# Hyperplanes in matroids ans the Axiom of Choice

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# HYPERPLANES IN MATROIDS AND THE AXIOM OF CHOICE

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Abstract. We show that in set-theory without the axiom of choice **ZF**, the statement **sH**: "Every proper closed subset of a finitary matroid is the intersection of hyperplanes including it" implies $\mathbf{AC^{fin}}$, the axiom of choice for (nonempty) finite sets. We also provide an equivalent of the statement $\mathbf{AC^{fin}}$ in terms of "graphic" matroids. Several open questions stay open in **ZF**, for example: does **sH** imply the Axiom of Choice?

## 1. Introduction

A *choice function* for a family $(A_i)_{i\in I}$ of nonempty sets is a family $(x_i)_{i\in I}$ such that for every $i \in I$, $x_i \in A_i$. The Axiom of Choice (**AC**) is the following statement: *"Every family of nonempty sets has a choice function."* We work in set theory without the axiom of choice **ZF**. We shall also consider the more general set theory **ZFA** (see [8, p. 44-45]), a modified version of set theory, in which "atoms" (*i.e.* nonempty objects which are not sets) are allowed. Consider the statement **VB** (Vector Basis): *"Every vector space has a basis"* (see [7, Note 75 p. 271]). It is known that in **ZFA**, **VB** implies the Multiple Choice axiom **MC** ([7, form 67]), and that in **ZF**, **MC** is equivalent to **AC**, but it is an open question to know whether **VB** imply **AC** in **ZFA**. In this paper, we discuss various statements about "finitary matroids" (which can been seen as generalisations of vector spaces, see Section 2.3.3) and their links with **AC**. We show that the statement *"Every finitary matroid has a basis"* is equivalent to **AC** in **ZFA** (see Proposition 5). We then consider the three following consequences of **AC** involving hyperplanes in finitary matroids, possibly satisfying the "binary elimination property" (see Section 3.2):

**sH**: *"Every proper flat in a finitary matroid is the intersection of hyperplanes including it."*

$\mathbf{sH}_{bep}$: *"Every proper flat in a finitary matroid with the binary elimination property is the intersection of hyperplanes including it."*

**H**: *"Every nonempty finitary matroid has an hyperplane."*

It is known that $\mathbf{AC} \Rightarrow \mathbf{sH}$ and of course $\mathbf{sH} \Rightarrow \mathbf{H}$ and $\mathbf{sH} \Rightarrow \mathbf{sH}_{bep}$. In this paper, we shall prove that $\mathbf{sH}_{bep}$ implies the following axiom of choice for finite sets:

$\mathbf{AC^{fin}}$: (form 62 of [7]) *Every nonempty family of finite nonempty sets has a choice function.*

It is known (see [7]) that $\mathbf{AC^{fin}}$ does not imply **AC** and that $\mathbf{AC^{fin}}$ is not provable in **ZF**. We do not know whether **H** implies **sH** or $\mathbf{sH}_{bep}$ or $\mathbf{AC^{fin}}$ nor do we know whether **H** or **sH** implies **AC** (see Figure 2 at the end of the paper). For every natural number $k \geq 2$ we consider the following consequence of $\mathbf{AC^{fin}}$:

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> $\mathbf{AC}^k$: *"For every nonempty family $(A_i)_{i\in I}$ of finite sets with $k$-elements, $\prod_{i\in I} A_i$ is nonempty."*

We also denote by $\forall k\mathbf{AC}^k$ the following statement, which is form 61 of [7]:

> *For every natural number $k \geq 2$, for every nonempty family $(A_i)_{i\in I}$ of finite sets with $k$-elements, $\prod_{i\in I} A_i$ is nonempty.*

In **ZF**, for every natural number $n \geq 2$, $\mathbf{AC} \Rightarrow \mathbf{AC^{fin}} \Rightarrow \forall k\mathbf{AC}^k \Rightarrow \mathbf{AC}^n$, and it is known (see [7]) that in **ZF**, none of these implications is reversible, and that $\mathbf{AC}^n$ is not provable.

Using the natural structure of finitary matroid over a vector space (see Example 1), **H** implies the following statement **D**: *"Given a commutative field $\mathbb{K}$ and a non null vector space $E$ over $\mathbb{K}$, there exists a non null linear form $f : E \to \mathbb{K}$"*. For every commutative field $\mathbb{K}$, we denote by $\mathbf{D}_{\mathbb{K}}$ the previous statement restricted to vector spaces over $\mathbb{K}$: *"For every non null $\mathbb{K}$-vector space $E$, the algebraic dual of $E$ is non null."* In [10, Corollary 2], we proved that for every prime number $p$, the statement $\mathbf{D}_{\mathbb{F}_p}$ (where $\mathbb{F}_p$ is the finite field $\mathbb{Z}/p\mathbb{Z}$) implies the statement $\mathbf{C}(p)$: *"For every family $(A_i)_{i\in I}$ of nonempty finite sets, there exists a family $(B_i)_{i\in I}$ such that for every $i \in I$, $B_i \subseteq A_i$ and $p$ does not divide the cardinal of $B_i$"*. Denoting by $\forall p\mathbf{C}(p)$ the statement $\forall p \in \mathbb{P}\ \mathbf{C}(p)$ where $\mathbb{P}$ is the set of prime natural numbers, then $\forall p\mathbf{C}(p)$ implies (and thus is equivalent to) the statement $\forall k\mathbf{AC}^k$ (see [10, Remarks 3 and 4]). It follows that $\mathbf{sH} \Rightarrow \mathbf{H} \Rightarrow \mathbf{D} \Rightarrow \forall k\mathbf{AC}^k$. However, we do not know whether **D** implies **H**. Notice that in **ZFA**, **D** does not imply $\mathbf{AC^{fin}}$, since the statement $\forall p\mathbf{MC}(p)$ (see [7, form 218]) implies the Ingleton statement **I** (the ultrametric counterpart of the Hahn-Banach statement, see [11]) which implies **D**, but $\forall p\mathbf{MC}(p)$ does not imply $\mathbf{AC^{fin}}$ (see Figure 2 at the end of the paper).

The paper is organized as follows. In Section 2 we review in set theory **ZF** some definitions and results about operators on finite or infinite sets in the sense of Higgs ([3]) and Klee ([9]): finitary operators, matroidal operators with particular emphasis on circuits and hyperplanes. We introduce the three notions of "circuit-accessibility", "hyperplane-accessibility" and "symmetric circuits". In Section 3, we formulate an equivalent of **AC** is terms of hyperplanes in a certain (non finitary) matroid, and we prove that the statement **sH** restricted to certain binary matroids implies $\mathbf{AC^{fin}}$. Finally, in the last section, we prove that $\mathbf{AC^{fin}}$ is equivalent to various statements about "graphic" matroids. We end with several questions about finitary matroids and **AC**.

## 2. Operators and the Axiom of choice

### 2.1. **Operators on a set.**

2.1.1. *Operators and their circuits.* An *operator* on a set $X$ (see [9, p. 138]) is a mapping $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$ which is *isotonic* (for every subsets $A$, $B$ of $X$, $(A \subseteq B \Rightarrow \phi(A) \subseteq \phi(B))$) and *enlarging* (for every subset $A$ of $X$, $A \subseteq \phi(A)$). Given an operator $\phi$ on a set $X$, a subset $D$ of $X$ is said to be *$\phi$-dependent* if there exists $x \in D$ such that $x \in \phi(D\backslash\{x\})$. A subset $I$ of $X$ is said to be *$\phi$-independent* if $I$ is not $\phi$-dependent *i.e.* if for every $x \in I$, $x \notin \phi(I\backslash\{x\})$. Minimal $\phi$-dependent subsets of $X$ are called *$\phi$-circuits*. A *loop* of the operator $\phi$ on $X$ is an element $x$ of $X$ such that $\{x\}$ is a $\phi$-circuit *i.e.* $\{x\}$ is $\phi$-dependent *i.e.* $x \in \phi(\varnothing)$. Two distinct elements $x, y$ of $X$ are *parallel* if $\{x, y\}$ is a $\phi$-circuit.

*Remark* 1. Given an operator $\phi$ on a set $X$:

(1) the collection $\mathcal{I}_\phi$ of $\phi$-independent subsets of $X$ contains $\varnothing$ and is *initial*: for all subsets $A$, $B$ of $X$, if $A \subseteq B$ and $B \in \mathcal{I}_\phi$, then $A \in \mathcal{I}_\phi$;
(2) the collection $\mathcal{D}_\phi$ of $\phi$-dependent subsets of $X$ does not contain $\varnothing$ and is *final*: for every subsets $A$, $B$ of $X$, if $A \subseteq B$ and $A \in \mathcal{D}_\phi$, then $B \in \mathcal{D}_\phi$.
(3) The collection $\mathcal{C}_\phi$ of $\phi$-circuits is an *antichain* of nonempty sets: no member of $\mathcal{C}_\phi$ includes another one.

2.1.2. *Finitary operators.* A operator $\phi$ on $X$ is said to be *finitary* if for every subset $Y$ of $X$ and every $x \in \phi(Y)$, there exists a finite subset $F$ of $Y$ satisfying $x \in \phi(F)$. If the operator $\phi$ is finitary, then every $\phi$-dependent set includes a (finite) $\phi$-circuit.

**Definition 1.** Given two *finitary* operators $\phi_1$ and $\phi_2$ on sets $X_1$ and $X_2$ and given a bijection $f : X_1 \to X_2$, the following statements are equivalent:

(1) for every subset $I$ of $X_1$, $I$ is $\phi_1$-independent if and only if $f[I]$ is $\phi_2$-independent
(2) for every subset $C$ of $X_1$, $C$ is a $\phi_1$-circuit if and only if $f[C]$ is a $\phi_2$-circuit.

Every bijection $f : X_1 \to X_2$ satisfying one of the two previous statements is called an *isomorphism* of finitary operators.

2.1.3. *Hyperplanes of an operator.* A subset $A$ of $X$ is said to be *$\phi$-spanning* if $\phi(A) = X$. Subsets of $X$ which are both $\phi$-independent and $\phi$-spanning are called *bases* of the operator $\phi$ (or $\phi$-bases). Maximal non-spanning subsets of $X$ are called *$\phi$-hyperplanes*. Subsets of $X$ which are fixed points of $\phi$ are called *flats* or *closed subsets* of the operator $\phi$.

*Remark* 2. Given an operator $\phi$ on a set $X$, for every nonempty family $(F_i)_{i\in I}$ of $\phi$-closed subsets of $X$, $\cap_{i\in I} F_i$ is $\phi$-closed, and thus, the *poset* $\mathcal{L}_\phi$ of $\phi$-closed subsets of $X$ endowed with the inclusion relation is a complete lattice (but it is not an induced sub-lattice of the lattice $(\mathcal{P}(X), \subseteq)$ in general).

## 2.2. **Minors of an operator.**

2.2.1. *Suboperators.* Given an operator $\phi$ on a set $X$, and a subset $Y$ of $X$, the mapping $\phi_Y : \mathcal{P}(Y) \to \mathcal{P}(Y)$ such that for every subset $Z$ of $Y$, $\phi_Y(Z) = \phi(Z) \cap Y$ is an operator on $Y$, called the *suboperator* induced by $\phi$ on $Y$, or *restriction operator* of $\phi$ to $Y$ (see [13, p. 263]). If the operator $\phi$ on $X$ is finitary, then the suboperator $\phi_Y$ is also finitary.

*Remark* 3. Given an operator $\phi$ on a set $X$, and a subset $Y$ of $X$, then:

(1) The $\phi_Y$-dependent subsets of $Y$ are the $\phi$-dependent sets that are included in $Y$;
(2) The $\phi_Y$-independent subsets of $Y$ are the $\phi$-independent sets that are included in $Y$.
(3) The $\phi_Y$-circuits are the $\phi$-circuits that are included in $Y$.

2.2.2. *Quotient operators.* Given an operator $\phi$ on a set $X$, and a subset $Y$ of $X$, the mapping $\phi^Y : \mathcal{P}(Y) \to \mathcal{P}(Y)$ associating to every subset $A$ of $Y$ the set $Y \cap \phi(A \cup (X \setminus Y))$ is an operator on $Y$. The operator $\phi^Y$ on $Y$ is called the *quotient operator* $\phi^Y$, or the *contraction* operator $\phi^Y$ (see [13, p. 263]). If the operator $\phi$ on $X$ is finitary, then the operator $\phi^Y$ is also finitary.

**Proposition 1.** *Given an operator $\phi$ on a set $X$ and a proper flat $F$ of $\phi$, then:*

*(1) $\phi$-flats including $F$ are subsets $F \cup Z$ where $Z$ is a flat of the quotient operator $\phi^{X\setminus F}$ on $X \setminus F$.*
*(2) $\phi$-hyperplanes including $F$ are subsets $F \cup Z$ where $Z$ is a hyperplane of the operator $\phi^{X\setminus F}$.*

*Proof.* (1) Given a subset $Z$ of $X\backslash F$, the following sentences are equivalent: $F \cup Z$ is a $\phi$-flat; $\phi(F \cup Z) \subseteq F \cup Z$; $\phi(F \cup Z)\backslash F \subseteq Z$; $\phi^{X\backslash F}(Z) \subseteq Z$; $Z$ is a $\phi^{X\backslash F}$-flat subset of $X\backslash F$.
(2) Given a subset $Z$ of $X\backslash F$, the following sentences are equivalent: $F \cup Z$ is a $\phi$-hyperplane; $(F \cup Z)$ is a proper $\phi$-flat but for every $x \in X\backslash(F \cup Z)$, $\phi((F \cup Z) \cup \{x\}) = X$; $Z$ is a proper $\phi^{X\backslash F}$-flat but for every $x \in X\backslash(F \cup Z)$, $\phi^{X\backslash F}(Z \cup \{x\}) = X\backslash F$; the subset $Z$ of $X\backslash F$ is a $\phi^{X\backslash F}$-hyperplane. □

*Remark* 4. Proposition 1 implies that given a class $\mathcal{O}$ of operators which is closed by quotient operators, if every $\phi \in \mathcal{O}$ has an hyperplane, then for every $\phi \in \mathcal{O}$, every proper flat of $\phi$ is included in a $\phi$-hyperplane.

**Definition 2.** Given an operator $\phi$ on a set $X$, a *minor* of an operator $\phi$ on a set $X$ is an operator $\psi$ on a subset $Y$ of $X$ such that there exists a sequence of operators $(\phi_i)_{0\leq i\leq n}$ such that $\phi_0 = \phi$, $\phi_n = \psi$ and for each $i \in \{1, \dots, n\}$, $\phi_i$ is a suboperator or a quotient operator of $\phi_{i-1}$.

### 2.3. **Finitary matroidal operators.**

2.3.1. *Idempotency properties.* A *closure operator* on $X$ is an operator $\phi$ on $X$ which is *idempotent* (see [9, p. 140]): for every subset $A$ of $X$, $\phi(\phi(A)) = \phi(A)$.

If the operator $\phi$ on $X$ is idempotent, then for every subset $Y$ of $X$, the operators $\phi_Y$ and $\phi^Y$ are also idempotent.

**Proposition 2.** *Given an idempotent operator $\phi$ on a set $X$, a subset $H$ of $X$ is a $\phi$-hyperplane iff $H$ is a maximal proper $\phi$-closed subset of $X$.*

*Proof.* Given an operator $\phi$ on a set $X$, for every $\phi$-hyperplane $H$, then either $\phi(H) = H$, and thus $H$ is a maximal proper $\phi$-closed subset of $X$, or $\phi(H)$ is spanning (else $H \subsetneq \phi(H) \subseteq \phi(\phi(H)) \subsetneq X$ and $H$ would not be a $\phi$-hyperplane since $\phi(H)$ would be a non spanning subset of $X$ strictly including $H$). It follows that if $\phi$ is idempotent, then every $\phi$-hyperplane is a maximal proper $\phi$-closed subset of $X$ (else, $\phi(H)$ would be spanning *i.e.* $X = \phi(\phi(H)) = \phi(H)$ by idempotency, and thus $H$ would be spanning). Reciprocally, if $H$ is a maximal proper $\phi$-closed subset of $X$, then for every $x \in X\backslash H$, $\phi(H \cup \{x\})$ is closed and thus $\phi(H \cup \{x\}) = X$ whence $H$ is a $\phi$-hyperplane. □

**Definition 3.** An operator $\phi$ on $X$ is *circuit-accessible* if for every subset $Y$ of $X$ and every $x \in \phi(Y)\backslash Y$, there exists a $\phi$-circuit $C$ such that $x \in C \subseteq Y \cup \{x\}$.

*Remark* 5. Every finitary idempotent operator is circuit-accessible.

*Proof.* Let $\phi$ be a finitary idempotent operator on a set $X$. Given some subset $A$ of $X$, and some $x \in \phi(A)\backslash A$, let $I$ be a minimal finite subset of $A$ such that $x \in \phi(I)$. Then $I$ is independent, else there exists $y \in I$ such that $y \in \phi(I\backslash\{y\})$, whence, denoting by $G$ the set $I\backslash\{y\}$, $x \in \phi(G \cup \{y\})$ and thus, by idempotency of $\phi$ and since $y \in \phi(G)$, $x \in \phi(G)$ which contradicts the minimality of $I$. Since $I \cup \{x\}$ is finite and dependent, there exists a $\phi$-circuit $C$ such that $C \subseteq I \cup \{x\}$. Since $I$ is independent, $x \in C$ and finally, $x \in C \subseteq A \cup \{x\}$. It follows that $\phi$ is circuit-accessible. □

2.3.2. *Exchange properties.* An operator $\phi$ on a set $X$ is said to satisfy the *exchange property* (see property (E) in [9, p. 140]) if for every subsets $Y, Z$ of $X$ and every $x \in X$, if $x \in \phi(Y \cup Z)$ and $x \notin \phi(Y)$, then there exists $y \in Z$ such that $y \in \phi(((Y \cup Z)\backslash\{y\}) \cup \{x\})$.

**Definition 4.** Given an operator $\phi$ on a set $X$, a $\phi$-circuit $C$ is *symmetric* if for every $x \in C$, $x \in \phi(C\backslash\{x\})$.

*Remark* 6. If an operator $\phi$ on a set $X$ satisfies the exchange property, then every $\phi$-circuit is symmetric.

2.3.3. *Matroidal operators.* We say that an operator $\phi$ on a set $X$ is *matroidal* if $\phi$ is idempotent and satisfies the exchange property.

*Example* 1 (The operator $\mathrm{span}_X$ associated to a vector space $X$). Given a vector space $X$ over a commutative field $\mathbb{K}$, the operator span on $X$, associating to every subset $Y$ of $X$ the vector subspace generated by $Y$ in $X$ is a finitary matroidal operator on $X$. The span-independent subsets of $X$ are the $\mathbb{K}$-linearly independent subsets of $X$; the span-bases of $X$ are the bases of the $\mathbb{K}$-vector space $X$; the span-flats are the vector subspaces of $X$, and the span-hyperplanes of $X$ are the kernels of non null linear forms $f : X \to \mathbb{K}$. The only loop of this operator is $\{0_X\}$.

*Example* 2 (The matroidal operator associated to a family of vectors). Given a $\mathbb{K}$-vector space $X$ and a mapping $f : I \to X$, the mapping $\phi : \mathcal{P}(I) \to \mathcal{P}(I)$ associating to every subset $J$ of $I$ the set $\{i \in I : f(i) \in \mathrm{span}(f[J])\}$ is a finitary matroidal operator. Loops of this operator are elements $i \in I$ such that $f(i) = 0_X$. Two elements $i, j$ of $I$ are parallel iff $i$, $j$ are not loops and if $f(i)$ and $f(j)$ are colinear.

Given a (commutative) field $\mathbb{F}$, a finitary matroidal operator $\phi$ on a set $X$ is said to be $\mathbb{F}$-*representable* if there exist a $\mathbb{K}$-vector space $E$ and a mapping $f : I \to E$ such that the matroidal operator $\phi$ is isomorphic with the finitary matroidal operator associated to $f$.

*Remark* 7. There are many equivalent definitions for the notion of *matroid* on a finite set (see [15, Chapter 1] or [16, Chapter 2]). Given an infinite set $X$, the notion of finitary matroidal operator on $X$ is equivalent to the notion of "transitive dependence relation" on $X$ (see for example [17, p. 97], [1, Prop. 2.1 p. 253], [15, Chapter 20.5], [2, p. 2]). In **ZFC**, finitary matroids have bases, but infinite matroids do not haves bases in general.

2.3.4. *Hyperplane-accessibility.*

**Definition 5.** An operator $\phi$ on a set $X$ is *hyperplane-accessible* if every proper flat of $\phi$ is the intersection of the set of the $\phi$-hyperplanes including it.

Given a commutative field $\mathbb{K}$, the statement $\mathbf{D}_{\mathbb{K}}$: *"Every non null vector space has a non null linear form."* is equivalent to the statement *"For every $\mathbb{K}$-vector space $E$, the finitary matroidal operator is hyperplane-accessible."*

## 2.4. **Finitary operators and the Axiom of choice.**

2.4.1. *Axiom of Choice and finitary operators.*

**Proposition 3** ( [14, p. 95] and [4]). **AC** *is equivalent to each of the following statements:*

1. *$AL'_3$:* [14, p. 95] *"For every finitary closure operator $\phi$ on a set $X$, for every collection $\mathcal{F}$ of subsets of $X$ which has finite character (*i.e. *for every subset $Z$ of $X$, $Z \in \mathcal{F}$ iff for every finite subset $Y$ of $Z$, $Y \in \mathcal{F}$), for every proper $\phi$-flat $F$ of $X$ such that $F \in \mathcal{F}$, then there exists a maximal $\phi$-flat $G$ such that $F \subseteq G$ and $G \in \mathcal{F}$."*

(2) $AL_3''$: *"For every finitary closure operator $\phi$ on a set $X$, for every proper $\phi$-flat $F$ of $X$ and every $x \in X \backslash F$, then there exists a maximal $\phi$-flat $G$ such that $F \subseteq G$ and $x \notin G$."*

(3) $K$ *(Krull): "Every proper ideal of commutative unitary ring has a maximal proper ideal."*

*It follows that* **AC** *implies the statement* **sH***:* "Every finitary matroid is hyperplane-accessible."

*Proof.* **AC** $\Rightarrow AL_3'$: The set $P := \{Z \in \mathcal{F} : F \subseteq Z \text{ and } \phi(Z) = Z\}$ endowed with the order induced by $\subseteq$ is inductive (for every chain $C$ of $P$, $\cup C \in P$) and thus, Zorn's lemma implies a maximal element $G$ of $P$. $AL_3' \Rightarrow AL_3''$: given a proper $\phi$-flat $F$ and $x \in X \backslash F$, the collection $\mathcal{F}$ of subsets of $X$ which do not contain $x$ has the finite character, and thus $AL_3'$ implies a maximal $\phi$-flat including $F$ and not containing $x$. $AL_3'' \Rightarrow K$: Given a proper ideal $I$ of a commutative unitary ring $A$, consider the closure operator $\phi$ on $A$ associating to each subset $Z$ of $A$ the ideal of $A$ generated by $Z$. Then $\phi$ is finitary, and thus $AL_3''$ implies a maximal $\phi$-closed subset $M$ of $A$ including $I$ such that $1 \notin M$. $K \Rightarrow$ **AC**: this implication is due to Hodges (see [4]).
In the conditions of statement $AL_3''$, if moreover $\phi$ satisfies the exchange property, then $G$ is a $\phi$-hyperplane, so the statement **sH** is the restriction of statement $AL_3''$ to finitary matroids. It follows that **AC** $\Rightarrow AL_3'' \Rightarrow$ **sH**. □

2.4.2. *Axiom of choice and finitary matroids.*

**Definition 6.** An operator $\phi$ on a set $X$ is said to satisfy the *interpolation property (for bases)* if for every $\phi$-independent subset $I$ of $X$ and every $\phi$-generating subset $G$ of $X$ such that $I \subseteq G$, there exists a $\phi$-basis $B$ such that $I \subseteq B \subseteq G$.

A *B-matroidal* operator on a set $X$ (see [3, p. 217], [13, p. 264]) is a matroidal operator $\phi$ on $X$ such that for every subset $Y$ of $X$, the suboperator $\phi_Y$ satisfies the interpolation property. Of course, every suboperator of a B-matroidal operator is B-matroidal.

**Proposition 4** ([3, p. 219])**.** *Every B-matroidal operator is hyperplane-accessible and circuit-accessible.*

*Proof.* Higgs defines a "C-matroid" as a matroidal operator which is both hyperplane-accessible and circuit-accessible. He proves that every B-matroid is a "C-matroid". □

**Proposition 5.** *(1)* **AC** *is equivalent to each of the following statements:*

$FB_0$*:* "Every finitary matroid satisfies the interpolation property"

$FB_1$*:* "Every finitary matroid is a $B$-matroid"

$FB_2$*:* "Every finitary matroid has a basis"

$FB_3$ *(form [1A] of* [7]*):* "Given a vector space $E$, every generating subset of $E$ includes a basis of $E$."

$FB_4$ "Every connected graph has a spanning tree."

*(2) The statement* **H***:* "Every nonempty finitary matroid has an hyperplane." *is equivalent to the statement "Every proper flat of a finitary matroid is included in a hyperplane."*

*Proof.* (1) **AC** $\Rightarrow FB_0$. Given a finitary matroidal operator $\phi$ on a set $X$, a $\phi$-independent subset $I$ of $X$ and a $\phi$-generating subset $G$ of $X$ such that $I \subseteq G$, consider the set $\mathcal{J}$ of $\phi$-independent subsets $J$ such that $I \subseteq J \subseteq G$. Then the *poset* $(\mathcal{J}, \subseteq)$ is inductive (every chain

$(J_t)_{t\in T}$ of this poset is dominated by $\bigcup_{t\in T} J_t$), so with Zorn's lemma, one gets a maximal element $B$ of the *poset* $(\mathcal{J}, \subseteq)$, and $B$ is a $\phi$-basis such that $I \subseteq B \subseteq G$. $FB_0 \Rightarrow FB_1$ follows from the previous point and the fact that every submatroid of a finitary matroid is finitary. $FB_1 \Rightarrow FB_2$ is trivial. $FB_2 \Rightarrow FB_3$: Consider a vector space $E$ and a generating subset $G$ of $E$. The operator $\phi$ induced by span on $G$ is finitary and matroidal, and thus $FB_2$ implies a $\phi$-basis, which is a basis of the vector space $E$ included in $G$. $FB_3 \Rightarrow FB_4$: See [6]. $FB_4 \Rightarrow$ **AC**: See [5].
(2) Given a finitary matroidal operator $\phi$ on a set $X$, and a proper flat $F$ of $\phi$, the statement **sH** applied to the finitary operator $\phi^F$ provides a hyperplane $Z$ of $\phi^F$, and then $F \cup Z$ is a $\phi$-hyperplane using Proposition 1. $\square$

## 3. Hyperplanes in matroids and the axiom of choice

### 3.1. The operator associated to an antichain of nonempty sets.

**Proposition 6.** *Every circuit-accessible operator $\phi$ on a set $X$ such that $\phi$-circuits are symmetric satisfies the exchange property.*

*Proof.* Assume that $Y$, $Z$ are two subsets of $X$ and that for some $x \in X$, $x \in \phi(Y \cup Z)$ but $x \notin \phi(Y)$. Since $\phi$ is circuit-accessible, let $C$ be a $\phi$-circuit such that $x \in C \subseteq (Y \cup Z) \cup \{x\}$. Since the circuit $C$ is symmetric, $x \in \phi(C \backslash \{x\})$, and thus $C\backslash\{x\}$ meets $Z$ (else $C\backslash\{x\} \subseteq Y$ so $\phi(C\backslash\{x\}) \subseteq \phi(Y)$ whence $x \in \phi(Y)$, which is contradictory!). Let $z \in (C\backslash\{x\}) \cap Z$; then, since the circuit $C$ is symmetric, $z \in \phi(C\backslash\{z\}) \subseteq \phi(((Y \cup Z) \cup \{x\})\backslash\{z\})$. $\square$

**Lemma 1.** *Given an antichain $\mathcal{C}$ of nonempty subsets of a set $X$, denote by $\phi$ the operator on $X$ associating to each subset $Y$ of $X$ the set $Y \cup B$ where $B$ is the set of elements $x \in X$ such that there exists $C \in \mathcal{C}$ satisfying $x \in C \subseteq Y \cup \{x\}$.*

*(1) $\phi$ is an operator on $X$.*
*(2) Each element of $\mathcal{C}$ is a symmetric $\phi$-circuit.*
*(3) $\mathcal{C}$ is the set of $\phi$-circuits, and the operator $\phi$ on $X$ is circuit-accessible.*
*(4) The operator $\phi$ satisfies the exchange property.*
*(5) If elements of $\mathcal{C}$ are finite sets, then the operator $\phi$ is finitary.*

*Proof.* (1) By definition of $\phi$, the mapping $\phi$ is expansive; moreover $\phi$ is isotonic since if $Y_1 \subseteq Y_2 \subseteq X$, for every $x \in X$ and every $C \in \mathcal{C}$ such that $x \in C \subseteq Y_1 \cup \{x\}$, then $x \in C \subseteq Y_2 \cup \{x\}$, thus $\phi(Y_1) \subseteq \phi(Y_2)$.
(2) If $C \in \mathcal{C}$, then, by definition of $\phi$, for every $x \in \mathcal{C}$, $x \in \phi(C\backslash\{x\})$, thus $C$ is $\phi$-dependent; moreover, the set $I := C\backslash\{x\}$ is $\phi$-independent, else let $y \in I$ such that $y \in \phi(I\backslash\{y\})$; then there would exist $C' \in \mathcal{C}$ such that $y \in C' \subseteq I \subsetneq C$ which is contradictory since $\mathcal{C}$ is an antichain.
(3) Let $C$ be a $\phi$-circuit. Then there exists $x \in C$ such that $x \in \phi(C\backslash\{x\})$. By definition of $\phi$, let $C' \in \mathcal{C}$ such that $x \in C' \subseteq (C\backslash\{x\}) \cup \{x\} = C$; using Point (2), $C'$ is a $\phi$-circuit, and since the set of $\phi$-circuits is an antichain, $C' = C$, and thus $C \in \mathcal{C}$. Since $\mathcal{C}$ is the set of $\phi$-circuits, it follows by definition of $\phi$ that the operator $\phi$ is circuit-accessible .
(4) This follows from Proposition 6 using the fact that $\phi$ is circuit-accessible and has symmetric circuits.
(5) Trivial since $\phi$ is circuit-accessible. $\square$

3.2. **Binary matroids.** A family $\mathcal{C}$ of subsets of a set $X$ is said to satisfy the *binary elimination property* if for all distinct elements $C_1$, $C_2$ of $\mathcal{C}$, the symmetric difference $C_1 \Delta C_2$ is a union of pairwise disjoint elements of $\mathcal{C}$.

**Theorem 1** ([12, Th 9.1.2 p . 344])**.** *Given a matroidal operator $\phi$ on a* finite *set $X$ and denoting by $\mathcal{C}$ the set of $\phi$-circuits, the following statements are equivalent:*

(1) *The operator $\phi$ is representable over the two-element field $\mathbb{F}_2$*
(2) *The symmetric difference of any set of circuits is either empty or contains a circuit*
(3) *$\mathcal{C}$ satisfies the binary elimination property*
(4) *For all distinct circuits $C_1$, $C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ is a (finite) union of circuits*
(5) *For all distinct circuits $C_1$, $C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ includes a circuit.*

The following corollary holds in **ZF** for *infinite* finitary matroids.

**Corollary 1.** *Given a finitary matroidal operator $\phi$ on a (non necessarily finite) set $X$ and denoting by $\mathcal{C}$ the set of $\phi$-circuits, the following statements are equivalent:*

(1) *$\phi$ is $\mathbb{F}_2$-representable*
(2) *Every finite submatroid of $\phi$ is $\mathbb{F}_2$-representable*
(3) *$\mathcal{C}$ satisfies the binary elimination property*
(4) *For all distinct $\phi$-circuits $C_1$, $C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ is a (finite) union of circuits*
(5) *For all distinct $\phi$-circuits $C_1$, $C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ includes a circuit.*
(6) *The symmetric difference of any set of $\phi$-circuits is either empty or contains a circuit.*

*Proof.* (1) $\Rightarrow$ (2) is easy and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are consequences of Theorem 1. We prove (6) $\Rightarrow$ (1). We consider the vector space ${\mathbb{F}_2}^{(X)}$ and its canonical basis $(e_x)_{x \in X}$ where for every $x \in X$, $e_x : X \to \mathbb{F}_2$ is the indicator function of the singleton $\{x\}$. Let $V$ be the vector subspace of ${\mathbb{F}_2}^{(X)}$ generated by the set $\{v_C := \sum_{x \in C} x : C\ \phi\text{-circuit}\}$. Let $Q$ be the quotient vector space ${\mathbb{F}_2}^{(X)}/V$ and let $f : X \to Q$ be the quotient mapping $x \mapsto e_x + V$. The (finitary) matroidal operator $\psi$ associated to $f$ is isomorphic with $\phi$ since $\phi$ and $\psi$ have the same circuits: given a subset $C$ of $X$, $C$ is a $\psi$-circuit iff $\sum_{x \in C}(e_x + V) = 0_Q$ and for every proper subset $I$ of $C$, $\sum_{x \in I}(e_x + V) \neq 0_Q$; equivalently, $\sum_{x \in C} e_x \in V$ and for every proper subset $I$ of $C$, $\sum_{x \in I} e_x \notin V$; this means that there exist $\phi$-circuits $C_1, \ldots, C_m$ such that $C = C_1 \Delta \ldots \Delta C_m$ and that no proper subset $I$ of $C$ is the symmetric difference of a nonempty sequence of $\phi$-circuits; using (2) it means that $C$ is a $\phi$-circuit. $\square$

**Definition 7.** A finitary matroid is said to be *binary* if it satisfies one of the previous equivalent statements.

3.3. **The matroidal operator associated to a a family of pairwise disjoint nonempty sets.**

**Definition 8.** Given an integer $n \geq 2$, a family $\mathcal{C}$ of subsets of a set $X$ is said to satisfy the *$n$-binary elimination property* if for all distinct elements $C_1$, $C_2$ of $\mathcal{C}$, the symmetric difference $C_1 \Delta C_2$ is a union of at most $n$ elements of $\mathcal{C}$.

**Theorem 2.** *Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, consider the set $X = \bigcup_{i \in I} A_i \cup \{O\}$ where $O$ is some set such that $O \notin \bigcup_{i \in I} A_i$. For every $i \in I$, let $C_i^1 := A_i \cup \{O\}$, and for all distinct elements $i, j \in I$, let $C_{i,j}^2 = A_i \cup A_j$. Let $\mathcal{C} := \{C_i^1 : i \in I\} \cup \{C_{i,j}^2 : i, j \in I; i \neq j\}$.*

*(1) $\mathcal{C}$ is an antichain of nonempty subsets of $X$*
*(2) $\mathcal{C}$ satisfies the 2-binary elimination property.*
*(3) Let $\phi$ be the operator associated to the antichain $\mathcal{C}$. Then $\phi$ is finitary iff for every $i \in I$, the set $A_i$ is finite.*
*(4) The operator $\phi$ is idempotent (and thus matroidal).*

*Proof.* Points (1), (2) and (3) are easy to check.
(4) Let $Z$ be a subset of $X$. Let $I_1$ be the set of elements $i \in I$ such that $A_i \backslash Z$ has at least two elements. Let $I_2 = I \backslash I_1$. If $O \in Z$ then $\phi(Z) = Z \cup \bigcup_{i \in I_2} A_i$ and thus, $\phi(\phi(Z)) = \phi(Z)$. If $O \notin Z$ and if there exists $i_0 \in I_2$ such that $A_{i_0} \subseteq Z$, then $\phi(Z) = Z \cup \{O\} \cup \bigcup_{i \in I_2} A_i$ and thus, $\phi(\phi(Z)) = \phi(Z)$; if $O \notin Z$ and if for every $i \in I_2$, $A_i \backslash Z$ has exactly one element, then $\phi(Z) = Z$ and thus $\phi(\phi(Z)) = \phi(Z)$. □

**Definition 9.** In the conditions of the previous theorem, we call $\phi$ the *matroidal operator associated to $O$ and the family $(A_i)_{i \in I}$.*

**Definition 10.** Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, a *selector* for this family is a subset $S$ of $\bigcup_{i \in I} A_i$ such that for every $i \in I$, $S \cap A_i$ has at most one element; the selector $S$ is said to be *total* if for every $i \in I$, $S \cap A_i$ has exactly one element.

**Theorem 3.** *Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, consider the set $X = \bigcup_{i \in I} A_i \cup \{O\}$ where $O$ is some set such that $O \notin \bigcup_{i \in I} A_i$. Let $\phi$ be the matroidal operator associated to $O$ and the family $(A_i)_{i \in I}$.*
*(1) A subset $L$ of $X$ is $\phi$-independent iff either ($O \in L$ and $\forall i \in I A_i \nsubseteq L$), or ($O \notin L$ and there exists at most one element $i_0 \in I$ such that $A_{i_0} \subseteq L$).*
*(2) A subset $G$ of $X$ is $\phi$-generating iff $S := (\bigcup_{i \in I} A_i) \backslash G$ is a selector for the family $(A_i)_{i \in I}$, which is not total if $O \notin G$.*
*(3) A subset $B$ of $X$ is a $\phi$-basis iff there exists a total selector $S$ for the family $(A_i)_{i \in I}$ such that $B = ((\bigcup_{i \in I} A_i) \backslash S) \cup \{a\}$ where $a$ is some element of $\{O\} \cup S$.*
*(4) A proper subset $F$ of $X$ is a $\phi$-flat iff ($O \in F$ or $\exists i_0 \in I A_{i_0} \subseteq F$) $\Rightarrow \forall i \in I$ $A_i \backslash F$ is not a singleton*
*(5) A subset $H$ of $X$ is a $\phi$-hyperplane iff $H = (\bigcup_{i \in I} A_i) \backslash S$ where $S$ is a total selector for the family $(A_i)_{i \in I}$, or $H = X \backslash \{x, y\}$ where $i_0 \in I$ and $x, y \in A_{i_0}$ with $x \neq y$.*
*(6) The following statements are equivalent:*
  *(a) The operator $\phi$ is hyperplane-accessible.*
  *(b) Every family $(B_i)_{i \in I}$ such that for every $i \in I$, $\varnothing \subsetneq B_i \subseteq A_i$ has a total selector.*
  *(c) The operator $\phi$ is B-matroidal.*
  *(d) The operator $\phi$ satisfies the interpolation property for bases*

*Proof.* Points (1), (2), (3), (4) and (5) are consequences of the definitions. We prove Point (6).
(a) $\Rightarrow$ (b): Given a family $(B_i)_{i \in I}$ such that for every $i \in I$, $\varnothing \subsetneq B_i \subseteq A_i$, consider the proper $\phi$-flat subset $F := \bigcup_{i \in I}(A_i \backslash B_i)$ of $X$; since $\phi$ is hyperplane-accessible, let $H$ be a $\phi$-hyperplane such that $F \subseteq H$ and $O \notin H$; then $\bigcup_{i \in I}(A_i \backslash H)$ is a total selector for the family $(B_i)_{i \in I}$.
(b) $\Rightarrow$ (c): Let $Y$ be a subset of $X$. Let $L$ be a $\phi$-independent subset of $Y$ and let $G$ be a $\phi_Y$-generating subset of $Y$ such that $L \subseteq G$. Let $J := \{i \in I : Y \cap A_i \neq \varnothing\}$. Let $J_1 := \{i \in J : A_i \nsubseteq G\}$. Let $J_2 = \{i \in J : A_i \subseteq G$ and $A_i \nsubseteq L\}$. Let $J_3 = \{i \in J : A_i \subseteq L\}$: notice that $J = J_1 \cup J_2 \cup J_3$ and that $J_1$, $J_2$ and $J_3$ are pairwise disjoint. For each $i \in J_1$,

let $x_i$ be the element of $A_i \backslash G$. Using (b), consider a choice function $(x_i)_{i \in J_2}$ for the family $(A_i \backslash L)_{i \in J_2}$. If $J_3$ is nonempty, then $J_3$ has a unique element $i_0$ and let $x_{i_0} = O$ if $O \in Y$. If $O \in Y$, let $B := Y \backslash \{x_i : i \in J\}$, and if $O \notin Y$, let $B := Y \backslash \{x_i : i \in J_1 \cup J_2\}$. Then $B$ is a $\phi_Y$-basis such that $L \subseteq B \subseteq G$.
(c) $\Rightarrow$ (d) follows from the definitions.
(d) $\Rightarrow$ (a): Let $F$ be a proper subset of $X$ which is a $\phi$-flat and let $x \in X \backslash F$. If $x = O$, then for every $i \in I$, $A_i \backslash F$ has at least one element (else $O$ would belong to $F$), and thus $F$ is $\phi$-independent; then $G = \bigcup_{i \in I} A_i$ is $\phi$-spanning and $F \subseteq G$: using the interpolation property, there exists a $\phi$-basis $B$ such that $F \subseteq B \subseteq G$; it follows that there exists a total selector $S$ for $(A_i)_i$ and en element $i_0 \in I$ such that $B = A_{i_0} \cup (\bigcup_{i \neq i_0} A_i) \backslash S$; let $x_{i_0} \in A_{i_0} \backslash F$; then $H = B \backslash \{x_{i_0}\}$ is a $\phi$-hyperplane including $F$ such that $O \notin H$. If $x \neq O$, then let $i_0$ be the element of $I$ such that $x \in A_{i_0}$. If $A_{i_0} \backslash F$ contains an element $y$ distinct from $x$, then $H := X \backslash \{x, y\}$ is a $\phi$-hyperplane including $F$ and not containing $x$. If $A_{i_0} \backslash F = \{x\}$, then for every $i \in I \backslash \{i_0\}$, $A_i \backslash F \neq \varnothing$ and $O \notin F$ (else $x$ would belong to $F$); using the independent set $L = F \backslash \{O\}$ and the generating set $G = \bigcup_i A_i$, consider a $\phi$-basis $B$ such that $L \subseteq B \subseteq G$; then $B$ yields a selector $S$ for the family $(A_i \backslash F)_{i \in I}$ (and thus $x \in S$). It follows that $H := (\bigcup_{i \in I} A_i) \backslash S$ is a $\phi$-hyperplane including $F$. □

**Corollary 2.** **AC** *is equivalent to the following statement:* "For every nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, and for every set $O$ such that $O \notin \bigcup_{i \in I} A_i$, the matroidal operator associated to $O$ and the family $(A_i)_{i \in I}$ has an hyperplane not containing $O$."

3.4. **The axiom sH implies $\mathbf{AC^{fin}}$.** We denote by $\mathbf{sH}_{bep}$ the axiom **sH** restricted to finitary matroids with the binary elimination property. For every natural number $n \geq 2$, we denote by $\mathbf{sH}_{bep_n}$ the axiom **sH** restricted to finitary matroids with the $n$-binary elimination property. We denote by $\mathbf{H}_{bep}$ (*resp.* $\mathbf{H}_{bep_n}$) the axiom **H** restricted to finitary matroids with the binary elimination property (*resp.* $n$-binary elimination property).

*Remark* 8. The matroidal operator associated to a family $(A_i)_{i \in I}$ of pairwise finite disjoint nonempty sets satisfies the 2-binary elimination property (and hence is binary).

**Theorem 4.** *In* **ZF**, $\mathbf{sH} \Rightarrow \mathbf{sH}_{bep} \Rightarrow \mathbf{sH}_{bep_2} \Rightarrow \mathbf{AC^{fin}}$.

*Proof.* Notice that $\mathbf{AC^{fin}}$ is equivalent to the statement "For every nonempty family $(A_i)_{i \in I}$ of *pairwise disjoint* finite nonempty sets, $\prod_{i \in I} A_i$ is nonempty.": given a family $(A_i)_{i \in I}$ of nonempy sets, consider the family $(A_i \times \{i\})_{i \in I}$. Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint finite nonempty sets, consider the set $X = \bigcup_{i \in I} A_i \cup \{O\}$ where $O \notin \bigcup_{i \in I} A_i$, and consider the finitary matroidal operator $\phi$ on $X$ associated to the family $(A_i)_{i \in I}$ (see Theorem 2). Since $\phi$ has no loops, $\phi(\varnothing) = \varnothing$, so $\varnothing$ is a proper flat of $\phi$ and thus, $\mathbf{sH}_{bep_2}$ implies a $\phi$-hyperplane $H$ not containing $O$. It follows from Theorem 3 that for each $i \in I$, $A_i \backslash H$ is a singleton $\{x_i\}$ where $(x_i)_{i \in I}$ is a choice function for the family $(A_i)_{i \in I}$. □

**Question 1.** We have shown that $\mathbf{AC} \Rightarrow \mathbf{sH}_1 \Rightarrow \mathbf{sH} \Rightarrow \mathbf{sH}_{bep} \Rightarrow \mathbf{sH}_{bep_2} \Rightarrow \mathbf{AC}^{fin}$ and of course $\mathbf{sH} \Rightarrow \mathbf{H} \Rightarrow \mathbf{H}_{bep} \Rightarrow \mathbf{H}_{bep_2}$. Does $\mathbf{sH}_{bep}$ imply **sH**? Does **H** imply $\mathbf{AC^{fin}}$? Does **H** imply **sH**?

## 4. Graphic matroids and the finite axiom of choice

### 4.1. **Strong and weak elimination properties.**

**Definition 11.** A family $\mathcal{C}$ of subsets of a set $X$ is said to satisfy the *elimination property* if for all distinct elements $C_1$, $C_2 \in \mathcal{C}$, for every $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2$ and $x \notin C_3$. The family $\mathcal{C}$ is said to satisfy the *strong elimination property* if for every elements $C_1$, $C_2 \in \mathcal{C}$, for every $x \in C_1 \cap C_2$ and every $y \in C_1 \backslash C_2$, then there exists $C_3 \in \mathcal{C}$ such that $y \in C_3 \subseteq C_1 \cup C_2$ and $x \notin C_3$.

Notice that the binary elimination property implies the strong elimination property, which in turn implies the elimination property.

**Notation 1.** For every finite set $F$, we denote by $|F|$ the cardinal of $F$.

The following result is classical:

**Proposition 7** ([15], [2])**.** *Let $\mathcal{C}$ be an antichain of nonempty finite subsets of a set $X$, and let $\phi$ be the (finitary) operator associated to $\mathcal{C}$. If $\mathcal{C}$ satisfies the weak elimination property, then:*

*(1) $\mathcal{C}$ satisfies the strong elimination property.*
*(2) The operator $\phi$ is a closure operator.*
*(3) The operator $\phi$ is matroidal.*

*Proof.* (1) See [15, Theorem 2 p. 24] or [2, Lemme 4 p. 17].
(2) See [2, Théorème 8 p. 18]. We sketch the proof. Let $A$ be a subset of $X$ and let $x \in \phi(\phi(A))$. Let us show that $x \in \phi(A)$. Let $C \in \mathcal{C}$ such that $x \in C \subseteq \phi(A) \cup \{x\}$, and such that $C \cap (\phi(A) \backslash A)$ is minimal. If $(C \backslash \{x\}) \cap (\phi(A) \backslash A)$ is nonempty, let $y \in (C \backslash \{x\}) \cap (\phi(A) \backslash A)$; since $y \in \phi(A)$, let $C_1 \in \mathcal{C}$ such that $y \in C_1 \subseteq A \cup \{y\}$. Using the strong elimination property, let $C_2 \in \mathcal{C}$ such that $x \in C_2 \subseteq (C \cup C_1) \backslash \{y\}$: then $|C_2 \cap (\phi(A) \backslash A)| < |C \cap (\phi(A) \backslash A)|$, which contradicts the minimality of $C \cap (\phi(A) \backslash A)$. It follows that $(C \backslash \{x\}) \cap (\phi(A) \backslash A) = \varnothing$ and thus, $(C \backslash \{x\}) \subseteq A$ so $x \in \phi(A)$.
(3) Using Lemma 1, $\mathcal{C}$ is the set of $\phi$-circuits and $\phi$ satisfies the exchange property, whence the closure operator $\phi$ is a matroidal operator on $X$. □

4.2. **The binary matroid associated to a multigraph.**

4.2.1. *Multigraphs.* A *multigraph* on a set $V$ is given by a mapping $f : X \to [V]^1 \cup [V]^2$, where, for each natural number $n \geq 1$, $[V]^n$ is the set of $n$-element subsets of $V$. Elements of $X$ such that $f(x) \in [V]^1$ are called *loops* of the multigraph.

Denoting by $(e_v)_{v \in V}$ the canonical basis of the vector space $\mathbb{F}_2^{(V)}$, the *incidence matrix* of the multigraph $f$ is the mapping $\tilde{f} : X \to \mathbb{F}_2^{(V)}$ such that for every $x \in X$, $\tilde{f}(x)$ is $0_{\mathbb{F}_2^{(V)}}$ if $f(x) \in [V]^1$, and $\tilde{f}(x) = e_{v_1} + e_{v_2}$ if $f(x)$ is the two-element sets $\{v_1, v_2\}$. The *matroid associated to the multigraph* $f$ is the (binary) matroidal operator on $X$ associated to the incidence matrix $\tilde{f}$. Loops of this matroid correspond to loops of the multigraph. A matroidal operator which is isomorphic with the (binary hence finitary) matroid associated to a multigraph is said to be *graphic*.

4.2.2. *Simple graphs.* A simple graph on a set $V$ is a binary relation $R$ on $V$ which is irreflexive (for every $x \in V$, $x \not R x$) and symmetric (for every $x, y \in V$, $xRy \Rightarrow yRx$). Elements of $V$ are called the *vertices* of the graph, and pairs $\{x, y\}$ of vertices such that $xRy$ are the edges of the simple graph. A simple graph on a set $V$ with set $E$ of edges is also denoted by $(V, E)$. A (partial) subgraph of a simple graph $G$ on a set $X$ with set of edges $E$

is a simple graph $(X', E')$ such that $X' \subseteq X$ and $E' \subseteq E$. Two graphs $(V_1, E_1)$ and $(V_2, E_2)$ are *isomorphic* when there exists a bijection $f : V_1 \to V_2$ which respects the edges.

**Notation 2.** Given some integer $n \geq 3$, we denote by $C_n$ the simple graph on $\mathbb{Z}/n\mathbb{Z} = \{0, \dots, n-1\}$ with set of edges $E_n = \{\{i, i +_n 1\} : i \in \mathbb{Z}/n\mathbb{Z}\}$, where $+_n$ is the additive law on $\mathbb{Z}/n\mathbb{Z}$;

Given some integer $n \geq 3$, a simple graph is a *n-cycle* if it is isomorphic with the simple graph $C_n$. Given a simple graph $G = (V, E)$, a *cycle of the graph* $G$ is a (partial) subgraph of $G$ which is isomorphic with a $n$-cycle for some natural number $n \geq 3$.

4.2.3. *Graphic matroids.* Given a set $V$ and a multigraph $f : X \to [V]^1 \cup [V]^2$, if $E = f[X] \cap [V]^2$, then $(V, E)$ is called the simple graph underlying the multigraph $f$. Reciprocally, every simple graph $(V, E)$ underlies the multigraph $\mathrm{id}_E : E \to E$ on $V$.

**Proposition 8** ([12, Proposition 1.1.7])**.** *Let $G = (V, E)$ be a simple graph. Let $\mathcal{C}_G$ be the set of (finite) subsets $F$ of $E$ such that $F$ is the set of edges of a cycle of $G$. Then $\mathcal{C}_G$ is the set of circuits of the (binary) matroidal operator $\mathcal{M}_G$ associated to the multigraph $\mathrm{id}_E : E \to E$.*

*Proof.* Let $W$ be the $\mathbb{F}_2$-vector space $\mathbb{F}_2^{(V)}$. For every $v \in V$ we denote by $e_v$ the $v$-th vector of the canonical basis of $W$. We identify each edge $\{a, b\}$ of $G$ with the vector $e_a + e_b$ of $W$. A subset $F$ of $E$ is a circuit of the matroid $\mathcal{M}_G$ iff $F$ is nonempty, $\sum_{e \in F} e = 0_W$ and for every nonempty proper subset $G$ of $F$, $\sum_{e \in G} e \neq 0_W$; replacing each element $e = \{a, b\}$ of $E$ by $e_a + e_b$, this means that $F \neq \varnothing$, every vertex of the subgraph $(\cup F, F)$ has even degree, but for every proper subset $G$ of $F$, some vertex of the subgraph $(\cup G, G)$ has an odd degree; this means that $F$ is a nonempty finite union of cycles of $G$, and that no proper subset of $F$ is a cycle of $G$; equivalently, $F$ is a cycle of the graph $G$. □

*Remark* 9. If $f : X \to [V]^1 \cup [V]^2$ is a multigraph on a set $V$, if $\mathcal{M}_f$ is the matroid associated to the multigraph $f$, loops of $\mathcal{M}_f$ are the singletons $\{x\}$ such that $x \in X$ and $f(x)$ is a singleton; circuits of cardinal two of $\mathcal{M}_f$ are the pairs $\{x, y\}$ of distinct elements of $X$ such that $f(x) = f(y)$. Given some natural number $n \geq 3$, then the $n$-circuits of $\mathcal{M}_f$ are the $n$-element subsets $\{x_1, \dots, x_n\}$ of $X$ such that $\{f(x_1), \dots, f(x_n)\}$ is the set of edges of a $n$-cycle of the underlying simple graph of $f$.

### 4.3. An equivalent of $\mathbf{AC^{fin}}$ in terms of graphic matroids.

**Theorem 5.** *The following statements are equivalent:*

*(1)* $\mathbf{AC^{fin}}$
*(2) For every family $(A_i)_{i \in I}$ of pairwise disjoint nonempty finite sets with at least two elements, the (binary hence finitary) matroid associated to this family is graphic.*

*Proof.* (1) ⇒ (2) Let $(A_i)_{i \in I}$ be an infinite family of pairwise disjoint nonempty finite sets, such that for every $i \in I$, $n_i := |A_i| \geq 2$. Let $\mathcal{M}$ be the matroid associated to the family $(A_i)_{i \in I}$: the underlying set of $\mathcal{M}$ is $M := \bigcup_{i \in I} A_i \cup \{O\}$ where $O \notin \bigcup_{i \in I} A_i$. We consider a family $(V_i)_{i \in I}$ of pairwise disjoint linearly ordered finite sets such that for each $i \in I$, $|V_i| = n_i - 1$. We also consider two distinct elements $a$ and $b$ not belonging to $\bigcup_{i \in I} V_i$, and we define the set $V := \{a, b\} \cup \bigcup_{i \in I} V_i$. Since each $V_i$ is linearly ordered, for each $i \in I$, we consider a graph $G_i$ on $V_i \cup \{a, b\}$ which is a $n_i$-cycle and such that $\{a, b\}$ is an edge of this graph: we denote by $E_i$ the set of edges of $G_i$ which are not equal to the edge $\{a, b\}$ of $G_i$.

We consider the simple graph $G$ on $V$ which admits $E := \bigcup_{i\in I} E_i \cup \{a, b\}$ as set of edges (see Figure 1). Notice that every finite subgraph of $G$ is planar. We denote by $\mathcal{G}$ the matroid on $E$ associated to the graph $G$. Using $\mathbf{AC^{fin}}$, we consider a family $(f_i)_{i\in I}$ such that for every $i \in I$, $f_i : E_i \to A_i$ is a bijection. It follows that $f := \bigcup_{i\in I} f_i$ is a bijection from $\bigcup_{i\in I} E_i$ to $\bigcup_{i\in I} A_i$ and we extend it into a bijection from $E$ to $M$. Then the bijection $f$ respects circuits of $\mathcal{M}_G$ and $\mathcal{M}$ thus t the matroid $\mathcal{M}$ is graphic.

(2) $\Rightarrow$ (1) Let $(A_i)_{i\in I}$ be a family of pairwise disjoint nonempty finite sets with at least two

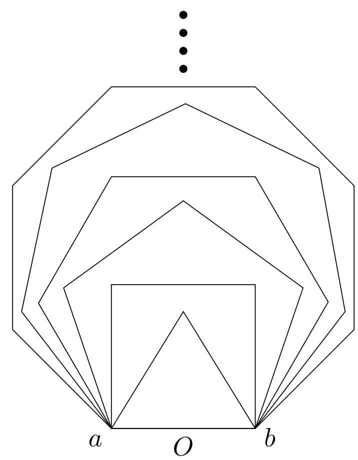


FIGURE 1. The graph $G$ associated to the matroid $\mathcal{M}$

elements. Let $\mathcal{M}$ be the finitary matroid on $\{O\} \cup \bigcup_{i\in I} A_i$ associated to this family. Let $G = (V, E)$ be a graph such that $\mathcal{M}$ is the graphic matroid associated to $G$. Let $a$, $b$ be the two extremities of the edge $O$ of $G$. Then, for every $i \in I$, $A_i \cup \{O\}$ is the set of edges of a cycle of the graph $G$: let $e_i$ be the the unique edge of $A_i$ which is incident to the vertex $a$. Then $(e_i)_{i\in I}$ is a choice function for the family $(A_i)_{i\in I}$. □

Consider the following well known consequences of $\mathbf{AC}$ imply $\mathbf{AC^{fin}}$:

$\mathbf{MG}_1$: "For every binary matroid $\mathcal{M}$, if every finite minor of $\mathcal{M}$ is graphic then $\mathcal{M}$ is graphic".

$\mathbf{MG}_2$: "For every binary matroid $\mathcal{M}$, if every finite submatroid of $\mathcal{M}$ is graphic and planar then $\mathcal{M}$ is graphic".

$\mathbf{MG}_3$: "For every binary matroid $\mathcal{M}$, if every finite minor of $\mathcal{M}$ is graphic and planar then $\mathcal{M}$ is graphic".

Notice that both statements $\mathbf{MG}_1$ and $\mathbf{MG}_2$ imply $\mathbf{MG}_3$. Moreover, every finite minor of the binary matroid used in the proof of Theorem 5 is graphic and planar, and thus, $\mathbf{MG}_3$ imply $\mathbf{AC^{fin}}$.

**Question 2.** Does $\mathbf{AC^{fin}}$ or $\mathbf{sH}$ imply one of the statements $\mathbf{MG}_1$, $\mathbf{MG}_2$ or $\mathbf{MG}_3$?

**Question 3.** Is the following statement provable in $\mathbf{ZF}$: *"Every (infinite) graphic matroid is hyperplane-accessible."*

In the diagram in Figure 2, we add the statement $\mathbf{D}_{\mathbb{Q}}$ which implies the statement $\mathbf{AC}^{\mathbb{Z}}$: *"Every family of posets isomorphic with the linear order $\mathbb{Z}$ has a nonempty product."* (see

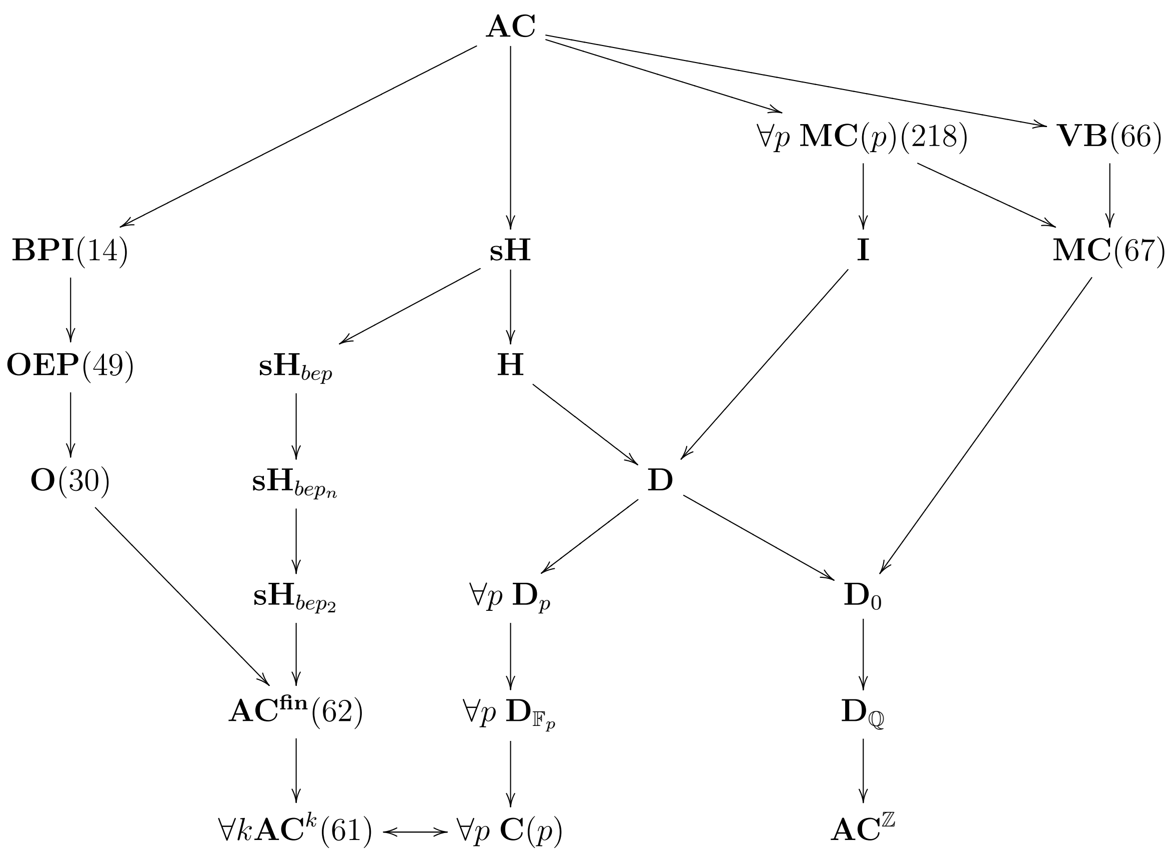


FIGURE 2. Summary diagram of the axioms

[10, Theorem 4]). We also add the statement $\mathbf{D}_0$ (*resp.* $\mathbf{D}_p$) which is $\mathbf{D}$ restricted to vector spaces over a commutative field $\mathbb{K}$ of characteristic 0 (*resp.* $p$).

**Question 4.** The statements **BPI** ("Every non trivial Boolean algebra has a maximal ideal"), **OEP** ("Every partial order on a set $X$ can be extended into a linear order on $X$") and **O** ("On every set $X$ there exists a linear order") (see forms 14, 49 and 30 of [7]) are well known consequences of **AC** which are stronger than $\mathbf{AC^{fin}}$. Are there implications between one of them and **H** or **sH** or $\mathbf{sH}_{bep}$ or $\mathbf{sH}_{bep_n}$ for some integer $n \geq 2$?

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