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► **To cite this version:**

Marianne Morillon. Ingleton's theorem and the Axiom of Choice. 33rd Summer Conference on Topology and its Applications, Western Kentucky University Bowling Green, Kentucky, USA, Jul 2018, Bowling Green (Kentucky), United States. hal-04553635

HAL Id: hal-04553635

<https://hal.univ-reunion.fr/hal-04553635>

Submitted on 21 Apr 2024

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Ingleton's axiom and the Axiom of Choice
33rd Summer Conference on Topology and its Applications

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Bowling Green, July 17-20, 2018

The Hahn-Banach axiom

Given a vector space E over the field \mathbb{R} of real numbers, a *semi-norm* on E is a mapping $N : E \rightarrow \mathbb{R}_+$ such that for every $\lambda \in \mathbb{R}$ and every $x, y \in E$, $N(\lambda \cdot x) = |\lambda|_{\mathbb{R}} N(x)$ and $N(x + y) \leq N(x) + N(y)$, where $|\cdot|_{\mathbb{R}}$ is the usual absolute value $x \mapsto \max(x, -x)$ on \mathbb{R} .

HB: *Given a \mathbb{R} -vector space E , a semi-norm $N : E \rightarrow \mathbb{R}_+$, a vector subspace V of E and a linear form $f : V \rightarrow \mathbb{R}$ such that for every $x \in V$, $|f(x)|_{\mathbb{R}} \leq N(x)$, there exists a linear form $\tilde{f} : E \rightarrow \mathbb{R}$ extending f such that for every $x \in E$, $|\tilde{f}(x)|_{\mathbb{R}} \leq N(x)$.*

Remark. In set-theory without the axiom of choice:

- **AC** \Rightarrow **HB** \Rightarrow “The Hausdorff-Banach-Tarski” paradox.
- None of these two arrows is reversible.

See Jech’s book “The Axiom of Choice” or Howard and Rubin’s book “Consequences of the Axiom of Choice”.

The Hahn-Banach Lemma (set theory without choice)

The usual proof of **HB** can be obtained by transfinitely iterating the following Lemma (for example using Zorn's lemma or a transfinite recursion and the Axiom of Choice).

Lemma (Hahn-Banach, 1932, "one step")

Let E be a \mathbb{R} -vector space, let $N : E \rightarrow \mathbb{R}_+$ be a semi-norm on E , let V be a vector subspace of E and let $f : V \rightarrow \mathbb{R}$ be a linear form such that $|f|_{\mathbb{R}} \leq N|_V$. For every $a \in E \setminus V$, there exists a linear form $\tilde{f} : V + \mathbb{R}.a \rightarrow \mathbb{R}$ extending f such that $|\tilde{f}|_{\mathbb{R}} \leq N|_{V + \mathbb{R}.a}$.

A similar result to Hahn-Banach's Lemma

The following result is choiceless:

Lemma (Ingleton, 1952, “one step”)

Let E be a vector space over a spherically complete ultrametric valued field $(\mathbb{K}, |\cdot|)$, let $N : E \rightarrow \mathbb{R}_+$ be a ultrametric semi-norm, let V be a vector subspace of E and let $f : V \rightarrow \mathbb{K}$ be a linear form such that $|f| \leq N|_V$. If $a \in E \setminus V$, then there exists a linear form $\tilde{f} : V + \mathbb{K}.a \rightarrow \mathbb{K}$ extending f such that $|\tilde{f}| \leq N|_{V+\mathbb{K}.a}$.

Valued fields

An *absolute value* on a (commutative) field \mathbb{K} is a mapping $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}_+$ satisfying the following properties for every $\lambda, \mu \in \mathbb{K}$: $|\lambda| = 0 \Leftrightarrow \lambda = 0$; $|\lambda\mu| = |\lambda||\mu|$ and $|\lambda + \mu| \leq |\lambda| + |\mu|$. Each valued field $(\mathbb{K}, |\cdot|)$ gives rise to a metric $d : \mathbb{K} \times \mathbb{K} \rightarrow \mathbb{R}_+$ defined by $d(x, y) = |x - y|$ for every $x, y \in \mathbb{K}$. An absolute value $|\cdot|$ on \mathbb{K} is said to be *ultrametric* if the associated metric d is ultrametric, equivalently if for every $\lambda, \mu \in \mathbb{K}$, $|\lambda + \mu| \leq \max(|\lambda|, |\mu|)$.

- For each commutative field \mathbb{K} , the mapping $|\cdot|_{triv} : \mathbb{K} \rightarrow \mathbb{R}_+$ associating to each $\lambda \in \mathbb{K}$ the real number 0 if $\lambda = 0$ and 1 otherwise is a ultrametric absolute value, called the *trivial* absolute value on \mathbb{K} . If \mathbb{K} is finite, $|\cdot|_{triv}$ is the only absolute value on \mathbb{K} .
- For each prime number p , the mapping $x \mapsto |x|_p := p^{-v_p(x)}$ is a ultrametric absolute value on the field \mathbb{Q} of rational numbers, where $v_p : \mathbb{Q} \rightarrow \mathbb{Z} \cup \{+\infty\}$ is the p -adic valuation on \mathbb{Q} .
- Every *non trivial* absolute value on \mathbb{Q} is of the form $|\cdot|_{\mathbb{R}}^\tau$ where $0 < \tau < 1$, or of the form $|\cdot|_p^\tau$ for some prime number p and some $\tau > 0$ (Ostrowski's theorem).

Spherically complete ultrametric valued fields

A ultrametric valued field $(\mathbb{K}, |\cdot|)$ is *spherically complete* if every chain of balls with “large inequalities” (i.e. of the form $\{x \in \mathbb{K} : |x - a| \leq r\}$ where $a \in \mathbb{K}$ and $r \in \mathbb{R}_+$) of the metric space (\mathbb{K}, d) has a non-empty intersection.

Examples

-Each commutative field \mathbb{K} endowed with the trivial absolute value is spherically complete.

-For each prime number p , the valued field $(\mathbb{Q}, |\cdot|_p)$ is not spherically complete, however, the Cauchy completion \mathbb{Q}_p of $(\mathbb{Q}, |\cdot|_p)$ is spherically complete (because the unit ball of \mathbb{Q}_p is compact).

Semi-normed vector spaces over a valued field

Given a vector space E over a valued field $(\mathbb{K}, |\cdot|)$, a *semi-norm* on E is a mapping $N : E \rightarrow \mathbb{R}_+$ satisfying for every $x, y \in E$ and $\lambda \in \mathbb{K}$ the properties $N(\lambda \cdot x) = |\lambda|N(x)$ and $N(x + y) \leq N(x) + N(y)$.

For a ultrametric valued field $(\mathbb{K}, |\cdot|)$, the semi-norm N is *ultrametric* if the semi-metric associated to N is ultrametric, equivalently if for every $x, y \in E$, $N(x + y) \leq \max(N(x), N(y))$.

AC implies Ingleton's statement

From Ingleton's Lemma and the Axiom of Choice, it follows for each spherically complete ultrametric valued field $(\mathbb{K}, |\cdot|)$:

Ingleton's statement

$\mathbf{I}_{\mathbb{K},|\cdot|}$: "Let E be a \mathbb{K} -vector space, let $N : E \rightarrow \mathbb{R}_+$ be a ultrametric semi-norm, let V be a vector subspace of E and let $f : V \rightarrow \mathbb{K}$ be a linear form such that $|f| \leq N|_V$. Then there exists a linear form $\tilde{f} : E \rightarrow \mathbb{K}$ extending f such that $|\tilde{f}| \leq N$."

- A.C.M. van Rooij (1992) asked whether the "full Ingleton theorem" (i.e. the conjunction of all statements $\mathbf{I}_{\mathbb{K},|\cdot|}$) implies **AC**.
- We shall show that in set theory **ZFA** (set theory without choice weakened to allow "atoms"), the "full Ingleton theorem" + **HB** does not imply **AC** (unless **ZFA** is inconsistent).

A model of $\mathbf{ZFA} + \neg \mathbf{AC}$ with “multiple choices”

Levy (1962) built a model of \mathbf{ZFA} in which there exists a sequence $(F_n)_{n \in \mathbb{N}}$ of finite sets such that for every $n \in \mathbb{N}$, $\#F_n = n + 1$ and $\prod_{n \in \mathbb{N}} F_n = \emptyset$: such a model does not satisfy \mathbf{AC} .

However, Levy showed that this model satisfies the following consequences of \mathbf{AC} :

- \mathbf{MC} : (“Multiple Choice”) *“For every family $(A_i)_{i \in I}$ of non-empty sets, there exists a family $(B_i)_{i \in I}$ of non-empty finite sets such that for every $i \in I$, $B_i \subseteq A_i$.”*

For every prime number $p \geq 2$, the following refined statement:

- $\mathbf{MC}(p)$: *“For every family $(A_i)_{i \in I}$ of nonempty sets, there exists a family $(B_i)_{i \in I}$ of finite sets such that for every $i \in I$, $B_i \subseteq A_i$ and $\#B_i$ is not a multiple of p .”*

Remark. In set-theory \mathbf{ZFA} , \mathbf{MC} does not imply \mathbf{AC} . In set-theory \mathbf{ZF} (without atoms), \mathbf{MC} implies \mathbf{AC} .

$\mathbf{MC}_{+\forall}^{\text{Prime}} p \mathbf{MC}(p)$ implies $\mathbf{HB} + \text{“Full Ingleton”}$

We shall prove the following Lemma:

Extension Lemma

Let $(\mathbb{K}, |\cdot|)$ be a spherically complete ultrametric valued field or the usual valued field \mathbb{R} . Let E be a \mathbb{K} -vector space endowed with a semi-norm N which is assumed to be ultrametric if $\mathbb{K} \neq \mathbb{R}$. Then $\mathbf{MC}_{+\forall}^{\text{Prime}} p \mathbf{MC}(p)$ implies the existence of a mapping associating to each ordered pair (V, f) where V is a proper vector subspace of E and $f : V \rightarrow \mathbb{K}$ is a linear form such that $|f| \leq N_{\upharpoonright V}$, an ordered pair (V', f') such that V' is a vector subspace of E strictly including V and $f' : V' \rightarrow \mathbb{K}$ is a linear mapping extending f with $|f'| \leq N_{\upharpoonright V'}$.

The “full Ingleton theorem” follows from this Lemma in set theory **ZFA**.

Proof of the Lemma in $\mathbf{ZFA} + \mathbf{MC} + \forall^{Prime} p \mathbf{MC}(p)$

With \mathbf{MC} , let Φ be a mapping associating to each non-empty subset X of $E \cup \mathbb{K}^E$ a finite non-empty subset of X . Given a proper vector subspace V of E and a linear form $f : V \rightarrow \mathbb{K}$ satisfying $|f| \leq N|_V$, let $F := \Phi(E \setminus V)$ and let $V_F := \text{span}(V \cup F)$. Using Hahn-Banach's lemma (for $\mathbb{K} = \mathbb{R}$) or Ingleton's lemma (otherwise), the set \mathcal{G} of linear forms $g : V_F \rightarrow \mathbb{K}$ extending f such that $|g| \leq N|_{V_F}$ is non-empty.

For the first two cases below, we let $G := \Phi(\mathcal{G})$.

- *Case $\mathbb{K} = \mathbb{R}$.* Consider the linear form $\tilde{f} := \frac{1}{\#G} \sum_{g \in G} g$ on V_F : then \tilde{f} extends f and $|\tilde{f}|_{\mathbb{R}} \leq N|_{V_F}$ (whence $\mathbf{MC} \Rightarrow \mathbf{HB}$).

- *Case \mathbb{K} has characteristic zero and the restriction $|\cdot|_{\mathbb{Q}}$ is the trivial absolute value.* Consider the same linear form

$$\tilde{f} := \frac{1}{\#G} \sum_{g \in G} g. \text{ Then } |\#G| = 1 \text{ thus for every } x \in V_F, |\tilde{f}(x)| = \frac{1}{|\#G|} \left| \sum_{g \in G} g(x) \right| = \left| \sum_{g \in G} g(x) \right| \leq \max_{g \in G} |g(x)| \leq N(x)$$

whence $|\tilde{f}| \leq N|_{V_F}$.

Proof of the Lemma in $\mathbf{ZFA} + \mathbf{MC} + \forall^{Prime} p \mathbf{MC}(p)$: cont'd

- *Other cases.*

-*Subcase a)*: The characteristic of the field \mathbb{K} is zero; then \mathbb{K} extends the field \mathbb{Q} of rational numbers; $|\cdot|_{\mathbb{Q}}$ is non-trivial. Using Ostrowski's theorem, the absolute value induced by $|\cdot|$ on \mathbb{Q} is equivalent to the p -adic absolute value for some prime number p .

-*Subcase b)*: The characteristic of \mathbb{K} is not zero. Then, this characteristic is a prime number p .

With $\mathbf{MC}(p)$, let Φ_p be a mapping associating to each non-empty subset X of \mathbb{K}^E a finite subset G of X such that p does not divide $\#G$. Let $G := \Phi_p(\mathcal{G})$: then G is a finite subset of \mathcal{G} such that p does not divide $\#G$. Let $n := \#G$. Then $|n| = 1$: in Subcase a), $|n| = |n|_p = 1$ because p does not divide n ; in Subcase b), $n \in \mathbb{F}_p \setminus \{0\} \subseteq \mathbb{K}$ thus $|n| = 1$.

Now we consider the linear form $\tilde{f} := \frac{1}{n} \sum_{g \in G} g$: this linear form extends f , and for every $x \in V_F$, $|\tilde{f}(x)| = \frac{1}{|n|} \left| \sum_{g \in G} g(x) \right| = \left| \sum_{g \in G} g(x) \right| \leq \max_{g \in G} (|g(x)|) \leq N(x)$, whence $|\tilde{f}| \leq N|_{V_F}$.

Some questions

- Are there links in set-theory without choice between the statements $\mathbf{I}_{\mathbb{K}}$ obtained for various spherically complete ultrametric valued fields \mathbb{K} ?
- Does the conjunction of the statements $\mathbf{I}_{\mathbb{Q}_p}$ for p prime number imply $\mathbf{I}_{\mathbb{Q}, |\cdot|_{triv}}$ or **HB**?
- Given two different prime numbers p and q , are the statements $\mathbf{I}_{\mathbb{Q}_p}$ and $\mathbf{I}_{\mathbb{Q}_q}$ equivalent?

Remark

For each ultrametric spherically complete valued field $(\mathbb{K}, |\cdot|)$, the statement $\mathbf{I}_{(\mathbb{K}, |\cdot|)}$ is equivalent to the following one (see MM-2017):

“For every vector subspace F of an ultrametric semi-normed \mathbb{K} -vector space (E, N) , there exists an isometric linear extender $T : BL(F, \mathbb{K}) \rightarrow BL(E, \mathbb{K})$.”

Here, given a vector subspace V of E , $BL(V, \mathbb{K})$ denotes the set of linear bounded mappings from V to \mathbb{K} .

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