For every vector subspace F of an ultrametric semi-normed \mathbb{K} -vector space (E, N) , there exists an isometric linear extender $T : BL(F, \mathbb{K}) \rightarrow BL(E, \mathbb{K})$.”

Here, given a vector subspace V of E , $BL(V, \mathbb{K})$ denotes the set of linear bounded mappings from V to \mathbb{K} .

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