# Elementary processes for Itô Integral against cylindrical Wiener process

Sylvain Dotti

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# Elementary processes for Itô Integral against cylindrical Wiener process

**Sylvain Dotti**[1]

[1] *PIMENT Laboratory, Reunion Island University, 15 Avenue René Cassin, CS 92003, 97744 Saint-Denis Cedex 9, La Réunion, France*
*e-mail:* sylvain.dotti@univ-reunion.fr

**Abstract:** In this article, I present different definitions of elementary processes which lead to definitions of Itô integrals against cylindrical Wiener processes. I prove the equivalence between those definitions by constructing this Itô integral using only Itô isometries and extensions by density. Then, from the perspective of white noise theory and Kondratiev spaces, I compare the definitions of cylindrical Wiener processes, and present a link between Itô integral and cylindrical white noise.

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## Contents

## 1. Introduction

Itô Integral against cylindrical Wiener process is a tool that has been developed to solve stochastic evolution equations in infinite dimension. One can quote E Mazel Khanin Sinai [9], Pardoux [27], Debussche Vovelle [5], Hairer Weber [19], Hairer [18], Dotti Vovelle [7], Galimberti Karlsen [12], for examples of recent studies of stochastic partial differential equations using cylindrical Wiener processes.

The first construction of this mathematical object is in the paper of Curtain and Falb [2]. Since this precursory article, many other presentations have been done by different authors: Métivier, Pellaumail [25], Pardoux, Grorud [16], Da Prato, Zabczyk [3], Prévôt Rökner, Frieler [28, 11], Gawarecki, Mandrekar, [14], Hairer [17].

A general comparison of stochastic integrals has already been done by Dalang, Quer-Sardanions [4]. We can also quote the comparison made by Protter [29], Bastons and Escudero [1]. In this paper I focus on Itô Integral against cylindrical Wiener process, and more precisely, the constructions using elementary processes. Indeed, we can find in the literature different definitions of elementary processes, and thus different definitions of Itô integrals, which lead to 'the Itô Integral'. It is interesting to prove that those constructions are equivalent. More than that, it is interesting to understand the links between those elementary processes, to understand why those definitions of elementary processes are natural, and from which idea they come from.

If we look in detail, we have four different definitions of elementary processes in the literature: the one given by Métivier, Pellaumail in [25], the one given by Da Prato, Zabczyk and Prévot, Röckner, Frieler in [3, 11, 28], the one given by Gawarecki, Mandrekar in [14], the one given by Hairer in [17]. For the construction of the integral, all authors give different methods: Métivier Pallaumail use Doléans measures, Hairer use Cameron-Martin spaces, Da Prato, Zabczyk announce the almost sure convergence, uniform with respect to the time variable, of the finite dimensional case to the infinite dimensional case without proving it, Prévot, Röckner use abstract Hilbert-Schmidt embedding, Gawarecki, Mandrekar announce that the proofs are similars to the case of Itô integral against $Q$-Wiener processes with $Q$ a symmetric, non-negative operator of finite trace.

In Section 2, I present the difficulties to generalize the standard Brownian motion in finite dimension to the infinite dimension. I also recall the definitions of $Q$-Wiener processes and Hilbert-Schmidt spaces.

In Section 3, I prove the equivalence of Itô integrals against $Q$-Wiener processes, starting by elementary processes of [25], of [3, 11, 28], of [14], of [17].

In Section 4, I explain the construction of Itô integral against cylindrical Wiener processes of Frieler, Prévôt, Röckner in [11, 28]. I give some corrections on the definition of cylindrical Wiener process and the announcements of Gawarecki, Mandrekar in [14]. I also present the ideas of Hairer in [17], and the ideas of Métivier and Pellaumail in [25] for their construction of the Itô integral against cylindrical Wiener process.

In Section 5, I give a new construction of this integral using the simplest tools, that is Itô isometries and extension theorem of operators by density. This construction proves the equivalence of Itô integrals against cylindrical Wiener processes, starting by elementary processes of [25], of [3, 11, 28], of [14], of [17].

In Section 6, I explain the ideas of Takeyuki Hida which lead to the Gaussian white noise theory, and then to the Kondratiev spaces in dimension one and in infinite dimension. As an element of those spaces, I present a definition of cylindrical Wiener process, then follows an example verifying the assumptions of the definition given by Gawarecki and Mandrekar in [14]. I end by a formula which is a bridge between Itô integral against cylindrical Wiener process and Lebesgue integral using cylindrical white noise. The bridge is made by the stochastic Wick product.

Throughout the article, we use the following notations:

- $\mathbb{N}$ is the set of natural numbers including 0, that is $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.
- $\mathbb{N}^*$ is the set of natural numbers without 0, that is $\mathbb{N}^* = \{1, 2, 3, \dots\}$.
- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{R}^*$ is the set of real numbers without 0.
- $\mathbb{R}_+$ is the set of nonnegative real numbers, that is $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$.
- $\mathbb{R}_+^*$ is the set of positive real numbers, that is $\mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}$.
- Let $N \in \mathbb{N}^* \backslash \{1\}$, $\mathbb{R}^N$ denotes the N-ary Cartesian power of $\mathbb{R}$.
- Let $H$ be a separable Hilbert space, $L(H, \mathbb{R})$ denotes the space of all bounded linear operators from $H$ to $\mathbb{R}$.

## 2. Wiener processes in infinite dimension

### *2.1. From the finite dimension to the infinite dimension*

In one dimension, a Brownian motion $(B_t)_{t \in \mathbb{R}_+}$ is a process taking its values in $\mathbb{R}$, adapted to some filtration, starting at some value $x \in \mathbb{R}$, whose increments are independent from the past and stationary. More precisely, there exists $q \in \mathbb{R}_+^*$ such that for any $0 < s < t < +\infty$, the random variable $B_t - B_s$ follows the normal distribution $\mathcal{N}(0, (t-s)q)$. The most commonly used is the version of Brownian motions with continuous paths, starting at $x = 0$, with $B_1$ which follows the standard normal distribution $\mathcal{N}(0, 1)$, that is $q = 1$ (see [30, p. 17]). This Brownian motion is then called the standard Brownian motion.

Let $N \in \mathbb{N}^*\backslash\{1\}$, a $N$-dimensional Brownian motion, which is a process taking its values in $\mathbb{R}^N$, has the same definition, replacing $x \in \mathbb{R}$ by $x \in \mathbb{R}^N$ and $q \in \mathbb{R}_+^*$ by a symmetric, non-negative definite square matrix $Q$ of order $N$ with real coefficients. Then, we obtain the standard Brownian motion when $x = 0$ and $Q$ is the identity matrix.

Note that in this case (that is when $Q$ is the identity matrix), the $N$-dimensional standard Brownian motion can be written

$$B_t = \sum_{k=1}^{N} B_t^k e_k$$

where $(e_k)_{1 \leq k \leq N}$ is an orthonormal basis of $\mathbb{R}^N$ and the $\left(B^k\right)_k$ are independent real standard Brownian motions (see proposition 2.1.10 of [28] for more details).

In infinite dimension, if $(e_k)_{k\in\mathbb{N}^*}$ is an orthonormal basis of a separable Hilbert space, the problem of existence of the sum arise. In other words, the problem of existence of the variance of such a process at each time $t \in \mathbb{R}_+^*$ arise: the standard Brownian motion (which is usually called cylindrical Wiener process) can not be defined immediately, only $Q$-Wiener processes can be defined, generalizing $N$-dimensional Brownian motions when $Q$ are symmetric, non-negative definite matrices. Let us define them.

### 2.2. A formal definition for cylindrical Wiener processes, a rigorous definition for Q-Wiener processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $[0, T] \subset \mathbb{R}_+$ a bounded interval of time, and $(\mathcal{F}_t)_{t\in[0,T]}$ be a complete, right-continuous filtration associated with $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\beta_k)_{k\in\mathbb{N}^*}$ a sequence of independent real Brownian motions adapted to the filtration $(\mathcal{F}_t)$. A cylindrical Wiener process $W$ is a stochastic process taking its values in a real Hilbert space $(H, \langle . \, , . \rangle_H)$, endowed with an orthonormal basis $(e_k)_{k\in\mathbb{N}^*}$ and defined by

$$W \;:\; (\omega, t) \in \Omega \times [0, T] \mapsto W(\omega, t) := \sum_{k=1}^{+\infty} \beta_k(\omega, t)\, e_k. \tag{2.1}$$

Usually, the elementary event $\omega$ is omitted. We follow this rule in the sequel.

The previous definition is formal, $W(t)$ does not exist in $H$. Indeed,

$$\mathbb{E}\left(\|W(t)\|_H^2\right) = \mathbb{E}\left(\sum_{k=1}^{+\infty} \beta_k(t)^2\right) = \sum_{k=1}^{+\infty} t = +\infty.$$

To make it rigorous, we have to use $J : H \to U$ a Hilbert-Schmidt embedding, with $U$ a real separable Hilbert space.

**Definition 2.1.** Let $(H, \langle ., . \rangle_H)$ and $(U, \langle ., . \rangle_U)$ be two real separable Hilbert spaces, $(e_k)_{k\in\mathbb{N}^*}$ be an orthonormal basis of $H$ and $(\varepsilon_j)_{j\in\mathbb{N}^*}$ be an orthonormal basis of $U$. A bounded linear operator $P : H \to U$ with finite Hilbert-Schmidt norm, that is

$$\|P\|_{L_2(H,U)}^2 := \sum_{k=1}^{+\infty} \langle P(e_k), P(e_k) \rangle_U < +\infty,$$

is called a Hilbert-Schmidt operator. The space of all Hilbert-Schmidt operators from $H$ to $U$ is denoted by $L_2(H,U)$.

**Proposition 2.1.** *$L_2(H,U)$ endowed with the bilinear form*

$$\forall S, P \in L_2(H,U), \quad \langle P, S\rangle_{L_2(H,U)} := \sum_{k=1}^{+\infty} \langle P(e_k), S(e_k)\rangle_U$$

*is a separable Hilbert space for which $(e_k \otimes \varepsilon_j)_{k,j\in\mathbb{N}^*}$ defined by*

$$e_k \otimes \varepsilon_j := \langle e_k, .\rangle_H \ \varepsilon_j$$

*is an orthonormal basis.*

*Proof of Proposition 2.1.* See pages 155–156 of the thesis of Katja Frieler and Claudia Prévôt [11]. □

For example, let us take $U = H$ and $J$ the Hilbert-Schmidt operator from $H$ to $H$ defined by $Je_k = \frac{1}{k}e_k$. We can then define almost surely, $\forall t \in [0,T]$:

$$JW(t) := \sum_{k=1}^{+\infty} \beta_k(t)\, Je_k. \tag{2.2}$$

The series converges in $H$, almost surely and uniformly on $[0,T]$ (see the proof of Proposition 2.2. Note that $JW(t)$ is not an element of $J(H)$ (there is a slight abuse of notation) and that $J(H)$ is dense in $H$ because $(kJe_k)_{k\in N^*}$ is an orhtonormal basis of $H$.

Let us recall now what is a $Q$-Wiener process, $Q$ playing the role of the covariance (operator):

**Definition 2.2** (Q-Wiener process)**.** Let $Q : H \to H$ be a linear bounded operator, which is

1. nonnegative, that is $\langle h, Q(h)\rangle_H \geq 0, \quad \forall h \in H$
2. symmetric, that is $\langle Q(h_1), h_2\rangle_H = \langle h_1, Q(h_2)\rangle_H, \quad \forall h_1, h_2 \in H$
3. with finite trace, that is $\mathrm{tr}(Q) := \sum_{k=1}^{+\infty} \langle Q(e_k), e_k\rangle_H < +\infty$

Let $W : \Omega \times [0,T] \to H$ be a stochastic process which

4. is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted, that is $W(t)$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$,
5. has its increments independent of the past, that is $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t \leq T$
6. starts from zero $\mathbb{P}$-almost surely, that is $W(0) = 0$ $\mathbb{P}$-a.s.
7. has $\mathbb{P}$-almost surely continuous trajectories
8. has its increments having the following Gaussian laws:

$$W(t) - W(s) \rightsquigarrow \mathcal{N}(0, (t-s)Q), \quad \forall 0 \leq s \leq t \leq T$$

i.e. the characteristic function of $W(t) - W(s)$ can be written

$$\varphi(h) = \int_H \exp\left(i\langle h, x\rangle_H\right) \mathbb{P}_{W(t)-W(s)}(dx) = \exp\left(-\frac{1}{2}\langle (t-s)Q(h), h\rangle_H\right)$$

$\forall h \in H$, (see theorem 6.3.1 of [15])

then $W$ is called a $Q$-Wiener process in $H$.

**Proposition 2.2.** *$JW$ defined by (2.2) is a $Q$-Wiener process in $H$ with covariance operator $Q := J \circ J^* = J^2$ ($J^* = J$ is the adjoint operator of $J$).*

*Proof of Proposition 2.2.* As we have $\forall k, j \in \mathbb{N}^*$

$$\langle e_k, J^*(e_j)\rangle_H = \langle J(e_k), e_j\rangle_H = \frac{1}{k}\langle e_k, e_j\rangle_H,$$

we can deduce $J^*(e_k) = \frac{1}{k} e_k = J(e_k), \forall k \in \mathbb{N}^*$. Then, the linear bounded operator $Q = J^2$ is obviously nonnegative, symmetric, with finite trace.

The next step is to prove the $\mathcal{F}_t$-measurability of $JW(t)$, for each $t \in [0,T]$. To use the $\mathcal{F}_t$-measurability of the $(\beta_k)_{k\in\mathbb{N}^*}$, we need the almost sure convergence of the series $JW(t)$ on a subset $\tilde{\Omega} \subset \Omega$ which does not depend on $t$ such that $\mathbb{P}\left(\tilde{\Omega}\right) = 1$. For that, we can use the proof of [14] pages 20–21: partial sums of $\|JW(t)\|_H^2$ are sub-martingales which allows to use a Doob's maximal inequality and a Lévy-Itô-Nisio theorem to prove that the series JW converges $\mathbb{P}$-almost surely uniformly on $[0,T]$.

Then items 4, 5, 6, 7 and 8 of Definition 2.2 follow from the properties of the $(\beta_k)_{k\in\mathbb{N}^*}$. □

The theory of integrals against $Q$-Wiener processes is described for example in [3], in [28], in [14]. I give some ideas of its construction in the particular case of integrals against

$$JW(t) = \sum_{k=1}^{+\infty} \frac{1}{k}\beta_k(t)\, e_k$$

where the integrand $\Phi$ takes its values in $L_2(J(H), \mathbb{R})$ (see Proposition 3.1 for the details). In a more general case, the integrand $\Phi$ can take its values in $L_2(J(H), K)$ where $K$ is a separable Hilbert space. My aim is to discuss on the ideas of the construction that is why the scalar case is enough. The first step is to define Itô integral for elementary processes. Different definitions of elementary processes can be taken. I will explain why, and I will prove the equivalence of the definitions of the Itô integrals against $Q$-Wiener processes starting with the elementary processes of [25], of [3], of [14], of [17]. Note that elementary processes of [11, 28, 3] are the same.

## 3. Discussion on the construction of Itô integral against a $Q$-Wiener process

In this section, we will define the Itô integral of elementary processes $\Phi \in L(H, \mathbb{R})$ against the $Q$-Wiener process $JW$ defined in (2.2). The case of a general $Q$-Wiener process is not different from the case of $JW$ because such a $Q$ admits a unique square root operator which plays the role of $J$ for $JW$ (see proposition 2.3.4 of [28]). The aim is to be able to define

$$\int_0^T \Phi(t)\, dJW(t) := \sum_{k=1}^{+\infty} \frac{1}{k} \int_0^T \Phi(t)\, e_k d\beta_k(t)$$

with $\Phi : \Omega \times [0,T] \to L(H,\mathbb{R})$. First, each

$$\int_0^T \Phi(t)\, e_k \; d\beta_k(t)$$

must exist. It is the case (see e.g. [13] page 121 and chapter 4) if

$$\forall k \in \mathbb{N}^*, \quad \mathbb{E}\left(\int_0^T (\Phi(t)\, e_k)^2\, dt\right) < +\infty. \tag{3.1}$$

It means that the condition $\Phi : \Omega \times [0,T] \to L(H,\mathbb{R})$ is not sufficient to define the set of integrands, we also need (3.1). Second, we want that the series

$$\sum_{k=1}^{+\infty} \frac{1}{k} \int_0^T \Phi(s)\, e_k d\beta_k(s) \tag{3.2}$$

converges in $L^2(\Omega)$. It is satisfied if

$$\mathbb{E}\left(\int_0^T \sum_{k=1}^{+\infty} \frac{1}{k^2} (\Phi(t)\, e_k)^2\, dt\right) < +\infty \tag{3.3}$$

because for all $M, N \in \mathbb{N}^*$,

$$\begin{aligned}\mathbb{E}\left(\left(\sum_{k=N}^{M} \frac{1}{k} \int_0^T \Phi(t)\, e_k d\beta_k(t)\right)^2\right) &= \sum_{k=N}^{M} \frac{1}{k^2}\left(\mathbb{E}\left(\int_0^T \Phi(t)\, e_k d\beta_k(t)\right)^2\right) \\ &= \sum_{k=N}^{M} \frac{1}{k^2} \mathbb{E} \int_0^T (\Phi(t)\, e_k)^2\, dt. \end{aligned} \tag{3.4}$$

The first equality is due to the independence of the $(\beta_k)_{k\in\mathbb{N}^*}$ and the second equality is due to the one dimensional Itô isometry.

Thus, trying to find necessary conditions to define Itô integral for elementary processes, we found that the Itô integral must satisfy the Itô isometry by taking $N = 1$ and $M \to +\infty$ in (3.4):

$$\mathbb{E}\left(\left(\int_0^T \Phi(t)\, dJW(t)\right)^2\right) = \sum_{k=1}^{+\infty} \frac{1}{k^2} \mathbb{E} \int_0^T (\Phi(t)\, e_k)^2\, dt.$$

Starting by defining Itô integral for elementary processes belonging to a normed space $\mathcal{E}$, the construction requires that the Itô integral belongs to a Banach space $M$ so that the Itô isometry allows to define an isometry (from $\mathcal{E}$ to $M$), which can be extended to the completion of $\mathcal{E}$.

**Definition 3.1.** Da prato, Zabczyk [3], Frieler, Prévôt [11], Prevôt, Röckner [28] define an elementary processes $\Phi : \Omega \times [0, T] \to L(H, \mathbb{R})$ if there exists $p \in \mathbb{N}^*,\ 0 = t_0 < t_1 < \cdots < t_i < \cdots < t_p = T$ such that

$$\Phi(t) = \sum_{i=0}^{p-1} \Phi_i \mathbf{1}_{]t_i, t_{i+1}]}(t)$$

where

1. $\forall i,\ \Phi_i : \Omega \to L(H, \mathbb{R})$ is $\mathcal{F}_{t_i}$-measurable, $L(H, \mathbb{R})$ being endowed with its Borel $\sigma$-algebra
2. $\forall i,\ \Phi_i$ takes only a finite number of values in $L(H, \mathbb{R})$

and define naturally the integral

$$\int_0^T \Phi\left(t\right) dJW\left(t\right) := \sum_{i=0}^{p-1} \Phi_i \left(JW(t_{i+1}) - JW(t_i)\right) \tag{3.5}$$

*Remark* 3.1. We can notice that the second item is a sufficient condition to have (3.3):

$$\begin{aligned} \mathbb{E}\left(\int_0^T \sum_{k=1}^{+\infty} \frac{1}{k^2} \left(\Phi\left(t\right) e_k\right)^2 dt\right) &= \sum_{k=1}^{+\infty} \frac{1}{k^2} \mathbb{E}\left(\sum_{i=0}^{p-1} \left(\Phi_i e_k\right)^2 \left(t_{i+1} - t_i\right)\right) \\ &= \sum_{k=1}^{+\infty} \frac{1}{k^2} \sum_{i=0}^{p-1} \left(t_{i+1} - t_i\right) \mathbb{E}\left(\Phi_i e_k\right)^2 . \end{aligned}$$

The values of $\Phi_i$ are for example $\Phi_i^1, \ldots, \Phi_i^r$ on the subsets $\Omega^1, \ldots, \Omega^r$ of $\Omega$. Those subsets can be the same for all $\Phi_i$, that is independent of $i$. Thus

$$\mathbb{E}\left(\left(\Phi_i e_k\right)^2\right) = \sum_{j=1}^{r} \left(\Phi_i^j e_k\right)^2 \mathbb{P}\left(\Omega^j\right) \leq \max_{i,j} \|\Phi_i^j\|^2_{L(H,\mathbb{R})}$$

and

$$\mathbb{E}\left(\int_0^T \sum_{k=1}^{+\infty} \frac{1}{k^2} \left(\Phi\left(t\right) e_k\right)^2 dt\right) \leq \max_{i,j} \|\Phi_i^j\|^2_{L(H,\mathbb{R})}\ T \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty.$$

□

**Definition 3.2.** Gawarecki, Mandrekar in [14] (see combination of p25 and p28), define an elementary processes $\Phi : \Omega \times [0, T] \to L(H, \mathbb{R})$ if there exists $p \in \mathbb{N}^*,\ 0 = t_0 < t_1 < \cdots < t_i < \cdots < t_p = T$ such that

$$\Phi(t) = \Phi_{-1} \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{p-1} \Phi_i \mathbf{1}_{]t_i, t_{i+1}]}(t)$$

where

1. $\forall i,\ \Phi_i : \Omega \to L(H,\mathbb{R})$ is $\mathcal{F}_{t_i}$-measurable, $L(H,\mathbb{R})$ being endowed with its Borel $\sigma$-algebra and denoting $\mathcal{F}_{-1} := \mathcal{F}_0$.
2. $(\Phi_i)_{i\in\{-1,\dots,p-1\}}$ is uniformly bounded in $L_2\left(J(H),\mathbb{R}\right)$ i.e. there exists a $M \in [0,+\infty)$ independent of $i$ and $\omega \in \Omega$ such that $\forall i$:

$$\|\Phi_i\|^2_{L_2(J(H),\mathbb{R})} := \sum_{k=1}^{+\infty} \frac{1}{k^2}\left(\Phi_i(e_k)\right)^2 \le M < +\infty, \quad a.s.$$

and define naturally the integral

$$\int_0^T \Phi\left(t\right) dJW\left(t\right) := \sum_{i=0}^{p-1} \Phi_i\left(JW(t_{i+1}) - JW(t_i)\right)$$

*Remark* 3.2. We can notice that the second item is a sufficient condition to have (3.3):

$$\begin{aligned}\mathbb{E}\left(\int_0^T \sum_{k=1}^{+\infty} \frac{1}{k^2}\left(\Phi\left(t\right)e_k\right)^2 dt\right) &= \sum_{k=1}^{+\infty}\frac{1}{k^2}\mathbb{E}\left(\sum_{i=0}^{p-1}\left(\Phi_i e_k\right)^2\left(t_{i+1}-t_i\right)\right)\\ &= \sum_{k=1}^{+\infty}\frac{1}{k^2}\sum_{i=0}^{p-1}\left(t_{i+1}-t_i\right)\mathbb{E}\left(\Phi_i e_k\right)^2 \le MT.\end{aligned}$$

□

*Remark* 3.3. Let us denote $\mathcal{E}_{PR}$ the set of elementary processes defined in Definition 3.1 and $\mathcal{E}_{GM}$ the set of elementary processes defined in Definition 3.2. Then

$$\mathcal{E}_{PR} \subset \mathcal{E}_{GM}.$$

Indeed, Let $\Phi \in \mathcal{E}_{PR}$ such that

$$\Phi(t) = \bar{0} \times \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{p-1} \Phi_i \mathbf{1}_{]t_i,t_{i+1}]}(t)$$

where $\bar{0} : \Omega \to L(H,\mathbb{R})$ is a constant random variable for which the unique value is the null function of $L(H,\mathbb{R})$. For each $i$, we have

$$\sum_{k=1}^{+\infty}\frac{1}{k^2}\left(\Phi_i(e_k)\right)^2 \le \max_{\substack{0\le i\le p-1\\ 1\le j\le r}} \|\Phi_i^j\|^2_{L(H,\mathbb{R})}\sum_{k=1}^{+\infty}\frac{1}{k^2}$$

where the values of $\Phi_i$ are $\Phi_i^1,\dots,\Phi_i^r$ on the subsets $\Omega^1,\dots,\Omega^r$ of $\Omega$. Those subsets are the same for all $\Phi_i$, i.e. independent of $i$, thus the item 2 of Definition 3.2 is satisfied by $\Phi$. □

**Proposition 3.1.** *$\mathcal{E}_{PR}$ and $\mathcal{E}_{GM}$ endowed with the norm*

$$\|\Phi\|_{\mathcal{E}} := \left(\mathbb{E}\left(\sum_{k=1}^{+\infty}\frac{1}{k^2}\int_0^T\left(\Phi\left(t\right)e_k\right)^2 dt\right)\right)^{1/2}, \quad \forall \Phi \in \mathcal{E}_{GM}$$

*are normed spaces. They are dense subspaces of* $\overline{\mathcal{E}_{PR}} = \overline{\mathcal{E}_{GM}}$ *which is equal to*

$$\left\{\Phi : \Omega\times[0,T] \to L_2\left(J(H),\mathbb{R}\right) \text{ progressively measurable such that } \|\Phi\|_{\mathcal{E}} < +\infty\right\}$$

*where* $J(H)$ *is the separable Hilbert space endowed with the scalar product*

$$\langle u, v\rangle_{J(H)} = \sum_{k=1}^{+\infty} k^2 \langle u, e_k\rangle_H \langle v, e_k\rangle_H \tag{3.6}$$

*as defined page 25 by [14], or equivalently defined page 42 by [28] as*

$$< J(x), J(y) >_{J(H)} = < x, y >_H \ .$$

*Proof of Proposition 3.1.* For [28], it is a consequence of their proposition 2.3.8. The processes $\Phi$ are first supposed to be predictable. But if we look carefully at the remark 2.5.3, the predictability can be replaced by progressive measurability.

For [14], it is a consequence of their proposition 2.2. The processes $\Phi$ are supposed to be measurable and adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. It implies that there exists a modification of $\Phi$ which is progressively measurable. That is why, we can replace measurable and adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ by progressively measurable. □

*Remark* 3.4. It is interesting to notice that if two processes $\Phi_1$ and $\Phi_2$ belonging to $\overline{\mathcal{E}_{GM}}$ are modifications one from the other, that is for each $t \in [0,T], \mathbb{P}(\Phi_1(t) = \Phi_2(t)) = 1$, then

$$\begin{aligned} \|\Phi_1 - \Phi_2\|_{\mathcal{E}}^2 &= \mathbb{E}\left(\sum_{k=1}^{+\infty} \frac{1}{k^2} \int_0^T \left((\Phi_1(t) - \Phi_2(t))e_k\right)^2 dt\right) \\ &= \int_0^T \left(\sum_{k=1}^{+\infty} \frac{1}{k^2} \mathbb{E}\left((\Phi_1(t) - \Phi_2(t))e_k\right)^2\right) dt = 0 \end{aligned}$$

by using Tonelli's theorem.

**Theorem 3.2.** *The bounded linear mapping*

$$\begin{aligned} \mathcal{E}_{PR} &\to L^2(\Omega) \\ \Phi \quad &\mapsto \int_0^T \Phi(t) dJW(t) \end{aligned}$$

*defined by* (3.5)*, can be extended to* $\overline{\mathcal{E}_{PR}}$*. Moreover, the extension satisfies the Itô isometry*

$$\|\Phi\|_{\mathcal{E}}^2 = \left\|\int_0^T \Phi(t) dJW(t)\right\|_{L^2(\Omega)}^2, \quad \forall \Phi \in \overline{\mathcal{E}_{PR}}$$

*that is*

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} \mathbb{E} \int_0^T \left(\Phi(t)\, e_k\right)^2 dt = \mathbb{E}\left(\left(\int_0^T \Phi(t)\, dJW(t)\right)^2\right).$$

*Proof of Theorem 3.2.* It is a well known result of functional analysis that can be found for example in [31], proposition 10.3. □

To finalize the comparison of elementary processes for the construction of Itô integrals against $Q$-Wiener processes, we are adding a definition related to the one given by Hairer in [17], page 26, for the construction of Itô integrals against cylindrical Wiener processes (see Definition 4.1 and the explanations about formula (4.1)), and a particular case of the definition given by Métivier and Pellaumail in [25](see Definition 4.4 for the particular case for the construction of Itô integral against cylindrical Wiener processes).

**Definition 3.3.** We denote $\mathcal{E}_{Hai}$ the set of elementary processes $\Phi : \Omega\times[0,T] \to L(H,\mathbb{R})$ for which there exists $p \in \mathbb{N}^*$, $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_p = T$ such that

$$\Phi(t) = \sum_{i=0}^{p-1} \Phi_i \mathbf{1}_{]t_i,t_{i+1}]}(t)$$

where

1. $\forall i \in \{0,\dots,p-1\}$, $\Phi_i : \Omega \to L(H,\mathbb{R})$ is $\mathcal{F}_{t_i}$-measurable, $L(H,\mathbb{R})$ being endowed with its Borel $\sigma$-algebra,
2. $\forall i \in \{0,\dots,p-1\}$, $\Phi_i$ is verifying

$$\mathbb{E}\left(\sum_{k=1}^{+\infty} \frac{1}{k^2}\left(\Phi_i e_k\right)^2\right) < +\infty. \tag{3.7}$$

**Definition 3.4.** We denote $\mathcal{E}_{MP}$ the set of elementary processes $\Phi : \Omega\times[0,T] \to L(H,\mathbb{R})$ for which there exists $p \in \mathbb{N}^*$, $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_p = T$ such that

$$\Phi(t) = \sum_{i=0}^{p-1} \Phi_i \mathbf{1}_{F_i\times]t_i,t_{i+1}]}(t)$$

where $\forall i \in \{0,\dots,p-1\}, F_i \in \mathcal{F}_{t_i}$ and $\Phi_i \in L(H,\mathbb{R})$.

We can define the Itô integral of $\Phi \in \mathcal{E}_{MP}$ with respect to the $Q$-Wiener process $JW$ defined in (2.2) by

$$\int_0^T \Phi(t)dW_t := \sum_{i=0}^{p-1} \mathbf{1}_{F_i}\Big(\Phi_i\left(JW_{t_{i+1}}\right) - \Phi_i\left(JW_{t_i}\right)\Big). \tag{3.8}$$

Noticing that $span(\mathcal{E}_{MP}) = \mathcal{E}_{PR}$ (the proof is the same as the proof of Lemma 5.1), we can extend the integral (3.8) by linearity to reach the integral (3.5) for all $\Phi \in \mathcal{E}_{PR}$.

**Proposition 3.3.** *The Itô integral against $Q$-Wiener processes, constructed starting from the elementary processes $\mathcal{E}_{PR}$ and extended to its completion $\overline{\mathcal{E}_{PR}}$ by the Itô isometry, is the same as the Itô integral against $Q$-Wiener processes constructed starting from the elementary processes $\mathcal{E}_{GM}$ or constructed starting from the elementary processes $\mathcal{E}_{Hai}$ or constructed starting from the elementary processes $\mathcal{E}_{MP}$.*

*Proof of Proposition 3.3.* We have

$$span(\mathcal{E}_{MP}) = \mathcal{E}_{PR} \subset \mathcal{E}_{GM} \subset \mathcal{E}_{Hai} \subset \overline{\mathcal{E}_{PR}},$$

thus

$$\overline{span\,(\mathcal{E}_{MP})} = \overline{\mathcal{E}_{PR}} = \overline{\mathcal{E}_{GM}} = \overline{\mathcal{E}_{Hai}}$$

□

## 4. Discussion on the construction of Itô integral against a cylindrical Wiener process

In this section, we will see how different authors have constructed the Itô integral against cylindrical Wiener processes, that is how they have overcome the difficulty that the series

$$\sum_{k=1}^{+\infty} \beta_k(t) e_k$$

does not converge in $L^2(\Omega; H)$ for each $t \in [0, T]$, to be able to define

$$\int_0^T \Phi\,(t)\, dW\,(t)$$

with $\Phi : \Omega \times [0, T] \to L\,(H, \mathbb{R})$. We have constructed in the previous section the Itô integral against the $Q$-Wiener process

$$JW(t) = \sum_{k=1}^{+\infty} \frac{1}{k} \beta_k(t) e_k$$

for integrands $\Phi$ satisfying

$$\mathbb{E}\left(\int_0^T \sum_{k=1}^{+\infty} \left(\frac{1}{k} \Phi\,(t)\, e_k\right)^2 dt\right) < +\infty. \tag{4.1}$$

By a transfer of the $\frac{1}{k^2}$ from $\Phi$ to $JW$, we can write (4.1):

$$\mathbb{E}\left(\sum_{k=1}^{+\infty} \int_0^T \left(\Phi\,(t)\, e_k\right)^2 \frac{dt}{k^2}\right) < +\infty$$

and remembering that the coefficients $\frac{1}{k}$ where taken to have an example of Hilbert-Schmidt embedding which is $J$ (and remembering that the coefficients $\frac{1}{k^2}$ became a particular case of eigenvalues of a covariance operator for $JW$ which is $Q = J^2$), it is then easy to understand that to integrate against the cylindrical Wiener process

$$W\,(t) = \sum_{k=1}^{+\infty} \beta_k\,(t)\, e_k,$$

the integrands $\Phi : \Omega \times [0, T] \to L_2\,(H, \mathbb{R})$ should verify the stronger condition

$$\mathbb{E}\left(\int_0^T \sum_{k=1}^{+\infty} \left(\Phi\,(t)\, e_k\right)^2 dt\right) < +\infty. \tag{4.2}$$

### 4.1. How Frieler, Prévôt and Röckner in [11, 28] have constructed the integral

Their idea is natural. As they have constructed Itô integrals for $Q$-Wiener processes, as they have transferred the formal cylindrical Wiener process $W$ defined in (2.1) into another Hilbert space $(U, \langle .,. \rangle_U)$ with the help of any Hilbert-Schmidt embedding $\tilde{J} : H \to U$, as it implies that $\tilde{J}W$ is a $Q$-Wiener process in $U$, they use the integral against $\tilde{J}W$ for integrands that are written $\Phi \circ \tilde{J}^{-1}$ to define

$$\int_0^T \Phi(t) dW(t) := \int_0^T \Phi(t) \circ \tilde{J}^{-1} d\tilde{J}W(t)$$

for all $\Phi \in L^2(\Omega \times [0,T]; L_2(H,K)$, progressively measurable).

*Remark* 4.1. In a way, we can say that the cylindrical Wiener process, which does not exist in $H$, exists in $U$. In another way, we can say that the formal cylindrical Wiener process $W$ is still not defined, only the $Q$-Wiener process $\tilde{J}W$ is well defined, which gives a definition for the Itô integral against cylindrical Wiener processes. The definition of this integral also raises the question of wether we can remove $\tilde{J}^{-1}$ and $\tilde{J}$. A possible answer is in the following section.

### 4.2. How Martin Hairer in [17] proposed to construct the integral

**Definition 4.1.** Martin Hairer in [17] page 26, defines an elementary process $\Phi : \Omega \times [0,T] \to L_2(H, \mathbb{R})$ if there exists $p \in \mathbb{N}^*,\ 0 = t_0 < t_1 < \cdots < t_i < \cdots < t_p = T$ such that

$$\Phi(t) = \sum_{i=0}^{p-1} \Phi_i \mathbf{1}_{]t_i, t_{i+1}]}(t)$$

where

1. $\forall i \in \{0, \ldots, p-1\},\ \Phi_i : \Omega \to L_2(H, \mathbb{R})$ is $\mathcal{F}_{t_i}$-measurable, $L_2(H, \mathbb{R})$ being endowed with its Borel $\sigma$-algebra.
2. $\forall i \in \{0, \ldots, p-1\},,\ \Phi_i$ is verifying

$$\mathbb{E} \sum_{k=1}^{+\infty} \left(\Phi_i(e_k)\right)^2 < +\infty.$$

We will denote the set of such processes as $\tilde{\mathcal{E}}_{Hai}$.

Actually, to solve stochastic partial differential equations involving a cylindrical Wiener process, we need only to integrate against it. So Martin Hairer proposed to let the cylindrical Wiener processes undefined (no need to use the Hilbert-Schmidt injection $J$) because the Itô integral against cylindrical Wiener process can be constructed without it. We just need to notice that the equalities

$$\int_0^T \Phi\left(t\right) dW\left(t\right) = \int_0^T \sum_{i=0}^{p-1} \Phi_i 1_{(t_i, t_{i+1}]}\left(t\right) dW\left(t\right)$$

$$= \sum_{i=0}^{p-1} \int_{t_i}^{t_{i+1}} \Phi_i dW\left(t\right) = \sum_{i=0}^{p-1} \left(\Phi_i W\left(t_{i+1}\right) - \Phi_i W\left(t_i\right)\right)$$

are well defined for $\Phi \in \tilde{\mathcal{E}}_{Hai}$ because each $\Phi_i$ takes its values in $L_2\left(H, \mathbb{R}\right)$, and each $\Phi_i W\left(t_i\right)$ is defined almost surely by

$$\Phi_i W\left(t_i\right) = \sum_{k=1}^{+\infty} \beta_k\left(t_i\right) \Phi_i e_k.$$

Then, using the theory of Gaussian measures on infinite dimensional spaces, he proves the itô isometry for his elementary processes, and extend this isometry by dense completion of $\tilde{\mathcal{E}}_{Hai}$ to the space

$$L^2(\Omega \times [0, T]; L_2(H, \mathbb{R}), \text{progressively measurable}).$$

It is in fact just a remark in his lecture [17]. He gives different ways to construct cylindrical Wiener processes. They are related to the two next constructions.

### *4.3. How Gawarecki and Mandrekar in [14] have constructed the integral*

Gawarecki and Mandrekar overcome the problem that the series

$$\sum_{k=1}^{+\infty} \beta_k(t) e_k$$

does not converge in $L^2(\Omega; H)$ for each $t \in [0, T]$ by defining a family (indexed by $H$) of real Brownian motions:

**Definition 4.2.** $\tilde{W} : \Omega \times [0, T] \times H \to \mathbb{R}$ is called a cylindrical Wiener process if

1. $\forall h \in H, \quad \tilde{W}(., ., h) : \Omega \times [0, T] \to \mathbb{R}$ are $\mathcal{F}_t$-real Brownian motions.
2. $\forall t \in \mathbb{R}, \quad \tilde{W}(., t, .) : H \to L^2(\Omega, \mathbb{R})$ is linear and continuous.
3. $\forall h, h' \in H, s, t \in [0, T], \quad \mathbb{E}\left(\tilde{W}(., t, h) \tilde{W}(., s, h')\right) = \min(s, t) < h, h' >_H$.

*Remark* 4.2. I have added the continuity in item 2 of Definition 4.2 as it is done in [16] or in [25]. In fact, Gawarecki and Mandrekar omitted the continuity and thus the measurability. But a linear operator is not necessarily measurable, and without measurability, we can not define the integral for elementary processes (see the next (4.6)). It is also obvious that they use the continuity just after their definition p19, they write:

$$\forall h \in H, \quad \tilde{W}(., t, h) = \sum_{k=1}^{+\infty} < h, e_k >_H \tilde{W}(., t, e_k), \tag{4.3}$$

where the series converges for each $t \in \mathbb{R}$ in $L^2(\Omega, \mathbb{R})$.

*Remark* 4.3. Instead of having a countable infinity of independent real Brownian motions $(\beta_k(t))_{k\in\mathbb{N}^*}$, they need an uncountable infinity of real Brownian motions $(\tilde{W}_h(t))_{h\in H}$. The existence of such a family is solved replacing $\{\tilde{W}(.,t,e_k)\}_{k\in\mathbb{N}*}$ by a sequence of independent standard real Brownian motions $\{\beta_k(t)\}_{k\in\mathbb{N}*}$ in (4.3) (see also Remark 6.7).

Following the definition of cylindrical Wiener process, the Itô cylindrical integral is also a family of stochastic processes indexed by $\mathbb{R}$. As explained in the beginning of the section, the integrands, and thus the elementary processes must take their values in $L_2(H,\mathbb{R})$:

**Definition 4.3.** Gawarecki and Mandrekar in [14] define an elementary process $\Phi : \Omega\times[0,T]\to L_2(H,\mathbb{R})$ for cylindrical Wiener processes on $H$ if there exists $p\in\mathbb{N}^*,\ 0=t_0<t_1<\dots<t_i<\dots<t_p=T$ such that

$$\Phi(t)=\Phi_{-1}\mathbf{1}_{\{0\}}(t)+\sum_{i=0}^{p-1}\Phi_i\mathbf{1}_{]t_i,t_{i+1}]}(t) \tag{4.4}$$

where

1. $\forall i,\ \Phi_i:\Omega\to L_2(H,\mathbb{R})$ is $\mathcal{F}_{t_i}$-measurable, $L_2(H,\mathbb{R})$ being endowed with its Borel $\sigma$-algebra and denoting $\mathcal{F}_{-1}:=\mathcal{F}_0$.
2. $(\Phi_i)_{i\in\{-1,\dots,p-1\}}$ is uniformly bounded in $L_2(H,\mathbb{R})$ i.e. there exists a $M\in[0,+\infty)$ independent of $i$ and $\omega\in\Omega$ such that $\forall i$:

$$\|\Phi_i\|^2_{L_2(H,\mathbb{R})}:=\sum_{k=1}^{+\infty}(\Phi_i(e_k))^2\le M<+\infty,\quad a.s. \tag{4.5}$$

We will denote $\tilde{\mathcal{E}}_{GM}$ the set of such elementary processes. They define the Itô cylindrical stochastic integral of $\Phi$ with respect to the cylindrical process $\tilde{W}$ defined in Definition 4.2 by the family

$$\left(\left(\int_0^T\Phi(t)d\tilde{W}_t\right)(x)\right)_{x\in\mathbb{R}}:=\left(\sum_{i=0}^{p-1}\tilde{W}_{t_{i+1}}\left(\Phi_i^*(x)\right)-\tilde{W}_{t_i}\left(\Phi_i^*(x)\right)\right)_{x\in\mathbb{R}} \tag{4.6}$$

where each $\Phi_i^*\in L(\mathbb{R},H)$ is the adjoint operator of $\Phi_i$ defined by

$$\Phi_i^*(x)=x\sum_{k=1}^{+\infty}\Phi_i(e_k)e_k,\quad\forall x\in\mathbb{R}. \tag{4.7}$$

*Remark* 4.4. It is important to notice that the stochastic process

$$(\omega,t)\mapsto\left(\phi_i^*(\omega,x),\tilde{W}(\omega,t,.)\right)\mapsto\tilde{W}(\omega,t,\phi_i^*(\omega,x))$$

does not have the same properties as $\tilde{W}(.,.,h)$ at fixed $h\in H$. It loses item 1 and 3 of Definition 4.2. It means that the following proposition (page 27 of [14]) can not be proved easily by using directly the assumptions of Definition 4.2:

**Proposition 4.1** (Itô Isometry)**.** *Let $\Phi$ be an elementary process as defined in Definition 4.3, then we have $\forall x \in \mathbb{R}$*

$$\left(\mathbb{E}\left(\left(\left(\int_0^T \Phi(t)d\tilde{W}_t\right)(x)\right)^2\right)\right) = \int_0^T \mathbb{E}\|\Phi^*(t)(x)\|_H^2 dt \tag{4.8}$$

*where*

$$\Phi(t)^* = \Phi_{-1}^* \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{p-1} \Phi_i^* \mathbf{1}_{]t_i,t_{i+1}]}(t)$$

*and the $\Phi_i^*$ as in (4.7).*

It is necessary to give another definition of elementary processes, the one given by Métivier and Pellaumail to prove the Proposition 4.1

### *4.4. How Métivier and Pellaumail in [25] have constructed the integral*

**Definition 4.4.** Métivier and Pellaumail define an elementary process (they call it simple process) $\Phi : \Omega \times [0,T] \to L_2(H,\mathbb{R})$ for cylindrical Wiener processes on $H$, if there exists $p \in \mathbb{N}^*,\ 0 = t_0 < t_1 < \cdots < t_i < \cdots < t_p = T$ such that

$$\Phi(t) = \sum_{i=0}^{p-1} \Phi_i \mathbf{1}_{F_i} \mathbf{1}_{]t_i,t_{i+1}]}(t)$$

where $\forall i \in \{0,\ldots,p-1\}, F_i \in \mathcal{F}_{t_i}$ and $\Phi_i \in L_2(H,\mathbb{R})$. We will denote $\tilde{\mathcal{E}}_{MP}$ the set of such elementary processes.

They define the Itô cylindrical stochastic integral of $\Phi$ with respect to the cylindrical process $\tilde{W}$ defined in Definition 4.2 by the family

$$\left(\left(\int_0^T \Phi(t)d\tilde{W}_t\right)(x)\right)_{x\in\mathbb{R}} := \left(\sum_{i=0}^{p-1} \mathbf{1}_{F_i}\left(\tilde{W}_{t_{i+1}}\left(\Phi_i^*(x)\right) - \tilde{W}_{t_i}\left(\Phi_i^*(x)\right)\right)\right)_{x\in\mathbb{R}} \tag{4.9}$$

where each $\Phi_i^* \in L(\mathbb{R},H)$ is the adjoint operator of $\Phi_i$ defined by

$$\Phi_i^*(x) = x\sum_{k=1}^{+\infty} \Phi_i(e_k)e_k, \quad \forall x \in \mathbb{R}. \tag{4.10}$$

**Proposition 4.2** (Itô Isometry)**.** *Let $\Phi$ be an elementary process as defined in 4.4, then we have $\forall x \in \mathbb{R}$*

$$\left(\mathbb{E}\left(\left(\left(\int_0^T \Phi(t)d\tilde{W}_t\right)(x)\right)^2\right)\right) = \int_0^T \mathbb{E}\|\Phi^*(t)(x)\|_H^2 dt \tag{4.11}$$

*where*

$$\Phi(t)^* = \sum_{i=0}^{p-1} \Phi_i^* \ \mathbf{1}_{F_i} \mathbf{1}_{]t_i,t_{i+1}]}(t)$$

*and the* $\Phi_i^*$ *as in* (4.10).

*Proof of the Itô Isometry* (4.11) *of Proposition 4.2 for elementary processes belonging to* $\tilde{\mathcal{E}}_{MP}$

For all $x \in \mathbb{R}$, $i \in \{0, \dots, p-1\}$, we have by items 1 and 2 of Definition 4.2:

$$\left( \mathbb{E}\ \mathbf{1}_{F_i} \left( \tilde{W}_{t_{i+1}} \left( x \sum_{k=1}^{+\infty} \Phi_i(e_k) e_k \right) - \tilde{W}_{t_i} \left( x \sum_{k=1}^{+\infty} \Phi_i(e_k) e_k \right) \right)^2 \right)$$
$$= x^2 \mathbb{P}(F_i) \left( \mathbb{E} \left( \sum_{k=1}^{+\infty} \tilde{W}_{t_{i+1}} \left( \Phi_i(e_k) e_k \right) - \tilde{W}_{t_i} \left( \Phi_i(e_k) e_k \right) \right)^2 \right)$$
$$= x^2 \mathbb{P}(F_i) \left( \mathbb{E} \left( \sum_{k=1}^{+\infty} \Phi_i(e_k) \left( \tilde{W}_{t_{i+1}} (e_k) - \tilde{W}_{t_i} (e_k) \right) \right)^2 \right),$$

then by continuity of the norm of $L^2(\Omega, \mathbb{R})$, we can write it

$$= x^2 \mathbb{P}(F_i) \left( \lim_{N \to +\infty} \mathbb{E} \left( \sum_{k=1}^{N} \Phi_i(e_k) \left( \tilde{W}_{t_{i+1}} (e_k) - \tilde{W}_{t_i} (e_k) \right) \right)^2 \right),$$

then by item 1 and 3 of Definition 4.2, we write it

$$= x^2 \mathbb{P}(F_i) \left( \lim_{N \to +\infty} \sum_{k=1}^{N} \left( \Phi_i(e_k) \right)^2 (t_{i+1} - t_i) \right)$$
$$= \mathbb{P}(F_i)\, (t_{i+1} - t_i)\, \|\Phi_i^*\|_H^2 .$$

For all $x \in \mathbb{R}, i, j \in \{0, \dots, p-1\}$ such that $i < j$, we have by properties of real Brownian motions that $\mathbf{1}_{F_i}\mathbf{1}_{F_j}(\tilde{W}_{t_{i+1}}(\Phi_i^*(x)) - \tilde{W}_{t_i}(\Phi_i^*(x)))$ is independent of $(\tilde{W}_{t_{j+1}}(\Phi_i^*(x)) - \tilde{W}_{t_j}(\Phi_i^*(x)))$ and

$$\mathbb{E}\left( \mathbf{1}_{F_i}\mathbf{1}_{F_j} \left( \tilde{W}_{t_{i+1}} (\Phi_i^*(x)) - \tilde{W}_{t_i} (\Phi_i^*(x)) \right) \left( \tilde{W}_{t_{j+1}} (\Phi_i^*(x)) - \tilde{W}_{t_j} (\Phi_i^*(x)) \right) \right) = 0$$

thus

$$\left( \mathbb{E} \ \left( \sum_{i=0}^{p-1} \mathbf{1}_{F_i} \left( \tilde{W}_{t_{i+1}} (\Phi_i^*(x)) - \tilde{W}_{t_i} (\Phi_i^*(x)) \right) \right)^2 \right)$$
$$= \sum_{i=0}^{p-1} \mathbb{P}(F_i)\, (t_{i+1} - t_i)\, \|\Phi_i^*\|_H^2 = \sum_{i=0}^{p-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\mathbf{1}_{F_i} \|\Phi_i^*(x)\|_H^2 dt$$
$$= \int_0^T \mathbb{E}\|\Phi^*(t)\|_H^2 dt.$$

□

*Remark* 4.5. In the previous proof, we can replace $\|\Phi_i^*\|_H^2$ by $x^2\|\Phi_i\|_{L_2(H,\mathbb{R})}^2$ because both are equal to $x^2\sum_{k=1}^{+\infty}(\Phi_i(e_k))^2$. It is then natural to replace $x$ by 1 to have the following definition of Itô integral against cylindrical Wiener processes and the Itô isometry for elementary processes:

**Definition 4.5.** We define the Itô integral of elementary processes $\Phi \in \tilde{\mathcal{E}}_{MP}$ with respect to the cylindrical process $\tilde{W}$ defined in Definition 4.2 by

$$\int_0^T \Phi(t)d\tilde{W}_t := \left(\int_0^T \Phi(t)d\tilde{W}_t\right)(1) \tag{4.12}$$

and have the following Itô isometry:

$$\left(\mathbb{E}\left(\int_0^T \Phi(t)d\tilde{W}_t\right)^2\right) = \int_0^T \mathbb{E}\|\Phi(t)\|_{L_2(H,\mathbb{R})}^2 dt \tag{4.13}$$

*Remark* 4.6. The interest of defining the family (4.9) is more visible for the general case of integrands $\Phi$ taking values in $L_2(H,K)$ where $K$ is a separable Hilbert space. Indeed, if $(f_j)_{j\in\mathbb{N}^*}$ is an orthonormal basis of $K$, then the formula (4.12) becomes:

$$\int_0^T \Phi(t)d\tilde{W}_t := \sum_{j=1}^{+\infty}\left(\int_0^T \Phi(t)d\tilde{W}_t\right)(f_j)\ f_j$$

*Remark* 4.7. Métivier and Pellaumail use Doléans measures and the space of 2-cylindrical martingales to finish their construction. I take an other way to finish the construction of the Itô integral against cylindrical Wiener processes: I slowly enlarge the space of integrands to the space, slightly modified, of elementary processes defined by [3, 28, 11], then to the space, slightly modified, of elementary processes defined by [14], then to the space of progressive processes taking values in $L_2(H,\mathbb{R})$.

## 5. A construction of the Itô integral against cylindrical Wiener process, using suitable elementary processes and Itô isometries

**Lemma 5.1.** *The linear span of* $\tilde{\mathcal{E}}_{MP}$ *is the linear space of elementary processes* $\Phi : \Omega\times[0,T]\to L_2(H,\mathbb{R})$ *for which there exists* $p\in\mathbb{N}^*$,
$0 = t_0 < t_1 < \cdots < t_i < \cdots < t_p = T$ *such that*

$$\Phi(t) = \sum_{i=0}^{p-1}\Phi_i \mathbf{1}_{]t_i,t_{i+1}]}(t) \tag{5.1}$$

*where*

1. *$\forall i,\ \Phi_i : \Omega \to L_2(H,\mathbb{R})$ is $\mathcal{F}_{t_i}$-measurable, $L_2(H,\mathbb{R})$ being endowed with its Borel $\sigma$-algebra*
2. *$\forall i,\ \Phi_i$ takes only a finite number of values in $L_2(H,\mathbb{R})$*

*We denote this linear space $\tilde{\mathcal{E}}_{PR}$.*

*Proof of Lemma 5.1.* Let $\Phi_1, \Phi_2 \in \tilde{\mathcal{E}}_{MP}$, we can always find a partition of $[0,T]$ such that

$$\Phi_1(t) = \sum_{i=0}^{p-1} \Phi_i^1 \mathbf{1}_{F_i^1} \mathbf{1}_{]t_i,t_{i+1}]}(t) \text{ and } \Phi_2(t) = \sum_{i=0}^{p-1} \Phi_i^2 \mathbf{1}_{F_i^2} \mathbf{1}_{]t_i,t_{i+1}]}(t)$$

thus

$$(\Phi_1 + \Phi_2)(t) = \sum_{i=0}^{p-1} \Phi_i \mathbf{1}_{]t_i,t_{i+1}]}(t)$$

with $\forall i$:

$$\Phi_i = (\Phi_i^1 + \Phi_i^2)\mathbf{1}_{F_i^1 \cap F_i^2} + \Phi_i^1 \mathbf{1}_{F_i^1 \setminus F_i^2} + \Phi_i^2 \mathbf{1}_{F_i^2 \setminus F_i^1}$$

Thus $\tilde{\mathcal{E}}_{MP} \subset \tilde{\mathcal{E}}_{PR}$.

Now, let $\Phi \in \tilde{\mathcal{E}}_{PR}$ such that $\Phi(t) = \sum_{i=0}^{p-1} \Phi_i \mathbf{1}_{]t_i,t_{i+1}]}(t)$. To prove that $\tilde{\mathcal{E}}_{PR} \subset$ *span* $\left(\tilde{\mathcal{E}}_{MP}\right)$, it suffices to prove that for each $i \in \{0,\dots,p-1\}, \Phi_i$ can be written as a linear combination of processes of the form $\varphi \mathbf{1}_F$ with $\varphi \in L_2(H,\mathbb{R})$ and $F \in \mathcal{F}_{t_i}$. But $\Phi_i$ takes only a finite number of values in $L_2(H,\mathbb{R})$, say $\varphi_j^i \in L_2(H,\mathbb{R})$ on $\Omega_j^i$ where $\left(\Omega_j^i\right)_{j\in J^i}$ is a finite partition of $\Omega$ and $\varphi_{j_1}^i \neq \varphi_{j_2}^i$, $\forall j_1 \neq j_2 \in J^i$. Then

$$\Phi_i = \sum_{j\in J^i} \varphi_j^i \mathbf{1}_{\Omega_j^i}. \tag{5.2}$$

$\Phi_i$ being $\mathcal{F}_{t_i}$-measurable, $\forall j \in J^i,\ \Omega_j^i = \Phi_i^{-1}\left(\left\{\varphi_j^i\right\}\right) \in \mathcal{F}_{t_i}$. □

**Definition 5.1.** We define the Itô integral of elementary processes $\Phi \in \tilde{\mathcal{E}}_{PR} =$ *span* $\left(\tilde{\mathcal{E}}_{MP}\right)$ with respect to the cylindrical process $\tilde{W}$ defined in Definition 4.2 by

$$\int_0^T \Phi(t) d\tilde{W}_t := \sum_{i=0}^{p-1} \sum_{j\in J^i} \int_0^T \varphi_j^i \mathbf{1}_{\Omega_j^i} \mathbf{1}_{]t_i,t_{i+1}]}(t) d\tilde{W}_t \tag{5.3}$$

with $p \in \mathbb{N}^*,\ 0 = t_0 < t_1 < \cdots < t_i < \cdots < t_p = T$ and

$$\Phi(t) = \sum_{i=0}^{p-1} \left( \sum_{j\in J^i} \varphi_j^i \mathbf{1}_{\Omega_j^i} \right) \mathbf{1}_{]t_i,t_{i+1}]}(t),$$

described in Lemma 5.1 and formula (5.2). The integrals on the right-hand side of the equality (5.3) are defined in Definitions 4.4 and 4.5.

**Lemma 5.2.** *For elementary processes $\Phi \in \tilde{\mathcal{E}}_{PR} = span\left(\tilde{\mathcal{E}}_{MP}\right)$ and the corresponding Itô integral defined in Definition 5.1 belonging to $L^2(\Omega, \mathbb{R})$, we have the following Itô isometry:*

$$\left(\mathbb{E}\left(\int_0^T \Phi(t) d\tilde{W}_t\right)^2\right) = \int_0^T \mathbb{E}\|\Phi(t)\|^2_{L_2(H,\mathbb{R})} dt \tag{5.4}$$

*Proof of Lemma 5.2.* Using the notations of Definition 5.1, we have

$$\begin{aligned}
&\left(\mathbb{E}\left(\sum_{i=0}^{p-1}\sum_{j\in J^i}\int_0^T \varphi_j^i \mathbf{1}_{\Omega_j^i}\mathbf{1}_{]t_i,t_{i+1}]}(t) d\tilde{W}_t\right)^2\right) \\
&\qquad = \mathbb{E}\sum_{i=0}^{p-1}\sum_{j\in J^i}\left(\int_0^T \varphi_j^i \mathbf{1}_{\Omega_j^i}\mathbf{1}_{]t_i,t_{i+1}]}(t) d\tilde{W}_t\right)^2 \\
&\quad + \sum_{i=0}^{p-1}\sum_{j\in J^i}\sum_{\substack{n=0 \\ n\neq i \text{ or } m\neq j}}^{p-1}\sum_{m\in J^n}\mathbb{E}\left(\left(\int_0^T \varphi_j^i \mathbf{1}_{\Omega_j^i}\mathbf{1}_{]t_i,t_{i+1}]}(t) d\tilde{W}_t\right)\right. \\
&\qquad\qquad \left.\times\left(\int_0^T \varphi_m^n \mathbf{1}_{\Omega_m^n}\mathbf{1}_{]t_n,t_{n+1}]}(t) d\tilde{W}_t\right)\right).
\end{aligned}$$

In the last term, if $i \neq n$, and e.g. $i < n$, then by independence and properties of real Brownian motions, we have

$$\begin{aligned}
&\mathbb{E}\left(\left(\int_0^T \varphi_j^i \mathbf{1}_{\Omega_j^i}\mathbf{1}_{]t_i,t_{i+1}]}(t) d\tilde{W}_t\right)\times\left(\int_0^T \varphi_m^n \mathbf{1}_{\Omega_m^n}\mathbf{1}_{]t_n,t_{n+1}]}(t) d\tilde{W}_t\right)\right) \\
&\qquad = \mathbb{E}\left(\mathbf{1}_{\Omega_j^i}\left(\tilde{W}_{t_{i+1}}\left(\varphi_j^{i\,*}(1)\right) - \tilde{W}_{t_i}\left(\varphi_j^{i\,*}(1)\right)\right)\right. \\
&\qquad\quad \left.\times\, \mathbf{1}_{\Omega_m^n}\left(\tilde{W}_{t_{n+1}}\left(\varphi_m^{n\,*}(1)\right) - \tilde{W}_{t_n}\left(\varphi_m^{n\,*}(1)\right)\right)\right) \\
&\qquad = \mathbb{E}\left(\mathbf{1}_{\Omega_j^i}\left(\tilde{W}_{t_{i+1}}\left(\varphi_j^{i\,*}(1)\right) - \tilde{W}_{t_i}\left(\varphi_j^{i\,*}(1)\right)\right)\right. \\
&\qquad\qquad \left.\times\, \mathbf{1}_{\Omega_m^n}\right)\mathbb{E}\left(\left(\tilde{W}_{t_{n+1}}\left(\varphi_m^{n\,*}(1)\right) - \tilde{W}_{t_n}\left(\varphi_m^{n\,*}(1)\right)\right)\right) = 0.
\end{aligned}$$

In the same last term, if $i = n$ and $j \neq m$, then $\mathbf{1}_{\Omega_j^i}(\omega)\mathbf{1}_{\Omega_m^i}(\omega) = 0,\ \forall \omega \in \Omega$. Thus

$$\left(\mathbb{E}\left(\sum_{i=0}^{p-1}\sum_{j\in J^i}\int_0^T \varphi_j^i \mathbf{1}_{\Omega_j^i}\mathbf{1}_{]t_i,t_{i+1}]}(t) d\tilde{W}_t\right)^2\right)$$

$$= \sum_{i=0}^{p-1} \sum_{j \in J^i} \mathbb{E}\left( \int_0^T \varphi_j^i \mathbf{1}_{\Omega_j^i} \mathbf{1}_{]t_i,t_{i+1}]}(t) d\tilde{W}_t \right)^2 + 0$$

$$= \sum_{i=0}^{p-1} \sum_{j \in J^i} \mathbb{E} \int_0^T \left\| \varphi_j^i \mathbf{1}_{\Omega_j^i} \mathbf{1}_{]t_i,t_{i+1}]}(t) \right\|^2_{L_2(H,\mathbb{R})} dt$$

$$= \sum_{i=0}^{p-1} \sum_{j \in J^i} \mathbb{E} \int_0^T \left\| \varphi_j^i \right\|^2_{L_2(H,\mathbb{R})} \mathbf{1}_{\Omega_j^i} \mathbf{1}_{]t_i,t_{i+1}]}(t) dt = \int_0^T \mathbb{E} \|\Phi(t)\|^2_{L_2(H,\mathbb{R})} dt.$$

□

**Lemma 5.3.** *$\tilde{\mathcal{E}}_{PR}$ is a dense linear subspace of $\tilde{\mathcal{E}}_{GM}$ defined in Definition 4.3 when the linear spaces are endowed with the norm*

$$\|\Phi\|_{\tilde{\mathcal{E}}_{MP}} := \left( \mathbb{E}\left( \sum_{k=1}^{+\infty} \int_0^T (\Phi(t) e_k)^2 \, dt \right) \right)^{1/2}, \quad \forall \Phi \in \tilde{\mathcal{E}}_{GM}. \tag{5.5}$$

*Proof of Lemma 5.3.* Let $\Phi \in \tilde{\mathcal{E}}_{PR}$, it can be written

$$\Phi(t) = \sum_{i=0}^{p-1} \Phi_i \mathbf{1}_{]t_i,t_{i+1}]}(t)$$

as in (5.1). Thus, using the decomposition (5.2) of $\Phi_i$, we obtain

$$\sum_{k=1}^{+\infty} (\Phi_i(e_k))^2 \leq \sum_{i=0}^{p-1} \sum_{j \in J^i} \left\| \varphi_j^i \right\|^2_{L_2(H,\mathbb{R})} < +\infty,$$

thus

$$\tilde{\mathcal{E}}_{PR} \subset \tilde{\mathcal{E}}_{GM}.$$

Now, let $\Phi \in \tilde{\mathcal{E}}_{GM}$ and $\Phi_i$ in the decomposition (4.4) of $\Phi$ with $i \neq -1$. As $L_2(H,\mathbb{R})$ is separable, there exists a countable dense subset of $\Phi_i(\Omega)$, that we will denote $\left\{h_k^i,\ k \in \mathbb{N}^*\right\}$.

Let us define for $m \in \mathbb{N}^*, \omega \in \Omega$:

- $d_m^i(\omega) := \min_{1 \leq k \leq m} \left\{ \left\| h_k^i - \Phi_i(\omega) \right\|^2_{L_2(H,\mathbb{R})} \right\}$
- $k_m^i(\omega) := \min \left\{ 1 \leq k \leq m \text{ such that } d_m^i(\omega) = \left\| h_k^i - \Phi_i(\omega) \right\|^2_{L_2(H,\mathbb{R})} \right\}$
- $H_m^i(\omega) := h_{k_m^i(\omega)}$

then

$$h_m^i(\Omega) \subset \left\{ h_1^i, \ldots, h_m^i \right\}.$$

To prove the $\mathcal{F}_{t_i}$-measurability of $H_m^i$, we start by

$$\begin{array}{rccl} d_m^i : & (\Omega, \mathcal{F}_{t_i}) & \longrightarrow & (\mathbb{R}, \mathcal{B}(\mathbb{R})) \\ & \omega & \longmapsto & \min_{1 \leq k \leq m} \left\{ \left\| h_k^i - \Phi_i(\omega) \right\|^2_{L_2(H,\mathbb{R})} \right\} \end{array}$$

which is measurable because $\Phi_i$ is $\mathcal{F}_{t_i}$-measurable, then

$$\begin{array}{rccl} k_m^i: & (\Omega, \mathcal{F}_{t_i}) & \longrightarrow & (\{1,\dots,m\}, \mathcal{P}(\{1,\dots,m\})) \\ & \omega & \longmapsto & \min\left\{1 \le k \le m \text{ such that } d_m^i(\omega) = \left\|h_k^i - \Phi_i(\omega)\right\|^2_{L_2(H,\mathbb{R})}\right\} \end{array}$$

which is measurable because, denoting $\varphi_k^i(\omega) := \left\|h_k^i - \Phi_i(\omega)\right\|^2_{L_2(H,\mathbb{R})}$, for any fixed $k \in \{2,\dots,m\}$, we have

$$\left(k_m^i\right)^{-1}(\{k\}) = \left(d_m^i - \varphi_k^i\right)^{-1}(\{0\}) \bigcap_{1 \le \tilde{k} < k} \left(d_m^i - \varphi_{\tilde{k}}^i\right)^{-1}(\mathbb{R}^*)$$

and

$$\left(k_m^i\right)^{-1}(\{1\}) = \left(d_m^i - \varphi_1^i\right)^{-1}(\{0\}).$$

We finish by proving the $\mathcal{F}_{t_i}$-measurability of each $H_m^i$, noticing that for all $s \in \{1,\dots,m\}$,

$$\left(H_m^i\right)^{-1}\left(\left\{h_s^i\right\}\right) = \left(k_m^i\right)^{-1}(\{s\}).$$

Hence the sequence of processes $(H_m)_{m\in\mathbb{N}^*}$ whose elements are defined for each $m \in \mathbb{N}^*$ by

$$H_m(\omega,t) := \sum_{i=0}^{p-1} H_m^i(\omega)\mathbf{1}_{]t_i,t_{i+1}]}(t), \forall(\omega,t) \in \Omega \times [0,T] \tag{5.6}$$

is a good candidate to converge towards $\Phi \in \tilde{\mathcal{E}}_{GM}$ with the norm $\|.\|_{\tilde{\mathcal{E}}_{MP}}$ because its elements are in $\tilde{\mathcal{E}}_{PR}$. Indeed, we have

$$\mathbb{E}\int_0^T \|H_m(t) - \Phi(t)\|^2_{L_2(H,\mathbb{R})}\,dt = \sum_{i=0}^{p-1}(t_{i+1} - t_i)\mathbb{E}\left\|H_m^i - \Phi_i\right\|^2_{L_2(H,\mathbb{R})}$$

and

$$m \mapsto \left\|H_m^i - \Phi_i\right\|^2_{L_2(H,\mathbb{R})}$$

is almost surely a decreasing sequence, uniformly bounded by $4M$ ($M$ being the bound in (4.5)). Moreover, by density of $\left\{h_k^i,\ k \in \mathbb{N}^*\right\}$ in $\Phi_i(\Omega)$, we have

$$\lim_{m\to+\infty} d_m^i = 0, \text{ almost surely}$$

that is

$$\lim_{m\to+\infty} \left\|H_m^i - \Phi_i\right\|^2_{L_2(H,\mathbb{R})} = 0, \text{ almost surely}$$

then the dominated convergence theorem allows us to conclude. □

**Definition 5.2.** By Lemma 5.3, the Itô isometry defined and proved in Lemma 5.2 has a unique extension to the linear space $\tilde{\mathcal{E}}_{GM}$ (see Definition 4.3) for which the Itô integral is an isometry. Thus we can define the Itô integral of elementary processes $\Phi \in \tilde{\mathcal{E}}_{GM}$ with respect to the cylindrical process $\tilde{W}$ defined in Definition 4.2 by

$$\int_0^T \Phi(t)d\tilde{W}_t := \lim_{n\to+\infty} \int_0^T \Phi_n(t)d\tilde{W}_t \tag{5.7}$$

where $(\Phi_n)_{n\in\mathbb{N}} \subset \tilde{\mathcal{E}}_{PR}$ is a sequence converging towards $\Phi \in \tilde{\mathcal{E}}_{GM}$ with the norm $\|.\|_{\tilde{\mathcal{E}}_{MP}}$ defined by (5.5)

**Proposition 5.4.** *$\tilde{\mathcal{E}}_{GM}$ is a dense linear subspace of*

$$\Lambda_2(H,\mathbb{R}) := \Big\{\Phi : \Omega\times[0,T] \to L_2(H,\mathbb{R}) \text{ progressively measurable such that } \|\Phi\|_{\tilde{\mathcal{E}}_{MP}} < +\infty\Big\}$$

*when they are endowed with the norm $\|.\|_{\tilde{\mathcal{E}}_{MP}}$ defined in* (5.5)

*Proof of Proposition 5.4.* It is an adaptation of the proof of proposition 2.2. page 28 in [14], which is a generalization of the proof pages 26–28 in [26].

Let $\Phi \in \Lambda_2(H,\mathbb{R})$, the sequence $(\Phi_n)_{n\in\mathbb{N}}$ defined for each $n \in \mathbb{N}$ by:

$$\Phi_n(\omega,t) = \begin{cases} \frac{n}{\|\Phi(\omega,t)\|_{L_2(H,\mathbb{R})}}\Phi(\omega,t) & \text{if } \|\Phi(\omega,t)\|_{L_2(H,\mathbb{R})} > n \\ \Phi(\omega,t) & \text{otherwise} \end{cases}$$

has each term bounded by $n$ in $L_2(H,\mathbb{R})$ and converges towards $\Phi$ with the norm $\|.\|_{\tilde{\mathcal{E}}_{MP}}$ by the dominated convergence theorem.

We now assume that $\Phi \in \Lambda_2(H,\mathbb{R})$ is bounded by $M \in \mathbb{R}_+$ for all $(\omega,t) \in \Omega\times[0,T]$. Then the sequence $(\Phi_n)_{n\in\mathbb{N}}$ defined for each $n \in \mathbb{N}$ by

$$\Phi_n(\omega,t) := 2\int_0^t \rho_n(t-s)\Phi(\omega,s)ds, \quad \forall t \in [0,T], \omega\in\Omega$$

where $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of mollifiers verifying for all $n \in \mathbb{N}$:

- $\rho_n \in C_c^\infty(\mathbb{R};\mathbb{R}_+)$
- The support of $\rho_n$ is included in $[-\frac{1}{n};\frac{1}{n}]$
- $\displaystyle\int_{\mathbb{R}} \rho_n(s)ds = 1$
- $\rho_n$ is even, thus $\displaystyle\int_0^{+\infty} \rho_n(s)ds = 0.5$,

is right-continuous for each fixed $\omega\in\Omega$, uniformely bounded by $M \in \mathbb{R}_+$, $\mathcal{F}_t$-measurable for each fixed $t \in [0,T]$, and converges towards $\Phi$ with the norm $\|.\|_{\tilde{\mathcal{E}}_{MP}}$. Let us prove each point of this assertion.

1. Let $(t_m)_{m\in\mathbb{N}} \subset [\bar{t};T]$ converging towards $\bar{t} \in [0,T[$, then for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\left\|\Phi_n(t_m) - \Phi_n(\bar{t})\right\|_{L_2(H,\mathbb{R})} \\
&= 2\left\|\int_0^T \left(\mathbf{1}_{[0,t_m]}(s)\rho_n(t_m - s) - \mathbf{1}_{[0,\bar{t}]}(s)\rho_n(\bar{t}-s)\right)\Phi(s)ds\right\|_{L_2(H,\mathbb{R})} \\
&\quad \leq 2\int_0^T \mathbf{1}_{[0,\bar{t}]}(s)\left|\rho_n(t_m - s) - \rho_n(\bar{t}-s)\right| \left\|\Phi(s)\right\|_{L_2(H,\mathbb{R})} ds \\
&\qquad + 2\int_0^T \mathbf{1}_{[\bar{t},t_m]}(s)\rho_n(t_m - s)\left\|\Phi(s)\right\|_{L_2(H,\mathbb{R})} ds.
\end{aligned}$$

The two last terms converge towards 0 when $m \to +\infty$ because of the uniform continuity of $\rho$, the fact that $\Phi$ is uniformly bounded, and the use of the dominated convergence theorem. Thus $\Phi_n$ is right-continuous.

2. For all $(\omega,t) \in \Omega \times [0,T]$, we have

$$\left\|\int_0^t \rho_n(t-s)\Phi(\omega,s)ds\right\|_{L_2(H,\mathbb{R})} \leq \int_0^T \rho_n(t-s)\left\|\Phi(\omega,s)\right\|_{L_2(H,\mathbb{R})} ds \leq M.$$

3. For any fixed $t \in [0,T]$, to show the $\mathcal{F}_t$-measurability, we use the fact that

$$(\omega,s) \in \Omega \times [0,t] \mapsto \mathbf{1}_{[0,t]}(s)\rho_n(t-s)\Phi(\omega,s)$$

is progressively measurable, thus $\mathcal{F}_t \otimes \mathcal{B}([0,t])$-measurable. It is integrable against the measure $d\mathbb{P}\otimes ds$, thus using Fubini's theorem (see [8], theorem 1.5.2), we can conclude that $\Phi_n$ is $\mathcal{F}_t$-measurable.

4. Let us fix $n \in \mathbb{N}, t \in [0,T], n \in \mathbb{N}$, then

$$\begin{aligned}
&\left\|\Phi_n - \Phi\right\|^2_{\tilde{\mathcal{E}}_{MP}} = \mathbb{E}\int_0^T \left\|\Phi_n(t) - \Phi(t)\right\|^2_{L_2(H,\mathbb{R})} dt \\
&\quad = \mathbb{E}\int_0^T \left\|2\int_0^t \rho_n(t-s)\Phi(s)ds - 2\int_0^{+\infty} \rho_n(s)\Phi(t)ds\right\|^2_{L_2(H,\mathbb{R})} dt \\
&\quad = 4\mathbb{E}\int_0^T \left\|\int_0^t \rho_n(s)\Phi(t-s)ds - \int_0^{+\infty} \rho_n(s)\Phi(t)ds\right\|^2_{L_2(H,\mathbb{R})} dt \\
&\quad \leq 4\mathbb{E}\int_0^T \left(\int_0^{+\infty} \left\|\left(\mathbf{1}_{[0,t]}(s)\Phi(t-s) - \Phi(t)\right)\rho_n(s)\right\|_{L_2(H,\mathbb{R})} ds\right)^2 dt \\
&\qquad \leq \mathbb{E}\int_0^T \int_0^{1/n} \left\|\mathbf{1}_{[0,t]}(s)\Phi(t-s) - \Phi(t)\right\|^2_{L_2(H,\mathbb{R})} 2\rho_n(s)dsdt
\end{aligned}$$

by Jensen inequality (because each $2\rho_n(s)ds$ is a probability measure on $(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$). Now we use corollary 1.3.1 of [8] to get sequences $(g_m(\omega,.))_{m\in\mathbb{N}} \subset C_c^\infty(]0;T[;L_2(H,\mathbb{R}))$ which converges towards $\Phi(\omega,.)$ in $L^2(]0,T[;L_2(H,\mathbb{R}))$ for each $\omega \in \Omega$. Then

$$
\begin{aligned}
&\int_0^T \int_0^{1/n} \left\|\mathbf{1}_{[0,t]}(s)\Phi(\omega,t-s)-\Phi(\omega,t)\right\|^2_{L_2(H,\mathbb{R})} 2\rho_n(s)dsdt \\
&\leq 6\int_0^{1/n}\int_0^T \left\|\mathbf{1}_{[0,t]}(s)\Phi(\omega,t-s)-\mathbf{1}_{[0,t]}(s)g_m(\omega,t-s)\right\|^2_{L_2(H,\mathbb{R})}\rho_n(s)dtds \\
&\quad + 6\int_0^{1/n}\int_0^T \left\|\mathbf{1}_{[0,t]}(s)g_m(\omega,t-s)-g_m(\omega,t)\right\|^2_{L_2(H,\mathbb{R})}\rho_n(s)dtds \\
&\quad + 6\int_0^T \|g_m(\omega,t)-\Phi(\omega,t)\|^2_{L_2(H,\mathbb{R})}dt \\
&\leq 6\int_0^{1/n}\int_0^T \|\Phi(\omega,t)-g_m(\omega,t)\|^2_{L_2(H,\mathbb{R})}dt\rho_n(s)ds \\
&\quad + 6\int_0^{1/n}\int_0^T \left\|\mathbf{1}_{[0,t]}(s)g_m(\omega,t-s)-g_m(\omega,t)\right\|^2_{L_2(H,\mathbb{R})}\rho_n(s)dtds \\
&\quad + 6\int_0^T \|g_m(\omega,t)-\Phi(\omega,t)\|^2_{L_2(H,\mathbb{R})}dt.
\end{aligned}
$$

The first and last terms of the last three terms will be less than any $\varepsilon > 0$ for $m$ large enough. Then, once the $m \in \mathbb{N}$ is fixed, we use the fact that each $g_m(\omega,.)$ is uniformly continuous on $]0,T[$ to see that the second term will be less than $\varepsilon > 0$ for $n$ large enough. □

**Definition 5.3.** By Proposition 5.4, the Itô isometry defined in Lemma 5.2 and extended in Definition 5.2 has a unique extension to the linear space $\Lambda_2(H,\mathbb{R})$ (see Proposition 5.4 for the definition of $\Lambda_2(H,\mathbb{R})$) for which the Itô integral is an isometry. Thus we can define the Itô integral of $\Phi \in \Lambda_2(H,\mathbb{R})$ with respect to the cylindrical process $\tilde{W}$ defined in Definition 4.2 by

$$\int_0^T \Phi(t)d\tilde{W}_t := \lim_{n\to+\infty}\int_0^T \Phi_n(t)d\tilde{W}_t \tag{5.8}$$

where $(\Phi_n)_{n\in\mathbb{N}} \subset \tilde{\mathcal{E}}_{GM}$ is a sequence converging towards $\Phi \in \Lambda_2(H,\mathbb{R})$ with the norm $\|.\|_{\tilde{\mathcal{E}}_{MP}}$ defined by (5.5).

To conclude this section, we can assert the following

**Theorem 5.5.** *The Itô integral against cylindrical Wiener processes, constructed starting from the elementary processes $\tilde{\mathcal{E}}_{MP}$ and extended to integrands belonging to $\Lambda_2(H,\mathbb{R})$ by Itô isometries, is the same as the Itô integral against cylindrical Wiener processes constructed starting from the elementary processes $\tilde{\mathcal{E}}_{PR}$ or constructed starting from the elementary processes $\tilde{\mathcal{E}}_{GM}$ or constructed starting from the elementary processes $\tilde{\mathcal{E}}_{Hai}$.*

*Proof of Theorem 5.5.* We have

$$span(\tilde{\mathcal{E}}_{MP}) = \tilde{\mathcal{E}}_{PR} \subset \tilde{\mathcal{E}}_{GM} \subset \tilde{\mathcal{E}}_{Hai} \subset \overline{span\left(\tilde{\mathcal{E}}_{MP}\right)} = \Lambda_2(H,\mathbb{R}),$$

thus

$$\overline{span\left(\tilde{\mathcal{E}}_{MP}\right)} = \overline{\tilde{\mathcal{E}}_{PR}} = \overline{\tilde{\mathcal{E}}_{GM}} = \overline{\tilde{\mathcal{E}}_{Hai}}$$

□

*Remark* 5.1. We can take in (5.8), $(\Phi_n)_{n\in\mathbb{N}}$ in any of the spaces $span(\tilde{\mathcal{E}}_{MP}) = \tilde{\mathcal{E}}_{PR}$, $\tilde{\mathcal{E}}_{GM}, \tilde{\mathcal{E}}_{Hai}$.

## 6. White noise theory and Itô Integral against cylindrical Wiener process

### 6.1. *Hida's white noise in dimension one*

#### *6.1.1. The fundamental questions related to white noise theory*

If the question of the definition of cylindrical Wiener processes arise in infinite dimension, the question of definition of white noise arise already in dimension 1. Indeed, if $B(t)$ is a standard real Brownian motion, it is well known that it is almost surely nowhere differentiable with respect to the time variable (see Theorem 2.2.1 in [13]). The Gaussian white noise related to $B(t)$ is $\dot{B}(t)$, that is the time derivative of $B(t)$. Thus to see how the theory of white noise overcome the difficulty of the existence of cylindrical Wiener processes in infinite dimension, we should first see how they overcome the difficulty of the formal derivative $\dot{B}(t)$. To understand the philosophy, we can take the stochastic balance law with $A \in C^2(\mathbb{R}, \mathbb{R})$ and $\Phi : \mathbb{R} \to \mathbb{R}$ a sublinear function:

$$du(x,t) + \partial_x A(u(x,t))dt = \Phi(u(x,t))dB(t), \quad x \in \mathbb{R}, t \in [0,T] \subset \mathbb{R}_+,$$

whose solutions $u(x,t)$ are written with Itô integrals against $B(t)$ (see [6]). The white noise theory prefers to write the equation

$$\partial_t u(x,t) + \partial_x A(u(x,t)) = \Phi(u(x,t))\dot{B}(t), \quad x \in \mathbb{R}, t \in [0,T] \subset \mathbb{R}_+,$$

which is also the deterministic way to write balance laws (see [6]).

For the first equation, we use test functions with respect to the space variable $x$, while, as in the deterministic way, test functions of time and space variables should be used in the second equation.

For the first equation,

$$\int_0^T g(x)\Phi(u(x,t))dB(t)$$

has to be define for $g \in C_c^\infty(\mathbb{R})$, while for the second equation

$$\int_0^T g(x)\varphi(t)\Phi(u(x,t))\dot{B}(t)dt$$

could be, at first sight, defined for $g \in C_c^\infty(\mathbb{R})$ and $\varphi \in C_c^\infty[0,T)$ (with $T > 0$). An integration by parts would give

$$\int_0^T g(x)\varphi(t)\Phi(u(x,t))\dot{B}(t)dt =$$

$$-\int_0^T g(x)\dot{\varphi}(t)\Phi(u(x,t))B(t)dt - \int_0^T g(x)\varphi(t)\frac{\partial\Phi(u(x,t))}{\partial t}B(t)dt \quad (6.1)$$

but the solution $u(x,t)$ is in general not differentiable with respect to the time variable in the classical sense. So the question of the construction of the stochastic integral is replaced by the questions

- which space of test functions should we use?
- what is the topological dual of the space of test functions to which $\dot{B}(t)$ and $\Phi(u(x,t))\dot{B}(t)$ must belong?

to make the Lebesgue integral

$$\int_0^T g(x)\varphi(t)\Phi(u(x,t))\dot{B}(t)dt$$

rigorous. Let us take the constant function $\Phi = 1$, then we have a possible definition of $\dot{B}(t)$, that is the linear mapping

$$\varphi \in C_c^\infty([0,T)) \mapsto \langle \dot{B}, \varphi\rangle := -\int_0^T \dot{\varphi}(t)B(t)dt. \quad (6.2)$$

If we consider a non-constant function $\Phi : \mathbb{R} \to \mathbb{R}$ in (6.1), the deterministic space of test functions $C_c^\infty([0,T))$ is not working for the functional $\Phi(u(x,t))\dot{B}(t)$.

In his conference paper [20], Hida explains that he is taking a bigger space of test functions, that is $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing smooth functions taking values on $\mathbb{R}$ which includes $C_c^\infty([0,T))$, hence he considers $\dot{B}$ as a random variable taking its values in the dual space $\mathcal{S}'(\mathbb{R})$ of $\mathcal{S}(\mathbb{R})$.

He considers the linear mapping $\varphi \in \mathcal{S}(\mathbb{R}) \mapsto \langle \dot{B}, \varphi\rangle$ as a stochastic process indexed by the test functions $\varphi \in \mathcal{S}(\mathbb{R})$. He proves that it is a Gaussian stochastic process for which each random variable $\langle \dot{B}, \varphi\rangle$ has for characteristic function

$$x \in \mathbb{R} \mapsto \mathbb{E}\exp(i\langle \dot{B}, \varphi\rangle x) = \exp\left(-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R})} \times x^2\right), \quad (6.3)$$

and defines the characteristic functional of this stochastic process which is a generalization of the characteristic function as:

$$\varphi \in \mathcal{S}(\mathbb{R}) \mapsto \mathbb{E}\exp(i\langle \dot{B}, \varphi\rangle) := \int_{\mathcal{S}'(\mathbb{R})} \exp(i\langle X, \varphi\rangle)\mu(dX) \quad (6.4)$$

where $\mu$ is a probability measure on the Borel sets of $\mathcal{S}'(\mathbb{R})$. Taking $x = 1$ in (6.3), the equation of unknown $\mu$ follows:

$$\int_{\mathcal{S}'(\mathbb{R})} \exp(i\langle X, \varphi\rangle)\mu(dX) = \exp\left(-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R})}\right). \quad (6.5)$$

By the Bochner-Milnos theorem (see theorem 2.1.1. in [22]), it has a unique solution. This solution is a Gaussian measure which is the probability distribution of the stochastic process $\dot{B}$ indexed by $\mathcal{S}(\mathbb{R})$.

As each element $x \in \mathcal{S}'(\mathbb{R})$ can be viewed as a realization of the stochastic process $\dot{B}$, and each Borel set $A \subset \mathcal{S}'(\mathbb{R})$ can be measured by the Gaussian measure $\mu$ solution of (6.5), the functionals of $\dot{B}$ can be written $\varphi(x)$ with $\varphi : \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$. That is why the Hilbert space $L^2(\mathcal{S}'(\mathbb{R}), \mu)$ is central in the theory of white noise.

*Remark* 6.1. When we write $L^2(\mathcal{S}'(\mathbb{R}), \mu)$, we mean the square integrable random variables defined on the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)$ where $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$ denotes the Borel $\sigma$-algebra on $\mathcal{S}'(\mathbb{R})$ equipped with the weak star topology.

*Remark* 6.2. The one dimensional standard real Brownian motion at time $t \in \mathbb{R}_+$ is the element of $L^2(\mathcal{S}'(\mathbb{R}), \mu)$ defined as the action of $\omega \in \mathcal{S}'(\mathbb{R})$ on the indicator function $\mathbf{1}_{[0,t]}$ of the time interval $[0, t]$:

$$B(t)(\omega) := \omega\left(\mathbf{1}_{[0,t]}\right), \quad \forall \omega \in \mathcal{S}'(\mathbb{R}). \tag{6.6}$$

See the very pedagogical article by Melnikova and Alshanskiy [24] for more details.

### *6.1.2. The answers of the fundamental questions related to white noise theory*

According to Kondratiev and Streit in [23], according to Holden, Øksendal, Ubøe, Zhang in [22], the spaces of test functions to be used to solve stochastic (partial) differential equations are the Kondratiev spaces of stochastic test functions $(\mathcal{S})_\rho$ defined in 2.3.2 a) in [22], the corresponding topological dual spaces of test functions to be used to solve stochastic (partial) differential equations are the Kondratiev spaces of stochastic distributions $(\mathcal{S})_{-\rho}$ defined in 2.3.2 b) in [22]. Let us write their definitions using Hermite polynomials.

Let $\{\xi_k\}_{k\in\mathbb{N}^*} \subset \mathcal{S}(\mathbb{R})$ be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the Hermite functions defined by

$$\xi_k(x) := \pi^{-1/4}((k-1)!)^{-1/2} \exp(-x^2/2) h_{k-1}(x), \forall x \in \mathbb{R} \tag{6.7}$$

where $\{h_n\}_{n\in\mathbb{N}}$ are the Hermite polynomials defined by

$$h_n(x) := (-1)^n \exp(x^2/2) \frac{d^n}{dx^n} \exp(-x^2/2), \forall x \in \mathbb{R}.$$

As in [22], let us regard multi-indices as elements of the subspace $\mathcal{J} \subset \mathbb{N}^{\mathbb{N}^*}$ of sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ with elements $\alpha_i \in \mathbb{N}$ such that only finitely many $\alpha_i \neq 0$. Let $\{\mathbf{H}_\alpha\}_{\alpha\in\mathcal{J}}$ be the orthogonal basis of $L^2(\mathcal{S}'(\mathbb{R}), \mu)$, consisting of the stochastic Hermite polynomials defined $\forall \alpha \in \mathcal{J}$ by

$$\mathbf{H}_\alpha(\omega) := \prod_{k\in\mathbb{N}^*} h_{\alpha_k}(w(\xi_k)), \quad \forall \omega \in \mathcal{S}'(\mathbb{R}) \tag{6.8}$$

whose norms in $L^2(\mathcal{S}'(\mathbb{R}), \mu)$ are verifying

$$\|\mathbf{H}_\alpha\|^2 = \alpha! = \alpha_1!\alpha_2! \cdots$$

**Definition 6.1.** Let $\rho \in [0,1]$, the Kondratiev space $(\mathcal{S})_\rho$ of $\mathbb{R}$-valued stochastic test functions consists of those

$$f = \sum_{\alpha \in \mathcal{J}} c_\alpha \mathbf{H}_\alpha \in L^2(\mathcal{S}'(\mathbb{R}), \mu)$$

with $c_\alpha \in \mathbb{R}$ such that $\forall q \in \mathbb{N}^*$:

$$\sum_{\alpha \in \mathcal{J}} \left( (\alpha!)^{1+\rho} c_\alpha^2 \prod_{j \in \mathbb{N}^*} (2j)^{q\alpha_j} \right) < +\infty.$$

**Definition 6.2.** Let $\rho \in [0,1]$, the corresponding Kondratiev space $(\mathcal{S})_{-\rho}$ of stochastic distributions consists of the formal expansions

$$f = \sum_{\alpha \in \mathcal{J}} b_\alpha \mathbf{H}_\alpha$$

with $b_\alpha \in \mathbb{R}$ such that there exists $q \in \mathbb{N}^*$ which verifies:

$$\sum_{\alpha \in \mathcal{J}} \left( (\alpha!)^{1-\rho} b_\alpha^2 \prod_{j \in \mathbb{N}^*} (2j)^{-q\alpha_j} \right) < +\infty.$$

*Remark* 6.3. At fixed $\rho \in [0,1]$, we obtain the inclusions

$$(\mathcal{S})_\rho \subset L^2(\mathcal{S}'(\mathbb{R}), \mu) \subset (\mathcal{S})_{-\rho}$$

with $(\mathcal{S})_{\rho_1} \subset (\mathcal{S})_{\rho_2}$ and $(\mathcal{S})_{-\rho_2} \subset (\mathcal{S})_{-\rho_1}$ for any $0 \leq \rho_2 \leq \rho_1 \leq 1$. The spaces $(\mathcal{S})_0$ and $(\mathcal{S})_{-0}$ are called the Hida spaces and are denoted by him $(\mathcal{S})$ and $(\mathcal{S})^*$ respectively (see [21]).

## 6.2. *Hida's white noise in infinite dimension*

Let us consider a separable Hilbert space $(H, \langle ., . \rangle_H)$ endowed with an orthonormal basis $(e_k)_{k \in N^*}$.

### 6.2.1. *The Kondratiev spaces for $H$-valued stochastic test functions and stochastic distributions*

In [10], Filinkov and Sorensen start to generalize the one dimensional case by proving that the family $\{\mathbf{H}_\alpha \otimes e_k\}_{\alpha \in \mathcal{J}, k \in \mathbb{N}^*}$ is an orthogonal basis of $L^2(\mathcal{S}'(\mathbb{R}), \mu, H)$ which is the set of all square integrable random variables from $\mathcal{S}'(\mathbb{R})$ to $H$, where each $\mathbf{H}_\alpha \otimes e_k$ is defined by

$$\mathbf{H}_\alpha \otimes e_k(\omega) := \mathbf{H}_\alpha(\omega) e_k, \quad \forall \omega \in \mathcal{S}'(\mathbb{R}).$$

The two definitions follows naturally:

**Definition 6.3.** Let $\rho \in [0,1]$, the Kondratiev space $(\mathcal{S})_\rho(H)$ of $H$-valued stochastic test functions consists of those

$$f = \sum_{\alpha\in\mathcal{J}} \sum_{k=1}^{+\infty} c_{\alpha,k} \mathbf{H}_\alpha \otimes e_k \in L^2(\mathcal{S}'(\mathbb{R}), \mu, H)$$

with $c_{\alpha,k} \in \mathbb{R}$ such that $\forall q \in \mathbb{N}^*$:

$$\sum_{\alpha\in\mathcal{J}} \sum_{k=1}^{+\infty} \Big( (\alpha!)^{1+\rho}\, c_{\alpha,k}^2 \prod_{j\in\mathbb{N}^*} (2j)^{q\alpha_j} \Big) < +\infty.$$

**Definition 6.4.** Let $\rho \in [0,1]$, the corresponding Kondratiev space $(\mathcal{S})_{-\rho}(H)$ of stochastic distributions consists of the formal expansions

$$f = \sum_{\alpha\in\mathcal{J}} \sum_{k=1}^{+\infty} b_{\alpha,k} \mathbf{H}_\alpha \otimes e_k$$

with $b_{\alpha,k} \in \mathbb{R}$ such that there exists $q \in \mathbb{N}^*$ which verifies:

$$\sum_{\alpha\in\mathcal{J}} \sum_{k=1}^{+\infty} \Big( (\alpha!)^{1-\rho}\, b_{\alpha,k}^2 \prod_{j\in\mathbb{N}^*} (2j)^{-q\alpha_j} \Big) < +\infty. \tag{6.9}$$

*Remark* 6.4. For all $0 \le \rho_2 \le \rho_1 \le 1$, we obtain the inclusions

$$(\mathcal{S})_{\rho_1}(H) \subset (\mathcal{S})_{\rho_2}(H) \subset L^2(\mathcal{S}'(\mathbb{R}), \mu, H) \subset (\mathcal{S})_{-\rho_2}(H) \subset (\mathcal{S})_{-\rho_1}(H),$$

*Remark* 6.5. The notations $(\mathcal{S})_\rho(H)$ and $(\mathcal{S})_{-\rho}(H)$ are taken from [24]. We also take their construction of a sequence of independent Brownian motions in the next section.

#### 6.2.2. *The cylindrical Wiener process as an element of the Kondratiev-Hida space $(\mathcal{S})_{-0}(H) = (\mathcal{S})^*(H)$*

Let $n : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ be the Hopcroft and Ullman pairing function defined by

$$n(i,k) := \frac{1}{2}(i+k-2)(i+k-1) + k, \quad \forall (i,k) \in \mathbb{N}^* \times \mathbb{N}^*.$$

It is a bijection between $\mathbb{N}^*$ and the Cartesian product $\mathbb{N}^* \times \mathbb{N}^*$.

Let $L^2(\mathbb{R})_k$ be the closure of the linear span of the set $\{\xi_{n(i,k)}, i \in \mathbb{N}^*\}$ which gives the orthogonal direct sum

$$L^2(\mathbb{R}) = \bigoplus_{k=1}^{+\infty} L^2(\mathbb{R})_k.$$

Let $\mathcal{T}_k$ be the isometric isomorphism from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})_k$ defined by

$$\mathcal{T}_k(f) = \sum_{i=1}^{+\infty} \int_{\mathbb{R}} f(y)\xi_i(y)dy\ \xi_{n(i,k)}, \quad \forall f \in L^2(\mathbb{R}),$$

then, the sequence $\{\beta_k(t)\}_{k\in\mathbb{N}^*}$ defined for each $k \in \mathbb{N}^*, t \in \mathbb{R}_+$ by the action of $\omega \in \mathcal{S}'(\mathbb{R})$ on the image of the indicator function $\mathbf{1}_{[0,t]}$ under $\mathcal{T}_k$

$$\beta_k(t)(\omega) := \omega\left(\mathcal{T}_k\left(\mathbf{1}_{[0,t]}\right)\right), \quad \forall \omega \in \mathcal{S}'(\mathbb{R}) \tag{6.10}$$

is a sequence of independent standard real Brownian motions. (see [24] for the proof).

The formal Wiener process (2.1) can be rewritten with this sequence:

$$\begin{aligned} W(\omega,t) &= \sum_{k=1}^{+\infty} \beta_k(\omega,t)e_k = \sum_{k=1}^{+\infty} \omega\left(\mathcal{T}_k\left(\mathbf{1}_{[0,t]}\right)\right) e_k \\ &= \sum_{k=1}^{+\infty} \omega\left(\sum_{i=1}^{+\infty}\int_0^t \xi_i(y)dy\ \xi_{n(i,k)}\right) e_k = \sum_{k=1}^{+\infty}\sum_{i=1}^{+\infty}\int_0^t \xi_i(y)dy\ \omega\left(\xi_{n(i,k)}\right) e_k \\ &= \sum_{k=1}^{+\infty}\sum_{i=1}^{+\infty}\int_0^t \xi_i(y)dy\ H_{\epsilon_{n(i,k)}}(\omega)\ e_k \end{aligned} \tag{6.11}$$

with $\epsilon_{n(i,k)} = (0,0,\ldots,0,1,0,\ldots,0,\ldots)$, the 1 being at the $n(i,k)$th position. Then, denoting

$$\theta_k(j)(t) := \begin{cases} \int_0^t \xi_i(y)dy & \text{if } \exists i \in \mathbb{N}^* \text{ such that } n(i,k) = j \\ 0 & \text{otherwise.} \end{cases}$$

it can we rewritten

$$\begin{aligned} W(\omega,t) &= \sum_{k=1}^{+\infty}\sum_{j=1}^{+\infty} \theta_k(j)(t)\ H_{\epsilon_j}(\omega)\ e_k = \sum_{\epsilon_j\in\mathcal{J}}\sum_{k=1}^{+\infty} \theta_k(j)(t)\ H_{\epsilon_j} \otimes e_k(\omega) \\ &= \sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty} \theta_k(j)(t)\ H_{\epsilon_j} \otimes e_k(\omega) = \sum_{j=1}^{+\infty}\int_0^t \xi_{\tilde{i}(j)}(y)dy\ H_{\epsilon_j} \otimes e_{\tilde{k}(j)}(\omega) \end{aligned} \tag{6.12}$$

where $\forall j \in \mathbb{N}^*$, we denote $n^{-1}(j) := \left(\tilde{i}(j), \tilde{k}(j)\right)$ with $n^{-1}$ the inverse of the pairing function $n$. Using the fact that $\sup_{y\in\mathbb{R}} |\xi_i(y)| = O\left(i^{-1/12}\right)$, we can check that $\forall t \in \mathbb{R}_+$, the condition (6.9) is verified. Indeed, for $q = 2$, we have

$$\sum_{j=1}^{+\infty}\left(\int_0^t \xi_{\tilde{i}(j)}(y)dy\right)^2 (2j)^{-q} < +\infty,$$

which means that the cylindrical Wiener process $W(t)$ belong to the Kondratiev space $(\mathcal{S})_{-0}(H)$. As we already know, it does not belong to $L^2(\mathcal{S}'(\mathbb{R}), \mu, H)$.

*Remark* 6.6. If the formal cylindrical Wiener process (6.11) can be viewed as a particular case of the formal cylindrical Wiener process (2.1) for the probability space $(\mathcal{S}'(\mathbb{R}),\mu)$, we can not extend the comparison, because the process (2.1) lives in a bigger space by the use of a Hilbert-Schmidt embedding. But the inclusion $L^2(\mathcal{S}'(\mathbb{R}),\mu,H)\subset(\mathcal{S})_{-0}(H)$ is not Hilbert-Schmidt (see [10] for the topology of $(\mathcal{S})_{-0}(H)$).

*Remark* 6.7. We have an example of cylindrical Wiener process defined in Definition 4.2 by Gawarecki and Mandrekar with the process

$$\tilde{W}(\omega,t,h)=\sum_{k=1}^{+\infty}<h,e_k>_H\ \beta_k(\omega,t)$$

in the particular case where $\omega\in(\mathcal{S}'(\mathbb{R}),\mu)$. The series converges in $L^2(\mathcal{S}'(\mathbb{R}),\mu)$.

*Remark* 6.8. From the formal process (6.11), we can create a $Q$-Wiener process defined on the probability space $(\mathcal{S}'(\mathbb{R}),\mu)$ as defined in Definition 2.2, by giving weights to each term of the series as we have already done in (2.2): by Proposition 2.2

$$(\omega,t)\mapsto\sum_{k=1}^{+\infty}\beta_k(\omega,t)\frac{1}{k}e_k$$

is a $Q$-Wiener process for the bounded linear operator $Q:H\to H$ defined $\forall k\in\mathbb{N}^*$ by $Q(e_k):=\dfrac{1}{k^2}e_k$.

### *6.3. The Wick product and its link with Itô integral*

The wick product gives a bridge between Itô integral against cylindrical Wiener process and cylindrical white noise. We follow the definitions and results of [10] and [24].

**Definition 6.5.** Let $G\in(\mathcal{S})_{-1}(H)$ and $F\in(\mathcal{S})_{-1}(L_2(H,\mathbb{R}))$. $H$ being endowed with the orthonormal basis $(e_k)_{k\in\mathbb{N}^*}$, the separable Hilbert space $L_2(H,\mathbb{R})$ is endowed with the orthonormal basis $(<e_k,.>_H)_{k\in\mathbb{N}^*}$. $F$ and $G$ can be written

$$F=\sum_{\alpha\in\mathcal{J}}\sum_{k=1}^{+\infty}c_{\alpha,k}\mathbf{H}_\alpha\otimes<e_k,.>_H,\quad G=\sum_{\gamma\in\mathcal{J}}\sum_{k=1}^{+\infty}d_{\gamma,k}\mathbf{H}_\gamma\otimes e_k,$$

with $c_{\alpha,k},d_{\gamma,k}\in\mathbb{R}$. The stochastic Wick product of $F$ and $G$ is defined by the formal expansion

$$F\diamond G:=\sum_{\delta\in\mathcal{J}}\sum_{k=1}^{+\infty}\left(\sum_{\alpha+\gamma=\delta}c_{\alpha,k}d_{\gamma,k}\right)\mathbf{H}_\delta. \tag{6.13}$$

In [10], Filinkov and Sorensen prove that $F\diamond G\in(\mathcal{S})_{-1}$.

**Definition 6.6.** The derivative with respect to the time variable of the formal cylindrical Wiener process (6.11) is the $H$-valued process

$$\dot{W}(\omega,t) := \sum_{k=1}^{+\infty}\sum_{i=1}^{+\infty} \xi_i(t)\ H_{\epsilon_{n(i,k)}}(\omega)\ e_k \tag{6.14}$$

which can also be written with the notations of (6.12) as

$$\dot{W}(\omega,t) = \sum_{j=1}^{+\infty} \xi_{\tilde{i}(j)}(t)\ H_{\epsilon_j} \otimes e_{\tilde{k}(j)}(\omega). \tag{6.15}$$

This process is called the cylindrical white noise.

In [10], Filinkov and Sorensen prove that $\forall t \in \mathbb{R}_+, \dot{W}(t) \in (\mathcal{S})_{-0}(H)$. They also prove that if $F \in (\mathcal{S})_{-0}(L_2(H,\mathbb{R}))$ then $\forall t \in \mathbb{R}_+, F \diamond \dot{W}(t) \in (\mathcal{S})_{-0}$.

We end by this beautiful formula taken from [24] (see their theorem 5.2), which is not given in its most general form, but in a sufficient form to understand how the theories of Itô integral and white noise come together.

**Theorem 6.1.** *Let $W(t)$ the $H$-valued cylindrical Wiener process defined by (6.12) on the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)$ endowed with a complete, right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ (where $[0,T] \subset \mathbb{R}_+$) for which the sequence of independent real Brownian motions (6.10) is adapted. Let $Y(t)$ a process belonging to $\Lambda_2(H,\mathbb{R})$ defined in Proposition 5.4 in the particular case $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu)$. Then, we have*

$$\int_0^T Y(t)dW(t) = \int_0^T Y(t) \diamond \dot{W}(t)dt$$

*where the integral of the right-hand side of the equality is a Lebesgue integral with respect to the time variable.*

*Remark* 6.9. $Y(t) \in \Lambda_2(H,\mathbb{R})$ implies that for almost every (fixed) $t \in [0,T]$, $Y(t) \in L^2(\mathcal{S}'(\mathbb{R}), L_2(H,\mathbb{R})) \subset (\mathcal{S})_{-0}(L_2(H,\mathbb{R}))$. That is why the Wick product of $Y(t)$ and $\dot{W}(t)$ exists for almost every $t \in [0,T]$.

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