# Paradoxical decompositions of free F2 -sets and the Hahn-Banach axiom

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# PARADOXICAL DECOMPOSITIONS OF FREE $F_2$-SETS AND THE HAHN-BANACH AXIOM

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Abstract. Denoting by $F_2$ the free group over a two-element alphabet, we show in set-theory without the axiom of choice **ZF** that the existence of a $(2,2)$-paradoxical decomposition of free $F_2$-sets follows from the conjunction of a weakened consequence of the Hahn-Banach axiom and a weakened consequence of the axiom of choice for pairs. The existence in **ZF** of a paradoxical decomposition with 4 pieces of the sphere in the 3-dimensional euclidean space follows from the same two statements restricted to the set $\mathbb{R}$ of real numbers. Our result is linked to the $(m,n)$-paradoxical decompositions of free $F_2$-sets previously obtained by Pawlikowski ($m=n=3$, see [11]) and then by Sato and Shioya ($m=3$ and $n=2$, see [13]) with the sole Hahn-Banach axiom.

## 1. Introduction

One form of the Banach-Tarski paradox is the existence of a partition $S = S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4$ of the euclidean unit sphere $S$ of $\mathbb{R}^3$ and rotations $u$, $v$ of the euclidean space $\mathbb{R}^3$ such that $S = S_1 \sqcup u[S_2] = S_3 \sqcup v[B_4]$ where $S_1 \cap u[S_2] = S_3 \cap v[S_4] = \varnothing$. Here the formula $Z = X \sqcup Y$ means that $Z = X \cup Y$ and that $X \cap Y = \varnothing$. This paradox, which can be viewed as a duplication of the sphere using 4 pieces, can be obtained (see [15]) using two rotations $\rho_1$ and $\rho_2$ of $\mathbb{R}^3$ generating a free subgroup $F$ (for example the two Satô rotations, see [14]), a partition $F = X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4$ such that $F = X_1 \sqcup \rho_1.X_2 = X_3 \sqcup \rho_2 X_4$, the natural action of $F$ on the sphere $S$, and a certain amount of axiom of choice to replicate the previous paradoxical decomposition of $F$ on each orbit of the action.

**Notation 1.** Given a family $(A_i)_{i\in I}$ of sets, the notation $A = \sqcup_{i\in I} A_i$ means that the sets $A_i$ are pairwise disjoint and that $A = \cup_{i\in I} A_i$.

More generally, given a group $G$ and a left action $. : G\times X \to X$ of $G$ on a nonempty set $X$, given two integers $m, n \geq 2$, a $(m,n)$-*paradoxical decomposition* of the $G$-set $X$ is a partition $X = \sqcup_{1\leq i\leq m} A_i \sqcup \sqcup_{1\leq i\leq n} B_i$ of $X$ in $m+n$ (nonempty) sets together with $m+n$ elements $a_1, \dots, a_m, b_1, \dots, b_n \in G$ satisfying $X = \sqcup_{1\leq i\leq m} a_i.A_i = \sqcup_{1\leq i\leq n} b_i.B_i$: such a decomposition can be viewed as a duplication of $X$ using $m+n$ pieces. It is known in **ZFC** (set theory with the axiom of choice) that denoting by $F_2$ the free group on a two-element alphabet, every free $F_2$-set has a (2,2)-paradoxical decomposition: the classical proof consists in choosing an element in each orbit and, using this choice, replicating on each orbit the usual (2,2)-paradoxical decomposition of the (free) action of $F_2$ by translations on itself (see Example 2).

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In this paper, we work in $\mathbf{ZF}$, set-theory without the axiom of choice, and we denote by $\mathbf{AC}$ the Axiom of Choice. In [11], Pawlikowski obtained a $(3,3)$-paradoxical decomposition of every free $F_2$-set in $\mathbf{ZF}+\mathbf{HB}$ where $\mathbf{HB}$ is the Hahn-Banach axiom (see Section 2.1.2), a consequence of $\mathbf{AC}$ which does not imply $\mathbf{AC}$ in $\mathbf{ZF}$. In fact, Pawlikowski used a weak form of the Hahn-Banach axiom, namely the axiom $\mathbf{HB}_w$ (see Section 2.2.3): instead of choosing a point in each orbit, he chose a probability on each orbit and then ended his proof with a combinatoric argument. As a consequence, a duplication of the sphere of $\mathbb{R}^3$ using 6 pieces was obtained in $\mathbf{ZF}+\mathbf{HB}_w$.

Recently, Sato and Shioya (see [13]) enhanced Pawlikowski's result: using the same consequence $\mathbf{HB}_w$ of the Hahn-Banach axiom, they obtained a (3,2)-paradoxical decomposition of every free right $F_2$-set (and thus a duplication of the sphere using 5 pieces), leaving open the question of whether a (2,2)-paradoxical decomposition was possible with the sole axiom $\mathbf{HB}$.

Denoting by $F_2$ the free group on the alphabet $\{a,b\}$, and given a free action of $F_2$ on a nonempty (infinite) set $X$, then $X$ endowed with the Cayley graph associated to the two generators $a$ and $b$ of $F_2$ is a forest, and every orbit of this action is a 4-regular tree. In this paper, we prove (see Theorem 2) that the existence of a $(2,2)$-paradoxical decomposition "by letters" (see Definition 2 in Section 2.4) of a free $F_2$-set $X$ is equivalent to the choice of and "end" in each connected component of the Cayley graph of $X$. We then prove in $\mathbf{ZF}+\mathbf{HB}_w$ (see Corollary 1) that one can choose in every orbit $T$ of a free $F_2$-set $X$, a "bi-infinite path" or an "end" of the tree $T$. We then deduce in $\mathbf{ZF}+\mathbf{HB}_w+\mathbf{AC}^{2,end,F_2}$ a $(2,2)$-paradoxical decomposition by letters of every free $F_2$-set, where $\mathbf{AC}^{2,end,F_2}$ is the following weak consequence of $\mathbf{AC}^2$, the axiom of choice for pairs:

> *"For every free $F_2$-set $X$, for every set $\Omega$ of orbits of this action, and every family $(P_T)_{T\in\Omega}$ such that each $P_T$ is a bi-infinite path of the tree $T$, there exists a family $(o_T)_{T\in\Omega}$ such that for each $T\in\Omega$, $o_T$ is one of the two ends of the path $P_T$."*

As a consequence, we obtain in $\mathbf{ZF}+\mathbf{HB}_w+\mathbf{AC}^{2,end,\mathbb{R}}$ a $(2,2)$-paradoxical decomposition of the unit sphere $S$ of $\mathbb{R}^3$ where $\mathbf{AC}^{2,end,\mathbb{R}}$ is the statement $\mathbf{AC}^{2,end,F_2}$ restricted to free actions of $F_2$ on a set equipotent with the set $\mathbb{R}$ of real numbers. Notice that $\mathbf{AC}^{2,end,\mathbb{R}}$ is also a consequence of the following statement introduced by Truss (see [16, p. 188], [5, p. 103, form 140]):

> (form 140 of [5]): *"Let $\Omega$ be the set of all (undirected) infinite cycles of reals (graphs whose vertices are real numbers, connected, no loops and each vertex adjacent to exactly two others). Then there is a function $f$ on $\Omega$ such that for all $s\in\Omega$, $f(s)$ is a direction along $s$."*

On the way, using Corollary 1, we provide (see Section 5.4) another proof in $\mathbf{ZF}+\mathbf{HB}_w$ of the $(3,2)$-paradoxical decomposition of free $F_2$-sets obtained by Sato and Shioya.

The paper is organized as follows: in Section 2, we present various consequences of the Axiom of Choice which will be used in our paper; in Section 3, we present some results about Cayley graphs of actions of groups. In Section 4, we prove that a $(2,2)$-paradoxical decomposition by letters of a free $F_2$-set is equivalent to the choice of a 1-out orientation of each orbit; in Section 5, we prove that the axiom $\mathbf{HB}_w$ implies the existence of an end or a bi-infinite path in every orbit of a free $F_2$-set. In Section 6, we prove that the conjunction

$\mathbf{HB}_w + \mathbf{AC}^{2,end,F_2}$ implies the existence of a $(2,2)$-paradoxical decomposition by letters of every free $F_2$-set. We notice that the existence of a $(2,2)$-paradoxical decomposition by letters of every free $F_2$-set implies the weak statement $\mathbf{AC}^{2,end,F_2}$ (see Remark 7).

## 2. Weak forms of the Axiom of Choice and paradoxical decompositions

### 2.1. Some known consequences of the axiom of choice.

2.1.1. *Consequences of* **AC** *involving choice functions.* We work in set theory without the axiom of choice **ZF**. Given a family $(X_i)_{i\in I}$ of nonempty sets, a *choice function* for this family is a mapping $f : I \to \cup_{i\in I} X_i$ such that for every $i \in I$, $f(i) \in X_i$. The axiom of choice is the following statement:

> **AC** ("Axiom of Choice"): *"Every family $(X_i)_{i\in I}$ of nonempty sets has a choice function."*

It is known that **AC** is not provable in set-theory **ZF**. We shall consider various consequences of **AC**, accompanied by their assigned number in Howard and Rubin's book (see [5]), for example:

> $\mathbf{AC^{fin}}$ ("finite axiom of choice", form 62): *"Every family $(X_i)_{i\in I}$ of nonempty finite sets has a choice function."*

**Notation 2.** Given a set $X$, we denote by $\mathrm{fin}(X)$ the set of finite subsets of $X$, and we denote by $\mathrm{fin}^*(X)$ the set of nonempty finite subsets of $X$.

A set $X$ is said to satisfy the *finite choice property* if there exists a mapping $\Phi : \mathrm{fin}^*(X) \to X$ such that for every $F \in \mathrm{fin}^*(X)$, $\Phi(F) \in F$: such a mapping is called a *witness* of the finite choice property on $X$. The axiom $\mathbf{AC^{fin}}$ is thus equivalent to the following one: *"Every set satisfies the finite choice property."*

*Remark* 1. If $\Phi$ is a witness of the finite choice property on a set $X$, then there exists a mapping $\Psi$ definable from $\Phi$ associating to each finite subset $F$ of $X$ a linear order on $F$.

*Proof.* For every finite subset $F$ with at least $n \geq 2$ elements, the one-to-one sequence $(x_0 = \Phi(F), x_1 = \Phi(F\backslash\{x_0\}), \dots)$ is a one-to-one enumeration of $F$. □

For every natural number $n \geq 2$ we consider the following consequence of $\mathbf{AC^{fin}}$:

> $\mathbf{AC}^{\leq n}$ ("axiom of choice for at most $n$-element sets", form $45(n)$): *"Every family $(X_i)_{i\in I}$ of nonempty finite sets with at most $n$ elements has a choice function."*

In the particular case where $n = 2$, we get

> $\mathbf{AC}^2$ ("axiom of choice for pairs", form 88): *"Every family $(X_i)_{i\in I}$ of two-element finite sets has a choice function."*

It is known that in **ZF**, for every natural number $n \geq 3$, $\mathbf{AC} \Rightarrow \mathbf{BPI} \Rightarrow \mathbf{AC^{fin}} \Rightarrow \mathbf{AC}^{\leq n} \Rightarrow \mathbf{AC}^2$, that none of the reverse implications is provable (in particular $\mathbf{AC}^2$ does not imply $\mathbf{AC}^3$) and that $\mathbf{AC}^2$ is not provable (see [5]).

2.1.2. *Consequences of* **AC** *involving probabilities and filters on boolean algebras.* A *boolean algebra* is a (commutative) unitary ring $(\mathcal{B}, \oplus, \times, 0, 1)$ such that for every $x \in \mathcal{B}$, $x \times x = x$. Given a boolean algebra $(\mathcal{B}, \oplus, \times, 0, 1)$, then the binary relation $\leq_{\mathcal{B}}$ on $\mathcal{B}$ defined by $\forall x, y \in \mathcal{B}(x \leq_{\mathcal{B}} y \Leftrightarrow x \times y = x)$ is a lattice-order relation, with "sup" law $\vee : (x, y) \mapsto x \oplus y \oplus (x \times y)$ and inf law $\wedge = \times$. The lattice $(\mathcal{B}, \leq_{\mathcal{B}})$ is thus distributive, with smallest element 0 and greatest) element 1. Notice that for every $x \in \mathcal{B}$, $x^c := 1 \oplus x$ is the unique complement of $x$ in the lattice $(\mathcal{B}, \leq_{\mathcal{B}})$.

Given a boolean algebra $\mathcal{B}$ and an ordered abelian group $(R, +, 0)$ with positive part $R_+$, a mapping $\mu : \mathcal{B} \to R_+$ is *finitely additive* if for every $x, y \in \mathcal{B}$, $(x \times y = 0 \Rightarrow \mu(x \oplus y) = \mu(x) + \mu(y))$ (whence it follows $\mu(0_{\mathcal{B}}) = 0$ and $\forall x, y \in \mathcal{B}$ $(x \leq_{\mathcal{B}} y \Rightarrow \mu(x) \leq \mu(y))$). A *measure* on a non trivial boolean algebra $\mathcal{B}$ is a finitely additive mapping $\mu : \mathcal{B} \to \mathbb{R}_+$ where $\mathbb{R}_+$ is the positive part of the usually ordered abelian group $(\mathbb{R}, +)$. A *probability* on $\mathcal{B}$ is a measure $\mu : \mathcal{B} \to \mathbb{R}_+$ such that $\mu(1_{\mathcal{B}}) = 1$ (and thus $\forall x \in \mathcal{B}$ $\mu(x) \in [0, 1]$). A *two-valued probability* on $\mathcal{B}$ is a probability $\mu : \mathcal{B} \to \{0, 1\}$. A *filter* on a non trivial boolean algebra $\mathcal{B}$ is a nonempty subset $F$ of $\mathcal{B}$ such that $0 \notin F$ and satisfying the two following conditions:

(1) $\forall x, y \in F,\ x \wedge y \in F$
(2) $\forall x \in F\ \forall y \in \mathcal{B}\ (x \leq y \Rightarrow y \in F)$

A *ultrafilter* on $\mathcal{B}$ is a maximal filter $F$ on $\mathcal{B}$.

**Proposition 1** (Ultrafilters on a boolean algebra)**.** *Let $\mathcal{B}$ be a non trivial boolean algebra.*

*(1) If $\mu : \mathcal{B} \to \{0, 1\}$ is a two-valued probability, then $\mathcal{U}_\mu := \{x \in \mathcal{B} : \mu(x) = 1\}$ is a ultrafilter of $\mathcal{B}$.*
*(2) If $\mathcal{U}$ is a ultrafilter of $\mathcal{B}$, then $\mathcal{M}_{\mathcal{U}} := \{x^c : x \in \mathcal{U}\}$ is a maximal ideal of the ring $(\mathcal{B}, \oplus, \times, 0, 1)$.*
*(3) If $\mathfrak{M}$ is a maximal ideal of $\mathcal{B}$, then the quotient field $\mathcal{B}/\mathfrak{M}$ has only two elements* 0 *and* 1 *and the quotient mapping $\mu : \mathcal{B} \to \{0, 1\}$ is a two-valued probability.*

*Proof.* This is quite obvious. □

*Remark* 2. A filter $F$ on $\mathcal{B}$ is a ultrafilter iff $\forall x \in \mathcal{B}$ ($x \in F$ or $x^c \in F$); equivalently the filter $F$ is a ultrafilter of $\mathcal{B}$ iff for every $x_1, x_2 \in \mathcal{B}$, $(x_1 \vee x_2 \in F \Rightarrow (x_1 \in F$ or $x_2 \in F))$.

Given a nonempty set $X$, a *probability on the set* $X$ is a probability on the boolean algebra $(\mathcal{P}(X), \Delta, \cap)$. A probability on $X$ is thus a mapping $\mu : \mathcal{P}(X) \to [0, 1]$ such that $\mu(X) = 1$ and for every disjoint subsets $A$, $B$ of $X$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$.

We consider the following consequences of **AC** involving probabilities on boolean algebras:

**BPI** ("Boolean Prime Ideal", form 14): *For every non trivial boolean algebra $\mathcal{B}$ there exists a two-valued probability $\mu : \mathcal{B} \to \{0, 1\}$.*

**HB** ("Hahn-Banach", form 52): *For every non trivial boolean algebra $\mathcal{B}$ there exists a real-valued probability $\mu : \mathcal{B} \to [0, 1]$.*

It is known (see [5]) that in **ZF**, **AC** $\Rightarrow$ **BPI** $\Rightarrow$ **HB**, **BPI** $\Rightarrow$ $\mathbf{AC}^{\mathbf{fin}}$, that none of the reciprocal implications hold, and that **HB** is not provable.

2.2. **Multiple forms of BPI and HB.** The statements **BPI** and **HB** are respectively equivalent to their multiple forms:

(multiple form of **HB**:) *For every family $(\mathcal{B}_i)_{i\in I}$ of non trivial boolean algebras, there exists a family $(\mu_i)_{i\in I}$ such that for each $i \in I$, $\mu_i : \mathcal{B}_i \to [0,1]$ is a real-valued probability.*

(multiple form of **BPI**:) *For every family $(\mathcal{B}_i)_{i\in I}$ of non trivial boolean algebras, there exists a family $(\mu_i)_{i\in I}$ such that for each $i \in I$, $\mu_i : \mathcal{B}_i \to \{0,1\}$ is two-valued probability.*

The equivalence between **BPI** (*resp.* **HB**) and its multiple form is known (see for example [7]), however we provide a unified proof for sake of completeness. We shall also introduce the multiple countable forms $\mathbf{BPI}_w$ and $\mathbf{HB}_w$ of the statements **BPI** and **HB** (see Section 2.2.3).

2.2.1. *Measures with values in a reduced power of the field of real numbers.* Given a first order language $\mathbb{L}$ and a $\mathbb{L}$-structure $\mathfrak{M}$ on a (nonempty) set $M$, given a nonempty set $I$ we denote by $\mathfrak{M}^I$ the product $\mathbb{L}$-structure on $M^I$. If $\mathcal{F}$ is a filter on $I$, we denote by $\sim_I$ the equivalence relation on $M^I$ satisfying for every $x = (x_i)_{i\in I}, y = (y_i)_{i\in I} \in M^I$, $x \sim_I y$ iff$\{i \in I : x_i = y_i\} \in \mathcal{F}$. The quotient set $M^I/\mathcal{F}$ endowed with the natural $\mathbb{L}$-structure (see [7], [4, p. 442-444]) is called the *reduced power* of the structure $\mathfrak{M}$ with respect to the filter $\mathcal{F}$, and we denote by $\mathfrak{M}_{\mathcal{F}}$ this $\mathbb{L}$-structure. In particular, a reduced power $\mathbb{R}_{\mathcal{F}}$ of the ordered field $\mathbb{R}$ of real numbers is a lattice-ordered unitary real algebra such that the canonical one-to-one embedding $j_{\mathcal{F}} : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ is a morphism of $\mathbb{R}$-algebras. Moreover, the subset $\mathcal{L}_0(\mathbb{R}_{\mathcal{F}}) := \{x \in \mathbb{R}_{\mathcal{F}} : \exists \lambda \in \mathbb{R}\ x \leq j_{\mathcal{F}}(\lambda)\}$ is a subalgebra of $\mathbb{R}_{\mathcal{F}}$.

**Lemma 1.** *Given a family $(B_t, \oplus_t, \times_t, 0_t, 1_t)_{t\in T}$ of boolean algebras, there exists a filter $\mathcal{F}$ on a set $I$ and a family $(m_t)_{t\in T}$ such that for every $t \in T$, $m_t : \mathcal{B}_t \to (\mathbb{R}_{\mathcal{F}})_+$ is a finitely additive mapping satisfying the following extra conditions for all $x, y \in \mathcal{B}_t$:*

*(1) $m_t(1_t) = 1$*
*(2) $m_t(x \times_t y) = m_t(x) \cdot_{\mathbb{R}_{\mathcal{F}}} m_t(y)$ ($m_t$ is multiplicative)*

*Proof.* Without loss of generality, we assume that for distinct elements $s, t \in T$, $0_s = 0_t$ (which we denote by 0) and that $1_s = 1_t$ (which we denote by 1) and that $\mathcal{B}_s\backslash\{0,1\}$ and $\mathcal{B}_t\backslash\{0,1\}$ are disjoint. Let $B := \{0,1\} \sqcup \sqcup_{t\in T}(\mathcal{B}_t\backslash\{0,1\})$. Let $I := \mathbb{R}^B$. We consider the binary relation $R$ on $\mathrm{fin}(B) \times I$ such that for every finite subset $F$ of $B$ and every $m \in I$, $R(F, m)$ iff the following conditions are satisfied for every $t \in T$ and every $x, y \in F \cap \mathcal{B}_t$:

a) $m(0) = 0_{\mathbb{R}}$ and $m(1) = 1_{\mathbb{R}}$
b) $m(x \times y) = m(x) \cdot_{\mathbb{R}} m(y)$
c) $x \times y = 0 \Rightarrow m(x \oplus y) = m(x) +_{\mathbb{R}} m(y)$

The binary relation $R$ is concurrent (see [7, p. 125, Section 3]) which means that for every $F \in \mathrm{fin}(B)$ there exists $m \in I$ such that $\forall x \in F\ R(x, m)$: in fact, given a finite subset $F$ of $B$, the set $T_0 := \{t \in T : F \cap (B_t\backslash\{0,1\}) \neq \varnothing\}$ is finite; for each $t \in T_0$, let $\mathcal{C}_t$ be the (finite, non trivial) boolean subalgebra of $\mathcal{B}_t$ generated by $F \cap \mathcal{B}_t$; for every $t \in T_0$, consider a ultrafilter $\mathcal{U}_t$ of $\mathcal{C}_t$ and then, the mapping $m : B \to \{0,1\}$ such that $m(x) = 1$ if $x \in \cup_{t\in T_0}\mathcal{U}_t$ and 0 else. Then $R(F, m)$. For each $t \in T$, let $m_t : \mathcal{B}_t \to (\mathbb{R}_{\mathcal{F}})_+$ be the mapping associating to each $x \in \mathcal{B}_t$ the equivalence class of $(i(x))_{i\in I}$ in $\mathbb{R}_{\mathcal{F}}$: then $m_t$ is a finitely additive mapping satisfying conditions (1) and (2). □

2.2.2. *Multiple forms of* **BPI** *and* **HB**. The following result shows that **HB** and **BPI** are both "multiple" axioms. The case of **HB** was obtained by Luxemburg (see [7, Theorem 7.3]).

We provide a unified proof for the two statements **HB** and **BPI**, which relies on Lemma 1 and on the following result due to Luxemburg:

**Theorem 1** ([7])**.** *Let $\mathcal{F}$ be a filter on a set $I$. Let $\rho_{\mathcal{F}} : \mathcal{L}_0(\mathbb{R}_{\mathcal{F}}) \to \mathbb{R}_+$ be the mapping $x \mapsto \inf\{t \in \mathbb{R} : |x| \leq t\}$.*

*(1) The mapping $\rho_{\mathcal{F}} : \mathcal{L}_0(\mathbb{R}_{\mathcal{F}}) \to \mathbb{R}_+$ is a submultiplicative semi-norm and the set $\mathcal{L}_1(\mathbb{R}_{\mathcal{F}}) := \{x \in \mathcal{L}_0(\mathbb{R}_{\mathcal{F}}) : \rho_{\mathcal{F}}(x) = 0\}$ is a proper ideal of the ring $\mathcal{L}_0(\mathbb{R}_{\mathcal{F}})$.*

*(2)* **HB** *implies the existence of a positive linear form $l : \mathcal{L}_0(\mathbb{R}_{\mathcal{F}}) \to \mathbb{R}$ such that $l(1) = 1$ (and in particular, for every $x \in \mathcal{L}_0(\mathbb{R}_{\mathcal{F}})$, $|l(x)| \leq \rho_{\mathcal{F}}(x)$ whence $l$ is null on $\mathcal{L}_1(\mathbb{R}_{\mathcal{F}})$).*

*(3) If $\mathcal{U}$ is a ultrafilter on $I$ containing $\mathcal{F}$, then $\mathcal{L}_1(\mathbb{R}_{\mathcal{U}})$ is a maximal ideal of the ring (and thus a hyperplane of the vector space) $\mathcal{L}_0(\mathbb{R}_{\mathcal{U}})$, and the direct sum $\mathcal{L}_0(\mathbb{R}_{\mathcal{U}}) = \mathcal{L}_1(\mathbb{R}_{\mathcal{U}}) \oplus \mathbb{R}$ provides the (unique) multiplicative unitary linear form $l_{\mathcal{U}} : \mathcal{L}_0(\mathbb{R}_{\mathcal{U}}) \to \mathbb{R}$.*

*(4)* **BPI** *implies the existence of a positive multiplicative unitary linear form $l : \mathcal{L}_0(\mathbb{R}_{\mathcal{F}}) \to \mathbb{R}$.*

*Proof.* (1) See [7, Section 4]. (2) See [7, Theorem 6.1]. (3) Given $x \in \mathcal{L}_0(\mathbb{R}_{\mathcal{U}})$, for every bounded $(x_i)_{i\in I} \in \mathbb{R}^I$ in the equivalence class of $x$, the family $(x_i)_{i\in I}$ converges to some real number with respect to the ultrafilter $\mathcal{U}$, and this real number, which does not depend on the representative $(x_i)_{i\in I}$ of $x$, is equal to $\rho_{\mathcal{U}}(x)$. This implies that $\rho_{\mathcal{U}} : \mathcal{L}_0(\mathbb{R}_{\mathcal{U}}) \to \mathbb{R}$ is linear, multiplicative and unitary, so its kernel $\mathcal{L}_1(\mathbb{R}_{\mathcal{U}})$ is a maximal ideal of $\mathcal{L}_0(\mathbb{R}_{\mathcal{U}})$.
(4) Using **BPI**, let $\mathcal{U}$ be a ultrafilter on $I$ containing $\mathcal{F}$. By composing the natural quotient mapping $\mathrm{can} : \mathcal{L}_0(\mathbb{R}_{\mathcal{F}}) \to \mathcal{L}_0(\mathbb{R}_{\mathcal{U}})$ and the multiplicative linear unitary form $l_{\mathcal{U}} : \mathcal{L}_0(\mathbb{R}_{\mathcal{U}}) \to \mathbb{R}$ we get a multiplicative unitary (positive) linear form $l_{\mathcal{U}} \circ \mathrm{can} : \mathcal{L}_0(\mathbb{R}_{\mathcal{F}}) \to \mathbb{R}$. □

**Proposition 2.** *(1) The axiom* **HB** *is equivalent to its "multiple form".*
*(2) The axiom* **BPI** *is equivalent to its "multiple form".*

*Proof.* Given a family $(\mathcal{B}_t)_{t\in T}$ of non trivial boolean algebras, consider with Lemma 1 a filter $\mathcal{F}$ on a set $I$ and a family $(m_t)_{t\in T}$ of such that for every $t \in T$, $m_t : \mathcal{B}_t \to \mathbb{R}_{\mathcal{F}}$ is a finitely additive and multiplicative measure such that $m_t(1_{\mathcal{B}_t}) = 1$.
(1) Using Theorem 1-(2), **HB** implies a positive, unitary linear form $l : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}$. Then for each $t \in T$, $\mu_t := l \circ m_t : \mathcal{B}_t \to [0, 1]$ is a probability.
(2) Using Theorem 1-(4), **BPI** implies a multiplicative unitary linear form $l : \mathcal{L}_0(\mathbb{R}_{\mathcal{F}}) \to \mathbb{R}$. Then for each $t \in T$, the probability $\mu_t := l \circ m_t$ on $\mathcal{B}_t$ is $\{0, 1\}$-valued: given $t \in T$ and $x \in \mathcal{B}_t$, $y := m_t(x) \in \{0, 1\}_{\mathcal{F}}$ thus $y^2 = y$ so by multiplicativity of $l$, $l(y) \in \{0, 1\}$. □

*Remark* 3. It is also possible to prove Proposition 2 using coproducts of boolean algebras built in **ZF** (see [9, Section 3.3.1 p. 127]).

2.2.3. *Multiple countable forms of* **BPI** *and* **HB***.* We shall consider the following weakened forms of **BPI** and **HB**:

> $\mathbf{HB}_w$: *For every family $(\mathcal{B}_i)_{i\in I}$ of non trivial countable boolean algebras, there exists a family $(\mu_i)_{i\in I}$ such that for every $i \in I$, $\mu_i$ is a probability on $\mathcal{B}_i$.*
>
> $\mathbf{BPI}_w$: *For every family $(\mathcal{B}_i)_{i\in I}$ of non trivial countable boolean algebras, there exists a family $(\mu_i)_{i\in I}$ such that for every $i \in I$, $\mu_i : \mathcal{B}_i \to \{0, 1\}$ is a two-valued probability on $\mathcal{B}_i$.*

The axiom $\mathbf{BPI}_w$ (and thus $\mathbf{HB}_w$) is a consequence of the following axiom of choice for countable sets (which does not imply **BPI** in **ZF**, see for example Mathias' model $\mathcal{M}_3$ -[5, p. 149]-):

$\mathbf{AC}^\omega$ (form 85): *Every family $(X_i)_{i\in I}$ of countable nonempty sets has a choice function.*

**Proposition 3.** *(1) $\mathbf{AC}^\omega$ is equivalent to the following statement:* "For every family $(X_i)_{i\in I}$ of countable sets, there exists a family $(\leq_i)_{i\in I}$ such that for each $i \in I$, $\leq_i$ is a well order on $X_i$."
*(2)* $\mathbf{AC}^\omega \Rightarrow \mathbf{BPI}_w \Rightarrow \mathbf{HB}_w$.

*Proof.* (1) Let $(X_i)_{i\in I}$ be a family of countable sets. Using $\mathbf{AC}^\omega$, consider a choice function $\Phi$ for the family of countable subsets of $\cup_{i\in I} X_i$. For each $i \in I$, $\Phi$ induces a choice function $\Phi_i : \mathcal{P}(X_i)\backslash\{\varnothing\} \to X_i$ associating to each nonempty (countable) subset $S$ of $X_i$ an element of $S$. Each mapping $\Phi_i$ provides a well order $\leq_i$ on $X_i$.
(2) Let $(\mathcal{B}_i)_{i\in I}$ be a family of countable boolean algebras. Using $\mathbf{AC}^\omega$, we choose for each $i \in I$ a well order $\leq_i$ on $\mathcal{B}_i$. For each $i \in I$, using this well order $\leq_i$, define an ordinal $\alpha_i$ and a one-to-one family $(x^i_t)_{t\in\alpha_i}$ of $\mathcal{B}_i$ such for every $t \in \alpha_i$, $x^i_t \notin \mathrm{bool}(\{x^i_s : s < t\})$ and such that $\mathrm{bool}(\{x^i_s : s \in \alpha_i\}) = \mathcal{B}_i$. For each $t \in \alpha_i$, let $\mathcal{B}^i_t$ be the boolean algebra $\mathrm{bool}(\{x^i_s : s < t\}) = \cup_{s<t}\mathcal{B}^i_s$; then for each $i \in I$, one can define by recursion on $t \in \alpha_i$ a $\{0, 1\}$-probability $\mu_i$ on $\mathcal{B}_i$. $\square$

### 2.3. **Paradoxical decompositions of $G$-sets.**

2.3.1. *Left and right $G$-sets.* Given a group $(G, *, 1)$ and a nonempty set $X$, a *left action* (*resp. right action*) of $G$ on $X$ is a mapping $. : G \times X \to X$ (*resp.* $. : X \times G \to X$) such that for every $g_1, g_2 \in G$ and every $x \in X$, $e.x = x$ and $g_2.(g_1.x) = (g_2 * g_1).x$ (*resp.* $x.e = x$ and $(x.g_1).g_2 = x.(g_1 * g_2)$).

Given a left (*resp.* right) action . of a group $G$ on a set $X$, for every $g \in G$, the mapping $x \mapsto g.x$ (*resp.* $x \mapsto x.g$) is a permutation of $X$ which is called the *$g$-translation* and is denoted by $t_g$. A *left $G$-set* (*resp. right $G$-set*) is a nonempty set $X$ endowed with a left (*resp.* right) action of $G$ on $X$.

2.3.2. *Orbits and free actions.* Given an action . of a group $G$ on a nonempty set $X$, the binary relation $R := \{(x, t_g(x)) : x \in X; g \in G\}$ on $X$ is an equivalence relation on $X$ and the equivalence classes of $R$ are called the *orbits* of the action . on $X$. The action . of $G$ on $X$ is *free* if for every $g \in G\backslash\{e\}$, the translation $t_g : X \to X$ does not have any fixed point.

*Example* 1. Given a group $(G, *, 1)$, then $* : G \times G \to G$ is both a left and right action of $G$ on $G$ which admits only one orbit. This action is free and is called the *action by translations of $G$ on itself.*

2.3.3. *Paradoxical decompositions of right $G$-sets.*

**Definition 1.** Given a left (*resp.* right) action . of a group $G$ on a nonempty set $X$, given two integers $m, n \geq 2$, a *$(m, n)$-paradoxical decomposition* of the $G$-set $X$ is a partition $X = \sqcup_{1\leq i\leq m} A_i \sqcup \sqcup_{1\leq i\leq n} B_i$ of $X$ in $m + n$ (nonempty) sets together with $m + n$ elements $a_1, \ldots, a_m, b_1, \ldots, b_n \in G$ satisfying $X = \sqcup_{1\leq i\leq m} a_i.A_i = \sqcup_{1\leq i\leq n} b_i.B_i$ (*resp.* $X = \sqcup_{1\leq i\leq m} A_i.a_i = \sqcup_{1\leq i\leq n} B_i.b_i$).

A $(2, 2)$-paradoxical decomposition of a right $G$-set $X$ is thus a partition $X = A_1 \sqcup A_2 \sqcup B_1 \sqcup B_2$ of $X$ in four (nonempty) sets together with four elements $a_1, a_2, b_1, b_2 \in G$ satisfying $X = A_1.a_1 \sqcup A_2.a_2 = B_1.b_1 \sqcup B_2.b_2$. Equivalently, a $(2, 2)$-paradoxical decomposition of a right $G$-set $X$ is a partition $X = A_1 \sqcup A_2 \sqcup B_1 \sqcup B_2$ of $X$ in four (nonempty) sets endowed with two elements $c_1, c_2 \in G$ satisfying $X = A_1 \sqcup A_2.c_1 = B_1 \sqcup B_2.c_2$ (consider $c_1 = a_2 * a_1^{-1}$ and $c_2 = b_2 * b_1^{-1}$).

2.4. **Free groups.** Given a nonempty set $A$, we denote by $A^*$ the *free monoid* on $A$ *i.e.* the set of words (sequences) on the alphabet $A$ endowed with the usual *concatenation law* $*$: this law is associative and the empty word $\varepsilon$ is the neutral element of this law.

We denote by $F(A)$ the *free group* on the alphabet $A$ (see [2, Def. 2.1.7 p. 31]): this group is obtained as follows. Consider a one-to-one mapping $u$ of $A$ onto a subset $A^-$ which is disjoint from $A$. Denoting by $u^{-1} : A^- \to A$ the inverse bijection of $u$, then $\iota := u \cup u^-$ is an involution of the new alphabet $C := (A \sqcup A^-)$. For each $x \in C$, we denote by $x^-$ the letter $\iota(x)$ in the alphabet $C$. We consider on the monoid $C^*$ the congruence $\equiv$ generated by the binary relation $\{(c * c^-, \varepsilon) : c \in C\}$. Then the quotient monoid $C^*/\equiv$ is a group which is called the *free group* on the alphabet $A$.

*Remark* 4 (Another construction of the free group on an alphabet $A$). We consider on $C^*$ the equivalence relation $R$ generated by $\{(w_1 * c * c^- * w_2, w_1 * w_2) : w_1, w_2 \in C^*; c \in C\}$. A word $w \in C^*$ is said to be *reduced* if the word $w$ has no factor of the form $c * c^-$ where $c \in C$. Denoting by $R(A)$ the set of reduced words on the alphabet $C$, let $\rho : C^* \to R(A)$ be the reduction mapping associating to each $w \in C^*$ the unique element of $R(A)$ which is $R$-equivalent to $w$. We endow $R(A)$ with the binary operation . satisfying for every $w_1, w_2 \in R(A)$, $w_1.w_2 := \rho(w_1 * w_2)$. Then $(R(A), .)$ is a group which is isomorphic with the free group $F(A)$.

**Definition 2** ((2, 2)-paradoxical decomposition by letters of a free $F_2$-set)**.** If $F_2$ is the free group on the alphabet $\{a, b\}$, given a right (*resp.* left) action . of $F_2$ on a set $X$, a $(2,2)$-paradoxical decomposition *by letters* is a $(2,2)$-paradoxical decomposition $X = A_1 \sqcup A_2 \sqcup B_1 \sqcup B_2$ such that $X = A_1 \sqcup A_2.a^- = A_3 \sqcup A_4.b^-$ or, equivalently, $X = A_1.a \sqcup A_2 = A_3.b \sqcup A_4$ (*resp.* $X = A_1 \sqcup a^-.A_2 = A_3 \sqcup b^-.A_4$ or, equivalently, $X = a.A_1 \sqcup A_2 = b.A_3 \sqcup A_4$).

**Notation 3.** If $F$ is the free group on a (nonempty) alphabet $A$, for each $c \in A \sqcup A^-$, we denote by $\Gamma(c)$ the set of elements $w \in F(A)$ which correspond to reduced words beginning with the letter $c$.

*Example* 2 (The classical $(2,2)$-paradoxical decomposition by letters of the free group $F_2$). Let $F_2$ be the free group over the 2-element alphabet $A = \{a, b\}$. We consider the (free) left action . of $F_2$ by translations on itself. Then $F_2$ has the following $(2,2)$-paradoxical decomposition by letters: $F_2 = \Gamma(b) \sqcup \Gamma(b^-) \sqcup \big(\Gamma(a) \sqcup R\big) \sqcup \big(\Gamma(a^-) \backslash R\big)$ where $R := \{a^{-n} : n \in \mathbb{N}\}$. Then $F_2 = \big(b^-.\Gamma(b)\big) \sqcup \Gamma(b^-) = \big(a^-.(\Gamma(a) \sqcup R)\big) \sqcup \big(\Gamma(a^-) \backslash R\big)$.

*Proof.* See [15, p.28, Fig. 3.2]. □

## 3. Cayley graphs of actions

### 3.1. Graphs.

3.1.1. *Paths and cycles.* A *graph* on a set $V$ is a binary relation $R$ on $V$ which has *no loops* ($\forall x \in V\ x \not R x$) and which is *symmetric* ($\forall x, y \in V\ (xRy \Rightarrow yRx)$). Given a graph $R$ on a set $V$, elements of $V$ are called the *vertices* of the graph, and pairs of (distinct) elements $x, y \in V$ such that $xRy$ are called the *edges* of the graph. A graph $G$ on a set $V$ is thus defined by a subset $E$ of the set $[V]^2$ of two-element subsets of $V$: we also denote by $(V, E)$ such a graph $G$. Given some vertex $v$ of a graph $G = (V, E)$, the *neighbourhood* of $v$ is the set $N(v)$ of vertices $w$ such that $\{v, w\}$ is an edge of the graph; if $N(v)$ is finite, then the cardinal of $N(v)$ is the *degree* of $v$. If for every vertex $v$ of $G$ the neighbourhood of $v$ is finite, then the graph $G$ is said to be *locally finite*; moreover, if there exists some natural number $n$ such that the degree of every vertex of $G$ is $n$, then the graph $G$ is said to be *$n$-regular*.

A *subgraph* of a graph $G = (V, E)$ is a graph $(V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. A subgraph $G' = (V', E')$ of $G = (V, E)$ is *induced* if $E' = E \cap [V']^2$. Given two graphs $G = (V, E)$ and $G' = (V', E')$, a *graph morphism* from $G$ to $G'$ is a mapping $f : V \to V'$ such that for every edge $e = \{x, y\}$ of $G$, $\{f(x), f(y)\}$ is an edge of the graph $G'$. A *graph isomorphism* from $G$ to $G'$ is a bijective morphism $f : (V, E) \to (V', E')$ such that $f^{-1} : (V', E') \to (V, E)$ is also a graph morphism (equivalently, it is a bijection $f : V \to V'$ such that for every $x, y \in V$, $\{x, y\}$ is an edge of $G$ if and only if $\{f(x), f(y)\}$ is an edge of $G'$).

*Example* 3 (Paths). Given some natural number $n \geq 1$, the graph on the set $\{0, \ldots, n\}$ with set of edges $\{\{i, i+1\} : i \in \{0, \ldots, n-1\}\}$ is called the *path* $P_n$; every graph isomorphic with $P_n$ is called a *$n$-path*. A *path* is a $n$-path for some natural number $n \geq 1$. Given a path $(V, E)$, the two vertices with degree 1 are called the *extremities* of the path, the other vertices have degree 2. The *length* of a $n$-path is the number $n$ of edges of this path; notice that the number of vertices of a $n$-path is $n + 1$.

A *walk* in the graph $G = (V, E)$ is a sequence $(v_i)_{0 \leq i \leq n}$ of vertices such that $n \geq 1$ and such that for each natural number $i < n$, $\{v_i, v_{i+1}\}$ is an edge of $G$: the vertex $v_0$ is the origin of the walk and $v_n$ is the target of the walk. The graph $G$ is said to be *connected* if for every distinct vertices $x, y \in V$, there exists a walk in $G$ with origin $x$ and target $y$ (equivalently, there exists in $G$ a path with extremities $x$ and $y$). Given a connected graph $G = (V, E)$, consider the mapping $d_G : V \times V \to \mathbb{N}$ associating to each $(x, y) \in V \times V$ the natural number 0 if $x = y$ and the length of a shortest path with extremities $x$ and $y$ if $x \neq y$. Then $d_G$ is a distance on $V$ which is called the *graphic distance* associated to the connected graph $G$ on $V$.

*Example* 4 (Cycles). Given some natural number $n \geq 3$, the graph on $\{0, \ldots, n-1\}$ with set of edges $\{\{i, i+1\} : i \in \{0, \ldots, n-2\}\} \cup \{\{n-1, 0\}\}$ is called the *cycle* $C_n$; ; every graph isomorphic with $C_n$ is called a *$n$-cycle*. A *cycle* is a $n$-cycle for some natural number $n \geq 3$. Every vertex of a cycle has degree 2.

3.1.2. *Trees.* A *forest* is a graph $G = (V, E)$ such that no subgraph of $G$ is a cycle. A *tree* is a connected forest. A *leaf* of a tree is a vertex with degree 1 in this tree. Every finite tree with at least two vertices has at least two leaves. Given a tree $T = (V, E)$, if $L$ is a subset of the set of leaves of $T$, then the subgraph induced by the tree $T$ on $V \setminus L$ is still a tree. Notice that a finite tree with at least 3 vertices has at least one vertex which is not a leaf.

**Definition 3** (center of a finite tree)**.** Given a finite nonempty tree $T$, the process of removing leaves from the tree while it has at least 3 vertices leads to the choice of a vertex $x$ or an edge $\{y, z\}$ of $T$: we call this set $\{x\}$ or $\{y, z\}$ the *center* of the finite nonempty tree $T$.

**Notation 4** (segment in a tree)**.** If $x$ and $y$ are two distinct vertices of a tree $T = (V, E)$, there exists a unique path in $T$ with extremities $x$ and $y$: the set of vertices on this path is called the *segment* with extremities $x$ and $y$ and is denoted by $[x,y]$ or $[y,x]$; if $x=y$, then we define $[x,x] := \{x\}$.

**Notation 5** (generated subtree)**.** Given a tree $T$ on a set $V$, if $W$ is a subset of $V$, the graph induced by the tree $T$ on $\cup_{x,y\in W}[x,y]$ is a tree: we denote by $\mathrm{tr}(W)$ this tree which is the smallest subtree of $T$ including $W$.

Notice that if $W$ is a finite subset of vertices of a tree, then the generated subtree $\mathrm{tr}(W)$ is also finite; also notice that every leaf $v$ of $\mathrm{tr}(W)$ belongs to $W$.

3.2. **Oriented graphs.** Given a graph $G=(V,E)$, an *orientation* of the graph $G$ is a choice function for the set $E$ of edges *i.e.* a mapping $o : E \to V$ such that for every edge $e = \{x,y\}$ of $G$, $o(e) \in \{x,y\}$. The axiom $\mathbf{AC}^2$ implies (and is equivalent to the fact that) that every graph has an orientation. An *oriented graph* on a set $V$ is a graph $G=(V,E)$ endowed with an orientation $o : E \to V$. Equivalently, an oriented graph corresponds to a binary relation $R$ on $V$ which has no loops ($\forall x \in V\ x \not\!R x$) and is *antisymmetric* (for every $x,y \in V$, if $x \neq y$ and $xRy$ then $y \not\!R x$). Given an oriented graph $R$ on $V$, couples $(x,y)$ in $R$ are called the *arcs* of the oriented graph. If $v$ is a vertex of an oriented graph, the *out-degree* of $v$ is the number of vertices $w$ such that $(v,w)$ is an arc of the oriented graph, and the *in-degree* of $v$ is the number of vertices $w$ such that $(w,v)$ is an arc. A *source* of an oriented graph is a vertex $v$ with in-degree 0 and a *sink* is a vertex with out-degree 0.

**Proposition 4.** *If $(T_i = (V_i, E_i, o_i))_{i\in I}$ is a family of finite nonempty oriented trees, then $(V_i)_{i\in I}$ has a choice function.*

*Proof.* For each $i \in I$, consider the center $Z_i$ of the tree; if $Z_i$ is an edge $e$ of the tree $T_i$, then $o_i(e)$ is a vertex $v_i$ of $T_i$, else $Z_i$ is a singleton $\{v_i\}$, and $(v_i)_{i\in I}$ is a choice function for $(V_i)_{i\in I}$. □

3.3. **A sufficient condition for the finite choice property on an oriented tree.** We recall (see Section 2.1.1) that a set $X$ satisfies the finite choice property if $X$ has a witness of the finite choice property *i.e.* a mapping $\Phi : \mathrm{fin}^*(X) \to X$ such that for every $F \in \mathrm{fin}^*(X)$, $\Phi(F) \in F$.

**Definition 4.** Given a graph $G=(V,E)$, a *neighbour choice* in $G$ is a mapping $f : V \to V$ such that for each $v \in V$, if $N(v) \neq \varnothing$ then $f(v) \in N(v)$. A *double neighbour choice* on a graph $G=(V,E)$ is an ordered pair $(f_1, f_2)$ of neighbour choices on $G$ such that for each $v \in V$ which has at least two neighbours, $f_1(v) \neq f_2(v)$.

*Example* 5. If $X$ is a free $F_2$-set, every subgraph $G=(V,E)$ of the forest $X$ has a double neighbour choice.

*Proof.* We linearly order the set $\{n,s,e,w\}$ of cardinal points. We then define a first neighbour choice $f_1 : V \to V$: given some vertex $v \in V$, if $v$ has at least one neighbour in $G$, we define $f_1(v) := v.c$ where $c$ is the first element in $\{n,e,s,w\}$ such that $v.c$ is a neighbour of $v$ in $G$, else $f_1(v) := v$. We define a second neighbour choice $f_2 : V \to V$ in the same way: for each $v \in V$, if $v$ has at least two distinct neighbours in $G$, we define $f_2(v) := v.c$ where $c$ is the first element in $\{n,e,s,w\}$ such that $v.c$ is a neighbour of $v$ which is distinct from $f_1(v)$, else $f_2(v) := f_1(v)$. □

**Proposition 5.** *Let $T = (V, E, o)$ be an oriented tree with at least two vertices. If $T$ has a double neighbour choice $(f_1, f_2)$, then there exists on $V$ a witness of the finite choice property which is definable from $(f_1, f_2)$ and $o$.*

*Proof.* We define a witness $\Phi$ of the finite choice property on $V$. Given a nonempty finite subset $F$ of $V$, we consider the generated tree $\mathrm{tr}(F)$. Using Proposition 4, we choose a vertex $r \in \mathrm{tr}(F)$. Using $(f_1, f_2)$, we build a one-to-one walk $(v_i)_{0\leq i\leq n}$ of $\mathrm{tr}(F)$ with origin $r$ and target a leaf $v_n$ of $\mathrm{tr}(F)$: then $v_n \in F$ and we define $\Phi(F) := v_n$. $\square$

3.4. **Rays and ends of a graph.** Given a graph $G = (V, E)$, a *ray* of $G$ is a one-to-one sequence $(x_n)_{n\in\mathbb{N}}$ of vertices of the graph such that for each $n \in \mathbb{N}$, $\{x_n, x_{n+1}\}$ is an edge of the graph. Two rays $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ of $G$ are said to be *equivalent* if there exists a ray $z = (z_n)_{n\in\mathbb{N}}$ of $G$ such that $\{x_n : n \in \mathbb{N}\} \cap \{z_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\} \cap \{z_n : n \in \mathbb{N}\}$ are both infinite. This binary relation on the set of rays of $G$ is obviously reflexive and symmetric; it is also transitive (see [3]). Equivalence classes of this equivalence relation are called the *ends* of the graph $G$.

**Proposition 6.** *Let $T = (V, E)$ be a tree.*

*(1) Given two distinct rays $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ of $T$, then $I := \{x_n : n \in \mathbb{N}\} \cap \{y_n : n \in \mathbb{N}\}$ is either $\varnothing$ or a singleton or a segment joining two vertices $v_1, v_2$ of $T$ or there exists $m_0, n_0 \in \mathbb{N}$ such that $I = \{x_n : n \in \mathbb{N}, n \geq m_0\} = \{y_n : n \in \mathbb{N}, n \geq n_0\}$.*
*(2) Two equivalent rays of $T$ with the same origin are equal.*
*(3) If $(x_n)_{n\in\mathbb{N}}$ is a ray of $T$, for every vertex $y \in V$, there exists a (unique) ray with origin $y$ which is equivalent to $(x_n)_{n\in\mathbb{N}}$.*

*Proof.* (1) We first notice that the subgraph $T'$ induced by $T$ on $I$ is connected (because $T$ is a tree), and that every vertex of $T'$ has at most two neighbours. If $I$ is finite then $T'$ is empty or a singleton or a (finite) path; if $I$ is infinite and if $T'$ has a vertex $v$ of degree 1 then $T'$ is a ray with origin $v$; the case where every vertex of $T'$ has degree 2 does not occur since $T'$ is a subgraph of a ray.
(2) This follows from the fact that the tree has no cycles.
(3) Let $x_{n_0}$ be the element of $\{x_n : n \in \mathbb{N}\}$ closest to $y$. If $x_{n_0} = y$ then the ray $(x_k)_{k\geq n_0}$ has origin $y$ and is equivalent to the ray $x$. If $x_{n_0} \neq y$ then the ray with origin $y$ which is the concatenation of the one-to-one walk leading from $y$ to $x_{n_0}$ and of the ray $(x_k)_{k\geq n_0}$ is equivalent to the ray $\{x_n : n \in \mathbb{N}\}$. $\square$

3.5. **Right $G$-sets and their Cayley graphs.**

3.5.1. *The Cayley graph of a free right G-set.* Given a *free* right action . of a group $(G, *, 1)$ on a nonempty set $X$ and a subset $C$ of $G$ generating the group $G$, given two distinct elements $x, y$ in $X$, then, there exists at most one element $g \in C$ such that $x.g = y$. The binary relation $R_C = \{(x, x.c) : c \in C\}$ on $X$ is called the *$C$-Cayley relation* associated to the free action . of $G$. If $1 \notin C$, then $R_C$ has no loops and if $C$ is stable by inversion ($c \in C \Rightarrow c^{-1} \in C$ or equivalently, $C = C^{-1}$) then $R_C$ is symmetric; if $1 \notin C$ and if $C = C^{-1}$, then $R_C$ is a graph and is called the *$C$-Cayley graph* of the free right action . on $X$.

**Proposition 7.** *Let $(G, *, 1)$ be a group and let $C$ be a subset of $G$ generating $G$ such that $1 \notin C$ and $C = C^{-1}$. Given a* free *right action . of the group $G$ on a nonempty set $X$ and denoting by $R_C$ the $C$-Cayley graph of the action ., then:*

*(1) The connected components of the graph $R_C$ are the orbits of the action.*
*(2) For every $x \in X$, the mapping $f_x : C \to N(x)$ associating to each $c \in C$ the element $x.c$ is bijective.*
*(3) If there is a subset $A$ of $C$ such that $C = A \sqcup A^{-1}$, then the graph $R_C$ has an orientation which is definable from $A$ and the action . of $G$ on $X$.*

*Proof.* (1) is almost obvious.
(2) If $x \in X$, the neighbourhood of $x$ in the graph $R_C$ is $N_x = \{c.x : c \in C\}$ so the mapping $f_x : c \mapsto c.x$ from $C$ to $N(x)$ is one-to-one since $G$ acts freely on $X$.
(3) Denoting by $E$ the set of edges of the graph $R_C$, we define an orientation $o : E \to X$ of $R_C$: given an edge $e$ of the graph $R_C$, there exists exactly one element $x \in X$ and one element $a \in A$ and such that $e = \{x, x.a\}$: we define $o(e) := x$. Then $o : E \to V$ is a choice function for the family of edges of the graph $R_C$, which is definable from $A$ and the action . of $G$ on $X$. $\square$

3.5.2. *Free actions of free groups and regular trees.*

**Proposition 8.** *Let $A$ be a nonempty alphabet and let $F$ be the free group on $A$. Let . be a free right-action of $F$ on a nonempty (infinite) set $X$. Let $C := A \sqcup A^-$ and let $G = (X, E)$ be the $C$-Cayley graph of this action. Then,*

*(1) The graph $G$ is a forest, and thus the (connected) subgraph induced by $G$ on each orbit is a tree.*
*(2) The graph $G$ has an orientation definable from $A$ and the action . of $F$ on $X$.*
*(3) If $A$ satisfies the finite choice property (and in particular if $A$ is finite), the forest $G$ has a double neighbour choice and $X$ also satisfies the finite choice property.*

*Proof.* (1) If $G$ has a cycle for some natural number $n \geq 3$, let $V_1 := \{x_0, \dots, x_{n-1}\}$ be a $n$-element subset of $X$ such that for each $i \in \{0, \dots, n-1\}$, $\{x_i, x_{i+_n 1}\}$ is an edge of $G$, where $+_n$ is the additive law on $\{0, \dots, (n-1)\}$ *modulo* $n$. For every $i \in \{0, \dots, n-1\}$, let $c_i$ be the element of $C$ such that $x_i.c_i = x_{i+_n 1}$. Then $x_0.(c_0 * \cdots * c_{n-1}) = x_0$. Since the action . of $F$ on $X$ is free, the word $(c_0 * \cdots * c_{n-1})$ reduces to the neutral element in $F$; it follows that there exists $i \in \{0, \dots n-2\}$ such that $c_{i+1} = c_i^-$ and thus $x_{i+_n 2} = x_i$ which is contradictory.
(2) Using Proposition 7, the graph $G$ has an orientation (definable from $A$ and the action . of $F$ on $X$).
(3) For every $x \in X$, denote by $f_x : C \to N(x)$ the bijection $c \mapsto x.c$. Since $A$ (and thus $C$) satisfies the finite choice property, the family of bijections $(f_x)_{x \in X}$ allows to define a double neighbour choice $(f_1, f_2)$ on the forest $G$. With Proposition 5, the double neighbour choice $(f_1, f_2)$ on the oriented forest $G$ implies a witness of the finite property on $X$ (which is definable from $(f_x)_{x \in X}$ and a witness of the finite choice property on $X$). $\square$

## 4. Paradoxical decompositions of free $F_2$-sets, ends and 1-out orientations

4.1. **1-out oriented graphs.** An orientation $o$ of a graph $G$ is said to be *1-out* if every vertex of the oriented graph $(G, o)$ has out-degree 1. Notice that if a graph $G = (V, E)$ has a 1-out orientation, then $V$ is infinite (because every finite oriented graph with at least two vertices has at least one source and at least one sink).

**Notation 6** (ray $[x, o)$ w.r.t. a 1-out orientation $o$)**.** Given a 1-out orientation $o$ of a graph $G$ on a set $V$, and given $x \in V$, there exists a unique sequence $(x_n)_{n\in\mathbb{N}}$ of $V$ with origin $x_0 := x$ such that for every $n \in \mathbb{N}$, $(x_n, x_{n+1})$ is an arc of the oriented graph $(G, o)$. If the sequence $(x_n)_{n\in\mathbb{N}}$ is one-to-one (which is the case if $G$ is a forest), then $(x_n)_{n\in\mathbb{N}}$ is called the *oriented ray with origin $x$ with respect to the 1-out orientation $o$* and it is denoted by $[x, o)$.

**Proposition 9** (A natural bijection between ends and 1-out orientations of a tree)**.** *Let $T = (V, E)$ be a tree.*

*(1) If $o$ is a 1-out orientation on $T$, then, for all $x, y \in V$, the rays $[x, o)$ and $[y, o)$ are equivalent. The end corresponding to the (equivalent) rays $[x, o)$ for $x \in V$ is called the* end associated to the 1-out orientation $o$ *in the tree $T$.*

*(2) If $\beta$ is an end of $T$, then*

*-For every $y \in V$, the end $\beta$ contains a unique ray $\rho_{y,\beta}$ with origin $y$.*

*-For every edge $e = \{x, y\}$ of $T$, either $x \in \rho_{y,\beta}$ or $y \in \rho_{x,\beta}$*

*-The mapping $o : E \to V$ such that for every edge $e = \{x, y\}$, $o(e) = y$ if $x \in \rho_{y,\beta}$ and $o(e) = x$ if $y \in \rho_{x,\beta}$ is a 1-out orientation of $T$.*

*Proof.* (1) Let $x_{n_0}$ the vertex in $[x, o)$ closest to $y$. If $x_{n_0} = y$, $[y, o) = (x_n)_{n\geq n_0}$ is equivalent to the ray $[x, o)$, else, let $w = (v_0 = y, \ldots, v_p = x_{n_0})$ be the one-to-one walk from $y$ to $x_{n_0}$; then for each $i \in \{0, \ldots, p-1\}$, $o(\{v_i, v_{i+1}\}) = v_i$ (induction on $p$) and thus the ray $[y, o)$ is the concatenation of the walk $w$ and of the ray $(x_n)_{n\geq n_0}$ so $[x, o)$ and $[y, o)$ are equivalent.
(2) -Let $(x_n)_{n\in\mathbb{N}}$ be a ray in the equivalence class $\beta$. Using Proposition 6, there exists a unique ray $\rho_{y,\beta}$ with origin $y$ which is equivalent to $(x_n)_{n\in\mathbb{N}}$ *i.e.* which belongs to $\beta$.

-Given an edge $\{x, y\}$ of the tree $T$, assume that $y \notin \rho_{x,\beta}$. Then the vertex $x_{n_0}$ in $\rho_{x,\beta}$ which is closest to $y$ is not equal with $y$. If $x_{n_0} \neq x$, then there is a cycle of the graph $G$ containing the vertices $y$, $x_{n_0}$ and $x$, which is contradictory. So $x_{n_0} = x$ and thus $x \in \rho_{y,\beta}$.

-The mapping $o$ is an orientation of the graph $G$. Moreover, if $x \in V$, then, denoting by $(x_n)_n$ the ray $\rho_{x,\beta}$, then for every neighbour $y$ of $x$, $o(\{x, y\}) = x$ if $y = x_1$, and $o(\{x, y\}) = y$ if $y \neq x_1$, so the out-degree of $x$ with respect to the orientation $o$ is 1. $\square$

4.2. **(2,2)-paradoxical decompositions by letters of free $F_2$-sets and 1-out orientations of the Cayley-graphs.** From now on, we shall denote by $F_2$ the free group on the two-element alphabet $A := \{e, n\}$ (East, North). We denote by $s$ (South) the inverse of $n$ in $F_2$ and we denote by $w$ (West) the inverse of $e$ in $F_2$. Then $C := A \sqcup A^- = \{n, s, e, w\}$ ($C$ is the set of the four cardinal points). Denoting by $\varepsilon$ the neutral element of the group $F_2$, $F_2 = \{\varepsilon\} \sqcup \sqcup_{c\in C}\Gamma(c)$ (see Notation 3).

For each free $F_2$-set $X$, we separate edges of the forest $X$ into "horizontal" edges and "vertical" edges: an edge of $X$ is *horizontal* if it is of the form $\{x, x.e\}$ (or equivalently of the form $\{y, y.w\}$) and it is *vertical* if it is of the form $\{x, x.n\}$ (or equivalently of the form $\{y, y.s\}$). We then consider the following orientation $o$ of $X$: every horizontal edge $e = \{x, x.e\}$ of $X$ is oriented from west to east *i.e.* $o(e) := x$, and every vertical edge $e = \{x, x.n\}$ is oriented from south to north *i.e.* $o(e) := x$. Notice that with respect to this orientation, every vertex of $X$ has out-degree 2 and in-degree 2 and thus the orientation $o$ is not a 1-out orientation of the forest $X$.

**Theorem 2.** *Let . be a free right-action of the group $F_2$ on a nonempty (infinite) set $X$. Let $C = \{e, n, w := e^{-1}, s := n^{-1}\}$ and let $G$ be the $C$-Cayley graph of the action . on $X$. The followings data are equivalent:*

*(1) A $(2,2)$-paradoxical decomposition by letters of the free right $F_2$-set $X$;*
*(2) A 1-out orientation of the forest $G$ (or equivalently -see Proposition 9- the choice of an end in each connected component of the forest $G$)*

*Proof.* (1) $\Rightarrow$ (2). Assuming that $X = X_n \sqcup X_s \sqcup X_e \sqcup X_w$ is a partition of $X$ in four pieces such that $X = X_n \sqcup X_s.s = X_e \sqcup X_w.w$, we shall define a 1-out orientation $o$ of the forest $G$. For each vertex $x \in X$, if $x \in X_n$, we orient the vertical edge $a = \{x, x.n\}$ in the "north direction" *i.e* we define $o(a) := x$, else $x \in (X_s).s$ and we orient the edge $a$ in the "south direction" *i.e.* we define $o(a) := x.n$. In the same way, if $x \in X_e$ we orient the horizontal edge $a = \{x, x.e\}$ in the east direction and define $o(a) := x$, else $x \in (X_w).w$ and we define $o(a) := x.e$. We thus obtain an orientation $o$ of the forest $G$. Moreover, if $x \in X$, since $X = X_n \sqcup X_s \sqcup X_e \sqcup X_w$, there exists exactly one element $c \in \{n, s, e, w\}$ such that $x \in X_c$, and it is easy to check that $y := x.c$ is the only neighbour of $x$ such that $o(\{x, y\}) = x$. It follows that the orientation $o$ of $G$ is 1-out.
(2) $\Rightarrow$ (1). Assume that $o$ is a 1-out orientation of the graph $G$. Given a vertex $x \in V$, denote by $y$ the (unique) neighbour of $x$ such that $o(\{x, y\}) = x$; if $c$ is the element of $C$ such that $y = x.c$, then we place $x$ in $X_c$. Of course, $X = X_n \sqcup X_e \sqcup X_w \sqcup X_s$. Let us check that $X = X_n \sqcup X_s.s = X_e \sqcup X_w.w$ (whence we get a $(2,2)$-paradoxical decomposition by letters of $X$). First, $X_n \cap X_s.s = \varnothing$ because if $x \in X_n$, then $o(\{x, x.n\}) = x$ and thus $x.n \notin X_s$. Moreover, if $x \in X$, either $o(\{x, x.n\}) = x$ and then $x \in X_n$, or $o(\{x, x.n\}) = x.n$ and thus $x.n \in X_s$ so $x \in X_s.s$ whence $X = X_n \cup X_s.s$. The equality $X = X_e \sqcup X_w.w$ can be proved in the same way. □

*Remark* 5 (A new (2,2)-decomposition of the free group $F_2$). The (free) right action of $F_2$ by translations on itself is $(2,2)$-paradoxical by letters using the classical decomposition (see Example 2). Our Theorem 2 provides another $(2,2)$-paradoxical decomposition by letters of this action: consider the Cayley graph $T$ of the action (with respect to the set $C := \{e, n, w, s\}$ of generators), and consider a 1-out orientation $o$ of the tree $T$ (for example the 1-out orientation defined by the end containing the ray $(\varepsilon.(n^k))_{k \in \mathbb{N}}$ (with origin $\varepsilon$ and "full north" direction). The partition $F_2 = V_e \sqcup V_n \sqcup V_w \sqcup V_s$ is defined as follows: for every vertex $v \in F_2$, if $c$ is the element of $C$ such that $v.c$ is the successor of $v$ with respect to $o$, then put $v$ in $V_c$. Then $F_2 = V_n \sqcup V_s.s = V_e \sqcup V_w.w$ and thus a $(2,2)$-paradoxical decomposition of $F_2$ by letters has been defined.

## 5. $\mathbf{HB}_w$ IMPLIES AN END OR A BI-INFINITE PATH IN EVERY ORBIT OF A FREE $F_2$-SET

So far we have only defined finite paths. We now define a *bi-infinite path* as an infinite connected graph in which the degree of every vertex is 2. Notice that a bi-infinite path is a tree in which every vertex has degree 2. The aim of this Section is to prove that the consequence $\mathbf{HB}_w$ of the Hahn-Banach axiom implies the choice of an end or a bi-infinite path in every orbit of a free $F_2$-set (see Corollary 1).

### 5.1. **Conic probabilities on a tree.**

**Notation 7** (Cone $\Gamma(x,y)$ in a tree)**.** Given a tree $T=(V,E)$, for each edge $e=\{x,y\}$ of $T$, we denote by $\Gamma(x,y)$ the connected component of $y$ in the subgraph $(V,E\backslash\{e\})$. The set $\Gamma(x,y)$ is called the *cone of origin $x$ in the direction $y$.*

Notice that the origin $x$ of a cone $\Gamma(x,y)$ of a tree $T$ does not belong to the cone $\Gamma(x,y)$. If a tree $T$ has at least two vertices, for every vertex $x$ of $T$, the singleton $\{x\}$ is the intersection of the cones $\Gamma(n,x)$ with origin a neighbour $n$ of $x$.

**Notation 8** (the boolean algebra $\mathcal{B}_T$ associated to a tree $T$)**.** If a tree $T=(V,E)$ has at least two vertices, we denote by $\mathcal{B}_T$ the boolean subalgebra of $\mathcal{P}(V)$ generated by the cones of the tree $T$.

If a vertex $v$ of a tree $T$ has a finite degree, then $\{v\}=\cap_{y\in N(v)}\Gamma(y,v)$ is the intersection of a finite set of cones of $T$, so if the tree $T$ is locally finite, then $\mathcal{B}_T$ contains all finite subsets of $V$ and the atoms of $\mathcal{B}_T$ (*i.e.* the minimal non null element of the poset $\mathcal{B}_T$) are the singletons $\{v\}$ where $v\in V$. Notice that if the tree $T$ is countable (for example if $T$ is an orbit of a free $F_2$-set $X$), then the set of cones of the tree $T$ is also countable, so the boolean algebra $\mathcal{B}_T$ is also countable and thus one can define in **ZF** a ultrafilter on $\mathcal{B}_T$.

A ultrafilter $\mathcal{U}$ on a boolean algebra $\mathcal{B}$ is *principal* if there exists an atom $a\in\mathcal{B}$ such that $\{a\}\in\mathcal{U}$.

**Definition 5.** A *conic* probability on a tree $T$ is a probability on the boolean algebra $\mathcal{B}_T$.

**Proposition 10.** *Given a locally finite tree $T=(V,E)$, the following data are equivalent:*

*(1) a non principal ultrafilter on $\mathcal{B}_T$;*

*(2) a 1-out orientation of the tree $T$.*

*Proof.* (1) $\Rightarrow$ (2). Given a non principal ultrafilter $\mathcal{U}$ on $\mathcal{B}_T$, for every $x\in V$, $V=\{x\}\sqcup\sqcup_{y\in N(v)}\Gamma(x,y)$ thus there exists a (unique) neighbour $y$ of $x$ such that $\Gamma(x,y)\in\mathcal{U}$: we denote this neighbour by $n_x$. Moreover, if $\{x,y\}$ is an edge of $T$, either $y=n_x$ or $x=n_y$: in fact, if $y\neq n_x$ then $\Gamma(x,n_x)\subseteq\Gamma(y,x)$ and thus $\Gamma(y,x)\in\mathcal{U}$ so $n_y=x$. Let $o$ be the orientation on $T$ such that for each edge $\{x,y\}$ of $T$, $o(\{x,y\})=x$ iff $y=n_x$. Then the orientation $o$ is 1-out.

(2) $\Rightarrow$ (1). Given a 1-out orientation $o$ of $T$, let us denote by $s:V\to V$ the mapping associating to every $x\in V$ the neighbour $y$ of $x$ such that $o(\{x,y\})=x$. We consider the set $\mathcal{C}=\{\Gamma(x,s(x)):x\in V\}$. Then the set $\mathcal{C}$ of cones generates a filter $F$ on $\mathcal{B}_T$. This filter is a (non trivial) ultrafilter on $\mathcal{B}_T$: if a cone $\Gamma(x,y)$ of $\mathcal{B}_T$ does not belong to $F$, then $y\neq s(x)$ whence $s(y)=x$ so $\Gamma(y,x)\in F$; since $\Gamma(y,x)$ is the complement of $\Gamma(x,y)$ in $\mathcal{B}_T$ and since cones of the tree $T$ generate the boolean algebra $\mathcal{B}_T$, the filter $F$ of $\mathcal{B}_T$ is a ultrafilter on $\mathcal{B}_T$. □

### 5.2. **Balanced vertices of a tree endowed with a conic probability.**

**Notation 9.** Given a conic probability $\mu$ on a tree $T$, for every real number $\varepsilon>0$, we denote by $B_\varepsilon(T,\mu)$ the set $\{x\in T:\forall y\in N(x)\ \mu(\Gamma(x,y))\geq\varepsilon\}$.

Elements of $B_\varepsilon(T,\mu)$ are said to be *$\varepsilon$-balanced with respect to* $\mu$. Elements of $B(T,\mu):=\cup_{\varepsilon>0}B_\varepsilon(T,\mu)$ are said to be *balanced* with respect to $\mu$.

Notice that if $\mu$ is a conic probability on a tree $T$, if $x$ is a finite degree vertex of $T$ and if $x$ is not balanced, then $x$ has a neighbour $y$ such that $\mu(\Gamma(x,y))=0$.

**Lemma 2.** *Let $T = (V, E)$ be a locally finite tree such that there exists a witness $\Phi_V$ of the finite choice property on $V$. For every $a \in V$ and every infinite subset $A$ of $V$ there exists a ray $(x_n)_{n\in\mathbb{N}}$ of the tree $T$ with origin $a$ which is definable from $T$ and $\Phi_V$ and such that the set $A \cap \{x_n : n \in \mathbb{N}\}$ is infinite.*

*Proof.* We define by recursion a ray $(x_n)_{n\in\mathbb{N}}$ with origin $a$ such that for every $n \in \mathbb{N}$, $\Gamma(x_n, x_{n+1}) \cap A$ is infinite. We consider the rooted tree $T$ at $a$ endowed with its usual partial order $\preceq_a$: if $x, y \in T$, then $x \preceq_a y$ iff $x$ belongs to the path with extremities $a$ and $y$. Using $\Phi_V$, we define the following ray $(x_n)_{n\in\mathbb{N}}$: $x_0 = a$ and, for every $n \in \mathbb{N}$, $x_{n+1} = \Phi_V(F)$ where $F = \{y \in N(x_n) : x_n \prec_a y$ and $(\Gamma(x_n, y) \cap A)$ is infinite$\}$. □

**Theorem 3.** *Let $T = (V, E)$ be a tree such that for every $x \in V$, $N(x)$ is finite with at least 3 elements, and such that there exists a witness $\Phi_V$ of the finite choice property on $V$. Assume that $\mu$ is a conic probability on the tree $T$.*

*(1) For every $\varepsilon > 0$, the set $B_\varepsilon(T, \mu)$ is finite.*

*(2) If there exists some $\mu$-balanced vertex in $T$, then there exists a $\mu$-balanced vertex in $T$ which is definable from $T$, $\Phi_V$ and $\mu$.*

*Proof.* (1) Let $\varepsilon$ be some real number such that $\varepsilon > 0$. If $B_\varepsilon(T, \mu)$ is infinite, using Lemma 2, consider some ray $(x_n)_{n\in\mathbb{N}}$ with an infinity of vertices in $B_\varepsilon(T, \mu)$. Let $A$ be the infinite set $\{x_n : x_n \in B_\varepsilon(T, \mu)\}$. For every $x \in A$, the degree of $x$ is $\geq 3$ so there exists at least one cone with origin $x$ which is disjoint from the set $\{x_n : n \in \mathbb{N}\}$: let $\mathcal{C}$ be the (infinite) set of cones with origin at some point $x \in A$ and which are disjoint from the set $\{x_n : n \in \mathbb{N}\}$. Then $\mathcal{C}$ is an infinite set of pairwise disjoint cones. Consider $N$ cones in $\mathcal{C}$ for some integer such that $N > \frac{1}{\varepsilon}$. Then the $\mu$-measure of the union of these $N$ cones is $\geq N\varepsilon > 1$ and this is contradictory! It follows that $B_\varepsilon(T, \mu)$ is finite (with cardinal $\leq \frac{1}{\varepsilon}$).
(2) If $T$ contains some $\mu$-balanced vertex, let $n_0$ be the first element of $\mathbb{N}^*$ such that $B_{\frac{1}{n_0}}(T, \mu)$ is nonempty. Since $B_{\frac{1}{n_0}}(T, \mu)$ is finite, $\Phi_V$ allows to choose a $\mu$-balanced vertex $v$ in $B_{\frac{1}{n_0}}(T, \mu)$ (and $v$ is thus definable form $T$, $\mu$ and $\Phi_V$). □

### 5.3. **Conic probabilities on a tree without any balanced vertex.**

**Notation 10.** Given an (infinite) locally finite tree $T = (V, E)$ with at least two elements, and given a conic probability $\mu$ on $T$, for every $\varepsilon > 0$ we denote by $B'_\varepsilon(T, \mu)$ the set of vertices $x \in V$ having exactly one neighbour $y$ such that $\mu(\Gamma(x, y)) = 0$ and such that $\min(\{\mu(\Gamma(x, z)); z \in N(x)\backslash\{y\}\}) \geq \varepsilon$.

We also denote by $B'(T, \mu)$ the set $\cup_{\varepsilon>0} B_\varepsilon(T, \mu)$ of vertices $x \in V$ having exactly one neighbour $y$ such that $\mu(\Gamma(x, y)) = 0$.

**Lemma 3.** *Let $T = (V, E)$ be a tree such that for every $x \in V$, $N(x)$ is finite with at least four elements, and such that there exists a witness $\Phi_V$ of the finite choice property on $V$. Assume that $\mu$ is a conic probability on the tree $T$ without any balanced vertices.*

*(1) For every $\varepsilon > 0$, the set $B'_\varepsilon(T, \mu)$ is finite.*

*(2) If $B'(T, \mu)$ is nonempty, then there is a vertex in $B'(T, \mu)$ which is definable from $T$, $\mu$ and $\Phi_V$.*

*Proof.* (1) Assume that $B'_\varepsilon(T, \mu)$ is infinite. Using Lemma 2, there exists a ray $(x_n)_{n\in\mathbb{N}}$ such that the set $A := B'_\varepsilon(T, \mu) \cap \{x_n : n \in \mathbb{N}\}$ is infinite. For every $x \in A$, there are at least three

cones $C$ with origin $x$ such that $\mu(C) \geq \varepsilon$, so there exists at least one cone $C$ with origin $x$ which is disjoint from the ray $\{x_n : n \in \mathbb{N}\}$ and such that $\mu(C) \geq \varepsilon$: let $\mathcal{C}_x$ be the set of such cones $C$. The cones belonging to $\cup_{x\in A}\mathcal{C}_x$ are pairwise disjoint and have $\mu$-measure $\geq \varepsilon$. It follows that $A$ is finite (with cardinality $\leq \frac{1}{\varepsilon}$): this is contradictory!
(2) This follows from the fact that $B'(T,\mu) = \cup_{n\in\mathbb{N}^*} B'_{1/n}(T,\mu)$ where each $B'_{1/n}(T,\mu)$ is finite. □

**Theorem 4.** *Let $T = (V,E)$ be a 4-regular tree such that there exists a witness $\Phi_V$ of the finite choice property on $V$. Assume that $\mu$ is a conic probability on the tree $T$. Then there is an end or a vertex (and thus an end) or a bi-infinite path of $T$ which is definable from $T$, $\mu$ and $\Phi_V$.*

*Proof.* If $V$ contains at least one $\mu$-balanced vertex, then $\Phi_V$ allows to choose a $\mu$-balanced vertex in $B(T,\mu)$ (which is finite, see Theorem 3). Else, for every $x \in V$, at least one cone of origin $x$ has a null $\mu$-measure. If the set $B'(T,\mu)$ is nonempty, then, using Lemma 3, $\Phi_V$ chooses a vertex in $B'(T,\mu)$. We now assume that $B(T,\mu) = B'(T,\mu) = \varnothing$: then, for every vertex $x$ of $T$, there are at least two neighbours $y_1$ and $y_2$ of $x$ such that $\mu(\Gamma(x,y_1)) = \mu(\Gamma(x,y_2)) = 0$. If for every $x \in V$, exactly three distinct cones with origin $x$ have a null $\mu$-measure, then we get an 1-out orientation $o$ (and thus an end) of the tree $T$: in fact, for each edge $\{x,y\}$ of $T$, then either $\mu(\Gamma(x,y)) = 1$ or $\mu(\Gamma(y,x)) = 1$; we define $o(\{x,y\}) = x$ iff $\mu(\Gamma(x,y)) = 1$. Let $F_V$ be the finite subset $\{x \in V : \mu(\{x\}) > 0\}$: if $F_V$ is nonempty, then $\Phi_V$ allows to choose some vertex in $V$.

We now assume that $F_V$ is empty (and thus every finite subset of $V$ has a null $\mu$-measure), and that for every $x \in V$, at least two cones with origin $x$ have a null measure, and that the set $P$ of vertices $x$ such that there exist exactly two cones with origin $x$ with null measure is nonempty. If $x \in P$, then the two neighbours of $x$ which do not belong to the $x$-cones of null measure also belong to $P$: if one of these two neighbours $y$ does not belong to $P$, then there are 3 cones with origin $y$ and null measure so $\mu(\{y\}) = \mu(\Gamma(x,y)) > 0$ which is contradictory with $F_V = \varnothing$. It follows that the set $P$ is a bi-infinite path of $V$ (definable from $T$, $\mu$ and $\Phi_V$). □

**Corollary 1.** *(1) Let $G = (V,E)$ be a 4-regular forest swhich has a witness $\Phi_V$ of the finite choice property on $V$. Then $\mathbf{HB}_w$ implies the choice of an end or a bi-infinite path in each connected component of $G$.*
*(2) In particular, if . is a free right action of the free group $F_2$ on a nonempty (infinite) set $X$, then $\mathbf{HB}_w$ implies the choice of an end or a bi-infinite path in each orbit of $X$.*

*Proof.* (1) Let $\Omega$ be the set of connected components of the forest $G$. Using $\mathbf{HB}_w$, we consider a family $(\mu_T)_{T\in\Omega}$ such that for every orbit $T$, $\mu_T$ is a probability on $\mathcal{B}_T$. With Theorem 4 and $\Phi$, we deduce an end or a bi-infinite path in every orbit. (2) is a consequence of (1) and Proposition 8. □

5.4. **The axiom $\mathbf{HB}_w$ implies $(2,3)$-paradoxical decompositions of free $F_2$-sets.** In this Section, we use Corollary 1 to give a proof of Shioya and Sato's $(3,2)$-paradoxical decomposition of free $F_2$-sets in $\mathbf{ZF} + \mathbf{HB}_w$ (see [13]).

**Proposition 11** ([13])**.** *In $\mathbf{ZF}+\mathbf{HB}_w$, every free $F_2$-set $X$ has a $(3,2)$-paradoxical decomposition of the form $X = X_n \sqcup X_s \sqcup X_e \sqcup X_w \sqcup X_r$ such that $X = X_n \sqcup (X_s).s = X_e \sqcup (X_w).w \sqcup (X_r).e$.*

*Proof.* Given a free $F_2$-set $X$, $\mathbf{HB}_w$ provides (see Corollary 1) the choice of a bi-infinite path or an end in each orbit of $X$. Moreover, $X$ satisfies the finite choice property (see Proposition 8-(3)): let $\Phi$ be a witness of the finite choice property on $X$. Using Theorem 2, each orbit $\omega$ in which an end (or equivalently a 1-out orientation) has been chosen has a natural $(2,2)$-paradoxical decomposition by letters: $\omega = \omega_n \sqcup \omega_s \sqcup \omega_e \sqcup \omega_w = \omega_n \sqcup (\omega_s).s = \omega_e \sqcup (\omega_e).w$. We now consider the case of an orbit $\omega$ in which a bi-infinite path $P_\omega$ has been chosen. We consider the orientation $o$ of $P_\omega$ in which every horizontal edge is oriented from west to east and every vertical edge is oriented from south to north. If the oriented graph $(P_\omega, o)$ has no sources and no sinks, then the orientation $o$ of $P_\omega$ is 1-out so it defines an end of $P_\omega$ and thus an end of the tree $\omega$ so using again Theorem 2, this end provides a $(2,2)$-paradoxical decomposition by letters of $\omega$. If the set of sources of $P_\omega$ is nonempty and finite, then, using the witness $\Phi$, we choose one source $x$ and a neighbour $y$ of $x$ in $P_\omega$; we then consider the ray of $P_\omega$ with origin $x$ containing $y$: this ray defines again a 1-out orientation of $P_\omega$ and thus an end of $\omega$ and a $(2,2)$-paradoxical decomposition by letters of $\omega$. If the set of sources of $P_\omega$ is nonempty and infinite but contained in a ray of $P_\omega$, then we may consider the smallest ray containing the set of sources (the intersection of all rays containing the set of sources) and thus we get an end of $\omega$ (which provides a $(2,2)$-paradoxical decomposition by letters of $\omega$).
We now assume that the set of sources of the oriented bi-infinite path $P_\omega$ is infinite but contained in no ray of $P_\omega$: in this case, there is at least one sink between two distinct sources and thus the set of sinks is also infinite and not contained in any ray. In this case, we now describe a $(3,2)$-paradoxical decomposition $\omega = \omega_n \sqcup \omega_s \sqcup \omega_e \sqcup \omega_w \sqcup \omega_r$ such that $\omega = \omega_n \sqcup (\omega_s).s = \omega_e \sqcup (\omega_w).w \sqcup (\omega_r).e$:

(1) Each vertex $v$ of the bi-infinite path $P_\omega$ which is neither a sink neither a source is associated with its natural direction: east direction if $v$ is the source of an horizontal edge, north direction if $v$ is the source of an horizontal edge; we place $v$ in $\omega_e$ in the first case and $\omega_n$ in the second case;
(2) Sources of $P_\omega$ are placed in the set $\omega_n$;
(3) For each source $v$ of $P_\omega$, vertices strictly west of $v$ but on the same horizontal as $v$ (*i.e.* of the form $v.w^n$ where $n \geq 1$) are placed in the set $\omega_r$;
(4) Sinks of $P_\omega$ are also placed in $\omega_r$;
(5) each other vertex $v$ is outside $P_\omega$ and not of the form $s.(w^n)$ where $s$ is a source of $P_\omega$ and $n \in \mathbb{N}^*$: we consider the one-to-one walk $(v_0 = v, v_1, \dots, v_n)$ leading from $v$ to the path $P_\omega$, and we place $v$ in $\omega_c$ where $c$ is the cardinal point such that $v_1 = v.c$.

We finally, for each $c \in \{n, s, e, w\}$ we define the set $X_c := \cup_{\omega \in \Omega} \omega_c$, and we define $X_r := \cup_{\omega \in \Omega} \omega_r$ (where $\omega_r := \varnothing$ if the paradoxical decomposition of the orbit $\omega$ is $(2,2)$). It is easy to check that $X = X_n \sqcup X_s \sqcup X_e \sqcup X_w \sqcup X_r$ and that $X = X_n \sqcup (X_s).s = X_e \sqcup (X_w).w \sqcup (X_r).e$. ☐

## 6. $(2,2)$-paradoxical decompositions using $\mathbf{HB}_w$ and weak forms of $\mathbf{AC}^2$

6.1. **A bi-infinite path in a tree is equivalent to two distinct ends of this tree.** A bi-infinite path $P = (V, E)$ has exactly two 1-out orientations: each 1-out orientation of $P$ corresponds to one of the two linear orders on $V$ such that for each edge $e = \{x, y\}$ of $P$, $x$ and $y$ are consecutive in the linear order. Each of these two 1-out orientations is equivalent to one of the two ends of the bi-infinite path.

**Proposition 12.** *Given a tree $T = (V, E)$, the following data are equivalent:*

*(1) A bi-infinite path in $T$*
*(2) Two distinct ends (or equivalently, two distinct 1-out orientations) of the tree $T$.*

*Proof.* $\Rightarrow$ Given a bi-infinite path $P$ of the tree $T$, the path $P$ has exactly two distinct ends, which define two distinct ends of $T$. $\Leftarrow$ Assume that $\beta_1$ and $\beta_2$ are two distinct ends of $T$. They are equivalent to two 1-out orientations $o_1$ and $o_2$ of the tree $T$. Let $P$ be the set of vertices $x \in V$ such that $[x, o_1) \cap [x, o_2) = \{x\}$. Then $P$ is nonempty: consider some vertex $v$ of $T$; consider the two (non equivalent) rays $[v, o_1)$ and $[v, o_2)$; then $[v, o_1) \cap [v, o_2)$ is a segment $[v, x]$; and the tree $T$ induces on $[x, o_1) \cup [x, o_2)$ a bi-infinite path equal to $P$. □

### 6.2. Weak consequences of $\mathbf{AC}^2$.

6.2.1. *Choice of an end in bi-infinite paths.* We consider the following consequences of $\mathbf{AC}^2$ for bi-infinite paths:

> $\mathbf{AC}^{2,end}$: *Given a family $(P_i)_{i\in I}$ of bi-infinite paths, there is a family $(o_i)_{i\in I}$ such that for each $i \in I$, $o_i$ is a 1-out orientation of the bi-infinite path $P_i$.*
>
> $\mathbf{AC}^{2,end,o}$: *Given a family $(P_i)_{i\in I}$ of oriented bi-infinite paths, there is a family $(o_i)_{i\in I}$ such that for each $i \in I$, $o_i$ is a 1-out orientation of the bi-infinite path $P_i$.*
>
> $\mathbf{AC}^{2,end,o-n}$: *Given a family $(P_i)_{i\in I}$ of oriented bi-infinite paths, if there exists a family $(f_i)_{i\in I}$ such that for each $i \in I$, $f_i$ is a neighbour choice on $P_i$, then there is a family $(o_i)_{i\in I}$ such that for each $i \in I$, $o_i$ is a 1-out orientation of the bi-infinite path $P_i$.*
>
> $\mathbf{AC}^{2,end,F_2}$: *Given a free $F_2$-set $X$, for every family $(P_i)_{i\in I}$ of bi-infinite paths of the forest $X$, there is a family $(o_i)_{i\in I}$ such that for each $i \in I$, $o_i$ is a 1-out orientation of the bi-infinite path $P_i$.*

Of course, $\mathbf{AC}^2 \Rightarrow \mathbf{AC}^{2,end} \Rightarrow \mathbf{AC}^{2,end,o} \Rightarrow \mathbf{AC}^{2,end,o-n} \Rightarrow \mathbf{AC}^{2,end,F_2}$.

6.2.2. *The two halves of a bi-infinite path.* Given a bi-infinite path $P = (V, E)$ there exists a unique partition $\{V_0, V_1\}$ of $V$ such that for every $i \in \{0, 1\}$, every vertex $x \in V_i$ has its two neighbours in $V_{1-i}$. The two sets $V_0$ and $V_1$ are called the two *halves* of the bi-infinite path $P$. Notice that the two halves of a bi-infinite path $P = (V, E)$ are maximal independent subsets of the graph $P$. We now consider the following consequences of $\mathbf{AC}^2$ for bi-infinite paths:

> $\mathbf{AC}^{2,h}$: *For every family $(P_i)_{i\in I}$ of bi-infinite paths, there exists a family $(H_i)_{i\in I}$ such that for each $i \in I$, $H_i$ is a half of $P_i$.*
>
> $\mathbf{AC}^{2,h,o}$: *For every family $(P_i)_{i\in I}$ of oriented bi-infinite paths, there exists a family $(H_i)_{i\in I}$ such that for each $i \in I$, $H_i$ is a half of $P_i$.*
>
> $\mathbf{AC}^{2,h,o-n}$: *For every family $(P_i)_{i\in I}$ of oriented bi-infinite paths, if there exists a family $(f_i)_{i\in I}$ such that for each $i \in I$, $f_i$ is a neighbour choice on $P_i$, then there exists a family $(H_i)_{i\in I}$ such that for each $i \in I$, $H_i$ is a half of $P_i$.*
>
> $\mathbf{AC}^{2,h,F_2}$: *Given a free $F_2$-set $X$, for every family $(P_i)_{i\in I}$ of bi-infinite paths of the forest $X$, there is a family $(H_i)_{i\in I}$ such that for each $i \in I$, $H_i$ is a half of the bi-infinite path $P_i$.*

Of course, $\mathbf{AC}^2 \Rightarrow \mathbf{AC}^{2,h} \Rightarrow \mathbf{AC}^{2,h,o} \Rightarrow \mathbf{AC}^{2,h,o-n} \Rightarrow \mathbf{AC}^{2,h,F_2}$.

6.2.3. *Coherently oriented intervals of an oriented bi-infinite path.*

**Definition 6.** A *mono-infinite* path is an infinite tree with one point of degree 1 and all other vertices with degree 2: the vertex with degree 1 is called the *extremity* of the mono-infinite path.

Given a bi-infinite path $P = (V, E)$, a nonempty connected subgraph $(V', E')$ of $P$ is either a (finite) path of $P$ or a mono-infinite path or the bi-infinite path $P$ itself. Connected subgraphs of $P$ with at least two vertices are called the *intervals* of $P$.

If $(P = (V, E), o)$ is an oriented bi-infinite path, an interval $I$ of $P$ is said to be *coherently oriented with respect to o* or *o-coherent* if for every $v \in I$ which is not an extremity of $I$, $v$ has in-degree 1 (*i.e.* out-degree 1) with respect to the orientation $o$. The oriented bi-infinite path $(P = (V, E), o)$ is the union of maximal $o$-coherent intervals. For each maximal $o$-coherent interval $I$ of $P$, the $o$-coherent orientation of $I$ is induced by one of the two 1-out orientations of $P$; moreover, if $I$ is finite, then $I$ has exactly one source and one sink and the source and the sink of $I$ are the extremities of $I$. The intersection of two distinct maximal $o$-coherent intervals of $P$ is $\varnothing$ or a singleton $\{v\}$ where $v$ is a source or a sink of $P$. Two maximal $o$-coherent intervals $I$ and $J$ of $P$ are said to be *adjacent* if $I \cap J$ is a singleton (whence $I \neq J$).

**Lemma 4.** *Let $(P = (V, E), o)$ be an oriented bi-infinite path, such that each maximal o-coherent interval of $P$ is finite.*

1. *The set $Z_P$ of maximal o-coherent intervals of $P$ endowed with the adjacency relation is a bi-infinite path.*
2. *If $I$ and $J$ are two adjacent elements of $Z_P$, then their o-orientations come from the two distinct 1-out orientations of $P$; it follows that if $I$ and $J$ are at an even distance in the graph $Z_P$, then their o-orientations come from the same 1-out orientation of $P$. In particular, each half of the bi-infinite path $Z_P$ defines one of the two 1-out orientations of $P$.*
3. *The bi-infinite path $Z_P$ has a neighbour choice $g$ which is definable from $o$.*
4. *If $P$ has a neighbour choice $f_P : V \to V$, then the bi-infinite path $Z_P$ has an orientation definable from $o$ and $f_P$.*

*Proof.* (1) The adjacency relation is irreflexive and symmetric so it defines a graph on $Z_P$. Moreover, given an element $I$ of $Z_P$, there are exactly two intervals adjacent with $I$ so $Z_P$ endowed with the adjacency relation is a bi-infinite path.
(2) Trivial.
(3) We define a neighbour choice $g$ on the bi-infinite path $Z_P$ as follows: given some vertex $I$ of $Z_P$, and denoting by $v$ the source of the interval $I$ endowed with its $o$-coherent orientation, then we define $g(I)$ as the interval in $Z_P$ containing $v$ but distinct from $I$.
(4) We define an orientation $t$ of $Z_P$: given two adjacent maximal $o$-coherent intervals $I$ and $J$ of $P$, and denoting by $v$ the vertex in $I \cap J$, then $w := f_P(v) \in (I \cup J) \backslash (I \cap J)$: we define $t(\{I, J\}) := I$ if $w \in I$ and $J$ else. □

6.2.4. $\mathbf{AC}^{2,h,o-n}$ *implies* $\mathbf{AC}^{2,end,o-n}$.

**Proposition 13.** $\mathbf{AC}^{2,h,o-n} \Rightarrow \mathbf{AC}^{2,end,o-n}$.

*Proof.* Let $(P_i = (V_i, E_i), o_i, f_i)_{i \in I}$ be a family such that for each $i \in I$, $P_i = (V_i, E_i)$ is a bi-infinite path, $o_i$ is an orientation of the path $P_i$ and $f_i$ is a neighbour choice on the path $P_i$. Then, $P$ has a double neighbour choice, so using Proposition 5, there is a family $(\Phi_i)_{i \in I}$ such that each $\Phi_i$ is a witness of the finite choice property on $V_i$. Let $T$ be the set of $i \in I$ such that $P_i$ has at least one infinite maximal o-coherent interval.

-If $i \in T$, then either $P_i$ has exactly one maximal infinite $o$-coherent interval, and this interval defines an end of $P_i$, or $P_i$ has exactly two infinite maximal $o$-coherent intervals: let $v_1$ and $v_2$ be their extremities, let $v_3 = \Phi_i(\{v_1, v_2\})$ and let $v_4 = f_i(v_3)$: then the ray of $P$ with origin $v_3$ and containing $v_4$ defines an end of $P_i$.

-If $i \in I \backslash T$ then each maximal $o$-coherent interval of $P_i$ is finite: let $Z_i$ be the bi-infinite path of maximal $o_i$-coherent intervals of $P_i$ endowed with the adjacency relation. Then, the path $Z_i$ has a neighbour choice $g_i$, and since $P_i$ has a neighbour choice $f_i$, the path $Z_i$ has an orientation $t_i$ definable from $o_i$ and $f_i$. Applying $\mathbf{AC}^{2,h,o-n}$ to the family $(Z_i, s_i, g_i))_{i \in I \backslash T}$ of oriented bi-infinite paths endowed with neighbour choices, for each $i \in I \backslash T$ we choose a half $H_i$ of $Z_i$: this half defines a 1-out orientation of $P_i$ (or equivalently an end of $P_i$). $\square$

6.2.5. *The existence of $\mathbb{Z}$-chameleons imply $\mathbf{AC}^{2,end,o-n}$.* Given some integer $n \geq 2$, a *cyclic $n$-chameleon* on a set $X$ is a mapping $\chi : \mathcal{P}(X) \to \mathbb{Z}/n\mathbb{Z}$ such that for every subset $A$ of $X$ and every $x \in X \backslash A$, $\chi(A \sqcup \{x\}) = \chi(A) +_n 1$ where $+_n$ is the additive law of the (abelian) quotient group $\mathbb{Z}/n\mathbb{Z}$. Chameleons were introduced by Mathias (see [8]). One can also define in a similar way $\mathbb{Z}$-chameleons (see [9]): a *$\mathbb{Z}$-chameleon* on a set $X$ is a mapping $\chi : \mathcal{P}(X) \to \mathbb{Z}$ such that for every subset $A$ of $X$ and every $x \in X \backslash A$, $\chi(A \sqcup \{x\}) = \chi(A) + 1$ where $+$ is the additive law of the (abelian) group $\mathbb{Z}$. Given a commutative field $\mathbb{K}$, we consider the following statement:

> $\mathbf{D}(\mathbb{K})$ ("non null dual") *"For every non null vector space $E$ over $\mathbb{K}$, there exists a non null linear form $f : E \to \mathbb{K}$."*

The following statement is a consequence of $\mathbf{D}(\mathbb{Q})$ (see [9, Theorem 4]):

> $\mathbf{ch}(\mathbb{Z})$ ("existence of $\mathbb{Z}$-chameleons"): *"For every set $X$ there exists a $\mathbb{Z}$-chameleon on $X$."*

Notice that the statement $\mathbf{ch}(\mathbb{Z})$ is "multiple": given a family $(X_i)_{i \in I}$ of sets, $\mathbf{ch}(\mathbb{Z})$ implies a chameleon $\chi$ on $X := \cup_{i \in I} X_i$, and for each $i \in I$, the mapping $\chi_i : A \mapsto \chi(A)$ from $\mathcal{P}(X_i)$ to $\mathbb{Z}$ is still a chameleon on $X_i$.

*Remark* 6. It was pointed out by Andreas Blass (see [1]) that given a commutative field $\mathbb{K}$, the statement $\mathbf{D}(\mathbb{K})$ is equivalent to form 284 of [5]: *"A system of linear equations over a field $\mathbb{K}$ has a solution in $\mathbb{K}$ if and only if every finite sub-system has a solution in $\mathbb{K}$."*

**Proposition 14.** *Let $(P_i = (V_i, E_i))_{i \in I}$ be a family of bi-infinite paths.*

*(1) The axiom $\mathbf{ch}(\mathbb{Z})$ implies the existence of a family $(e_i)_{i \in I}$ such that for each $i \in I$, $e_i$ is a singleton or an edge of $P_i$.*
*(2) Moreover, if for each $i \in I$, the path $P_i$ is endowed with an orientation $o_i$, then $\mathbf{ch}(\mathbb{Z})$ implies the existence of a choice function $(v_i)_{i \in I}$ for the family $(V_i)_{i \in I}$.*
*(3) If in addition, for each $i \in I$ the path $P_i$ is endowed with an orientation $o_i$ and a neighbour choice $f_i$, then $\mathbf{ch}(\mathbb{Z})$ implies the existence of a family $(\rho_i)_{i \in I}$ such that for each $i \in I$, $\rho_i$ is a ray of $P_i$.*

*Proof.* (1) With $\mathbf{ch}(\mathbb{Z})$, for each $i \in I$, we consider a $\mathbb{Z}$-chameleon $\chi_i$ on the set $V_i$. Given an orientation $o$ of $P_i$, for every vertex $v$ of $P_i$ we denote by $I_{v,o}$ the set of vertices of the ray $[v, o)$ of $P_i$; then there exists a unique $v \in P_i$ such that $\chi(I_{v,o}) = 0$. Denoting by $o_1$ and $o_2$ the two orientations of $P_i$, let $x$ and $y$ be the elements of $P_i$ such that $\chi_i(I_{o_1,x}) = \chi_i(I_{o_2,y}) = 0$: if $x = y$ let $e_i = \{x\} = \{y\}$ else, let $e_i$ be the center of the finite path with extremities $x$ and $y$: then $e_i$ is a singleton or an edge of $P_i$.
(2) For each $i \in I$, if $e_i$ is a singleton then let $v_i$ be the element of $e_i$ else let $v_i := o_i(e_i)$: then $v_i \in V_i$.
(3) For each $i \in I$, consider the neighbour $w_i := f_i(v_i)$ and then the ray with origin $v_i$ which contains $w_i$. □

**Corollary 2.** $\mathbf{ch}(\mathbb{Z}) \Rightarrow \mathbf{AC}^{2,h,o-n}$.

*Proof.* Given a family $(P_i = (V_i, E_i), o_i, f_i)_{i\in I}$ of oriented bi-infinite paths $P_i$ endowed with a neighbour choice $f_i$, using Proposition 14, the axiom $\mathbf{ch}(\mathbb{Z})$ allows to choose a ray $\rho_i$ in each path $P_i$, and this ray defines an end of $P_i$. □

### 6.3. $\mathbf{HB}_w$+$\mathbf{AC}^{2,end,F_2}$ and $(2,2)$-paradoxical decompositions by letters of free $F_2$-sets.

**Corollary 3.** $\mathbf{HB}_w$+$\mathbf{AC}^{2,end,F_2}$ *implies the existence of a* $(2,2)$*-paradoxical decomposition by letters in every free* $F_2$*-set.*

*Proof.* With $\mathbf{HB}_w$, we choose in each orbit an end or a bi-infinite path (see Corollary 1). In case a bi-infinite path has been chosen, with $\mathbf{AC}^{2,end,F_2}$, we choose in this bi-infinite path one of the two ends of the path: this end yields an end of the orbit. Using Theorem 2, we deduce a $(2,2)$-paradoxical decomposition by letters of the free $F_2$-set $X$ from the choice of an end in each orbit. □

*Remark* 7. The axiom $\mathbf{AC}^{2,end,F_2}$ is a consequence of the existence of a $(2,2)$-paradoxical decomposition by letters of every free $F_2$-set.

*Proof.* Use Theorem 2. □

### 6.4. Paradoxical decompositions of the sphere and the solid ball of $\mathbb{R}^3$.

6.4.1. $(2,2)$*-paradoxical decompositions by letters of the sphere of* $\mathbb{R}^3$. The two Satô rotations $\rho_1$ and $\rho_2$ generate a subgroup $G$ of $\mathrm{SO}_3(\mathbb{R})$ which is isomorphic with $F_2$. This group acts naturally on the unit sphere $S = S(0,1)$ of $\mathbb{R}^3$. Each rotation $r$ of $\mathbb{R}^3$ has exactly two (antipodal) fixed points in $S$, so the set $D$ of fixed points of rotations in $G$ is countable: here, $S$ is linearly orderable (since $S$ and $\mathbb{R}$ are equipotent) so the family of pairs $(\mathrm{fix}(g))_{g\in G}$ has a choice function and thus $D = \sqcup_{g\in G} \mathrm{fix}(g)$ is also countable. It follows that the family of orbits of the $G$-set $D$ has a choice function and this implies in $\mathbf{ZF}$ a $(2,2)$-paradoxical decomposition of the $F_2$-set $D$ by letters (see [13] or [15, Theorem 5.5 p.64-65]). Using $\mathbf{HB}_w + \mathbf{AC}^{2,end,\mathbb{R}}$, we consider a $(2,2)$-paradoxical decomposition by letters of the free $F_2$-set $S\backslash D$. We get a $(2,2)$-paradoxical decomposition by letters of $S$ by joining these two decompositions. Likewise, a $(2,2)$-paradoxical decomposition by letters of the $F_2$-set $B\backslash\{(0,0,0)\} = \sqcup_{0<r\leq 1} S(0,r)$ can be obtained.

6.4.2. $(3,2)$-*paradoxical decompositions of the solid ball of* $\mathbb{R}^3$. The $(2,2)$-paradoxical decomposition of $B\backslash\{(0,0,0)\}$ can be transformed in the classic way and in **ZF** (see [13] or [15, Theorem 5.7 p.66-67]) into a $(3,2)$-paradoxical of the solid unit ball $B(0,1)$ (which is thus also a consequence of $\mathbf{HB}_w + \mathbf{AC}^{2,end,F_2}$).

6.5. **A diagram.** All the implications in Figure 1 hold in **ZFA** (set theory without the Axiom of Choice, weakened to allow "atoms"):

(1) The statement **MC** is the Multiple Choice axiom: *"Every set satisfies the finite choice property."* and is equivalent to **AC** in **ZF** (see [6], [5]). The implications $\mathbf{MC} \Rightarrow \mathbf{D}(\mathbb{Q}) \Rightarrow \mathbf{ch}(\mathbb{Z})$ hold in **ZFA** and can be found in [9].
(2) The statement **R** is Rado's selection Lemma (form 99 in [5]). The proof of $\mathbf{BPI} \Rightarrow \mathbf{R}$ is in [12] and the proof of $\mathbf{R} \Rightarrow \mathbf{HB}$ is in [10].
(3) **HM** is Hall's infinite marriage theorem (form 107 of [5]).

**Question 1.** Which reciprocal arrows hold between the consequences of $\mathbf{AC}^2$ of the diagram? Notice that in **ZFA**, **MC** does not imply $\mathbf{AC}^2$ whence $\mathbf{AC}^{2,h,o-n}$ does not imply $\mathbf{AC}^2$.

$$
\begin{tikzcd}
\mathbf{MC} \arrow[dr] \arrow[ddddd] & & & \mathbf{BPI} \arrow[dll] \arrow[ddl] \arrow[d] & \\
& \mathbf{R} \arrow[d] & \mathbf{AC}^{\omega} \arrow[d] & \mathbf{HM} \arrow[d] & \\
& \mathbf{HB} \arrow[d] & \mathbf{BPI}_w \arrow[dl] & \mathbf{AC}^{\mathbf{fin}} \arrow[d] & \\
& \mathbf{HB}_w & & \forall n \mathbf{AC}^n \arrow[d] & \\
& & & \mathbf{AC}^2 \arrow[dll] \arrow[dr] & \\
\mathbf{D}(\mathbb{Q}) \arrow[d] & \mathbf{AC}^{2,h} \arrow[d] & & & \mathbf{AC}^{2,end} \arrow[d] \\
\mathbf{ch}(\mathbb{Z}) \arrow[dr] & \mathbf{AC}^{2,h,o} \arrow[d] & & & \mathbf{AC}^{2,end,o} \arrow[d] \\
& \mathbf{AC}^{2,h,o-n} \arrow[rrr] \arrow[d] & & & \mathbf{AC}^{2,end,o-n} \arrow[d] \\
& \mathbf{AC}^{2,h,F_2} \arrow[d] & & & \mathbf{AC}^{2,end,F_2} \arrow[d] \\
& \mathbf{AC}^{2,h,\mathbb{R}} & & & \mathbf{AC}^{2,end,\mathbb{R}}
\end{tikzcd}
$$

Figure 1. Summary diagram

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