Hyperplanes in matroids ans the Axiom of Choice

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HYPERPLANES IN MATROIDS AND THE AXIOM OF CHOICE

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Abstract. We show that in set-theory without the axiom of choice **ZF**, the statement **sH**: "Every proper closed subset of a finitary matroid is the intersection of hyperplanes including it" implies $\mathbf{AC^{fin}}$, the axiom of choice for (nonempty) finite sets. We also provide an equivalent of the statement $\mathbf{AC^{fin}}$ in terms of "graphic" matroids. Several open questions stay open in **ZF**, for example: does **sH** imply the Axiom of Choice?

1. Introduction

A *choice function* for a family $(A_i)_{i\in I}$ of nonempty sets is a family $(x_i)_{i\in I}$ such that for every $i \in I$, $x_i \in A_i$. The Axiom of Choice (**AC**) is the following statement: *"Every family of nonempty sets has a choice function."* We work in set theory without the axiom of choice **ZF**. We shall also consider the more general set theory **ZFA** (see [8, p. 44-45]), a modified version of set theory, in which "atoms" (*i.e.* nonempty objects which are not sets) are allowed. Consider the statement **VB** (Vector Basis): *"Every vector space has a basis"* (see [7, Note 75 p. 271]). It is known that in **ZFA**, **VB** implies the Multiple Choice axiom **MC** ([7, form 67]), and that in **ZF**, **MC** is equivalent to **AC**, but it is an open question to know whether **VB** imply **AC** in **ZFA**. In this paper, we discuss various statements about "finitary matroids" (which can been seen as generalisations of vector spaces, see Section 2.3.3) and their links with **AC**. We show that the statement *"Every finitary matroid has a basis"* is equivalent to **AC** in **ZFA** (see Proposition 5). We then consider the three following consequences of **AC** involving hyperplanes in finitary matroids, possibly satisfying the "binary elimination property" (see Section 3.2):

sH: *"Every proper flat in a finitary matroid is the intersection of hyperplanes including it."*

$\mathbf{sH}_{bep}$: *"Every proper flat in a finitary matroid with the binary elimination property is the intersection of hyperplanes including it."*

H: *"Every nonempty finitary matroid has an hyperplane."*

It is known that $\mathbf{AC} \Rightarrow \mathbf{sH}$ and of course $\mathbf{sH} \Rightarrow \mathbf{H}$ and $\mathbf{sH} \Rightarrow \mathbf{sH}_{bep}$. In this paper, we shall prove that $\mathbf{sH}_{bep}$ implies the following axiom of choice for finite sets:

$\mathbf{AC^{fin}}$: (form 62 of [7]) *Every nonempty family of finite nonempty sets has a choice function.*

It is known (see [7]) that $\mathbf{AC^{fin}}$ does not imply **AC** and that $\mathbf{AC^{fin}}$ is not provable in **ZF**. We do not know whether **H** implies **sH** or $\mathbf{sH}_{bep}$ or $\mathbf{AC^{fin}}$ nor do we know whether **H** or **sH** implies **AC** (see Figure 2 at the end of the paper). For every natural number $k \geq 2$ we consider the following consequence of $\mathbf{AC^{fin}}$:

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> $\mathbf{AC}^k$: *"For every nonempty family $(A_i)_{i\in I}$ of finite sets with k-elements, $\prod_{i\in I} A_i$ is nonempty."*

We also denote by $\forall k\mathbf{AC}^k$ the following statement, which is form 61 of [7]:

> *For every natural number $k \geq 2$, for every nonempty family $(A_i)_{i\in I}$ of finite sets with k-elements, $\prod_{i\in I} A_i$ is nonempty.*

In **ZF**, for every natural number $n \geq 2$, $\mathbf{AC} \Rightarrow \mathbf{AC^{fin}} \Rightarrow \forall k\mathbf{AC}^k \Rightarrow \mathbf{AC}^n$, and it is known (see [7]) that in **ZF**, none of these implications is reversible, and that $\mathbf{AC}^n$ is not provable.

Using the natural structure of finitary matroid over a vector space (see Example 1), **H** implies the following statement **D**: *"Given a commutative field $\mathbb{K}$ and a non null vector space E over $\mathbb{K}$, there exists a non null linear form $f : E \to \mathbb{K}$"*. For every commutative field $\mathbb{K}$, we denote by $\mathbf{D}_{\mathbb{K}}$ the previous statement restricted to vector spaces over $\mathbb{K}$: *"For every non null $\mathbb{K}$-vector space E, the algebraic dual of E is non null."* In [10, Corollary 2], we proved that for every prime number p, the statement $\mathbf{D}_{\mathbb{F}_p}$ (where $\mathbb{F}_p$ is the finite field $\mathbb{Z}/p\mathbb{Z}$) implies the statement $\mathbf{C}(p)$: *"For every family $(A_i)_{i\in I}$ of nonempty finite sets, there exists a family $(B_i)_{i\in I}$ such that for every $i \in I$, $B_i \subseteq A_i$ and p does not divide the cardinal of B_i"*. Denoting by $\forall p\mathbf{C}(p)$ the statement $\forall p \in \mathbb{P}\ \mathbf{C}(p)$ where $\mathbb{P}$ is the set of prime natural numbers, then $\forall p\mathbf{C}(p)$ implies (and thus is equivalent to) the statement $\forall k\mathbf{AC}^k$ (see [10, Remarks 3 and 4]). It follows that $\mathbf{sH} \Rightarrow \mathbf{H} \Rightarrow \mathbf{D} \Rightarrow \forall k\mathbf{AC}^k$. However, we do not know whether **D** implies **H**. Notice that in **ZFA**, **D** does not imply $\mathbf{AC^{fin}}$, since the statement $\forall p\mathbf{MC}(p)$ (see [7, form 218]) implies the Ingleton statement **I** (the ultrametric counterpart of the Hahn-Banach statement, see [11]) which implies **D**, but $\forall p\mathbf{MC}(p)$ does not imply $\mathbf{AC^{fin}}$ (see Figure 2 at the end of the paper).

The paper is organized as follows. In Section 2 we review in set theory **ZF** some definitions and results about operators on finite or infinite sets in the sense of Higgs ([3]) and Klee ([9]): finitary operators, matroidal operators with particular emphasis on circuits and hyperplanes. We introduce the three notions of "circuit-accessibility", "hyperplane-accessibility" and "symmetric circuits". In Section 3, we formulate an equivalent of **AC** is terms of hyperplanes in a certain (non finitary) matroid, and we prove that the statement **sH** restricted to certain binary matroids implies $\mathbf{AC^{fin}}$. Finally, in the last section, we prove that $\mathbf{AC^{fin}}$ is equivalent to various statements about "graphic" matroids. We end with several questions about finitary matroids and **AC**.

2. Operators and the Axiom of choice

2.1. Operators on a set.

2.1.1. *Operators and their circuits.* An *operator* on a set X (see [9, p. 138]) is a mapping $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$ which is *isotonic* (for every subsets A, B of X, $(A \subseteq B \Rightarrow \phi(A) \subseteq \phi(B))$) and *enlarging* (for every subset A of X, $A \subseteq \phi(A)$). Given an operator ϕ on a set X, a subset D of X is said to be *ϕ-dependent* if there exists $x \in D$ such that $x \in \phi(D\backslash\{x\})$. A subset I of X is said to be *ϕ-independent* if I is not ϕ-dependent *i.e.* if for every $x \in I$, $x \notin \phi(I\backslash\{x\})$. Minimal ϕ-dependent subsets of X are called *ϕ-circuits.* A *loop* of the operator ϕ on X is an element x of X such that $\{x\}$ is a ϕ-circuit *i.e.* $\{x\}$ is ϕ-dependent *i.e.* $x \in \phi(\varnothing)$. Two distinct elements x, y of X are *parallel* if $\{x, y\}$ is a ϕ-circuit.

Remark 1. Given an operator ϕ on a set X:

(1) the collection $\mathcal{I}_\phi$ of ϕ-independent subsets of X contains $\varnothing$ and is *initial*: for all subsets A, B of X, if $A \subseteq B$ and $B \in \mathcal{I}_\phi$, then $A \in \mathcal{I}_\phi$;
(2) the collection $\mathcal{D}_\phi$ of ϕ-dependent subsets of X does not contain $\varnothing$ and is *final*: for every subsets A, B of X, if $A \subseteq B$ and $A \in \mathcal{D}_\phi$, then $B \in \mathcal{D}_\phi$.
(3) The collection $\mathcal{C}_\phi$ of ϕ-circuits is an *antichain* of nonempty sets: no member of $\mathcal{C}_\phi$ includes another one.

2.1.2. *Finitary operators.* A operator ϕ on X is said to be *finitary* if for every subset Y of X and every $x \in \phi(Y)$, there exists a finite subset F of Y satisfying $x \in \phi(F)$. If the operator ϕ is finitary, then every ϕ-dependent set includes a (finite) ϕ-circuit.

Definition 1. Given two *finitary* operators ϕ_1 and ϕ_2 on sets X_1 and X_2 and given a bijection $f : X_1 \to X_2$, the following statements are equivalent:

(1) for every subset I of X_1, I is ϕ_1-independent if and only if $f[I]$ is ϕ_2-independent
(2) for every subset C of X_1, C is a ϕ_1-circuit if and only if $f[C]$ is a ϕ_2-circuit.

Every bijection $f : X_1 \to X_2$ satisfying one of the two previous statements is called an *isomorphism* of finitary operators.

2.1.3. *Hyperplanes of an operator.* A subset A of X is said to be *ϕ-spanning* if $\phi(A) = X$. Subsets of X which are both ϕ-independent and ϕ-spanning are called *bases* of the operator ϕ (or ϕ-bases). Maximal non-spanning subsets of X are called *ϕ-hyperplanes*. Subsets of X which are fixed points of ϕ are called *flats* or *closed subsets* of the operator ϕ.

Remark 2. Given an operator ϕ on a set X, for every nonempty family $(F_i)_{i \in I}$ of ϕ-closed subsets of X, $\cap_{i \in I} F_i$ is ϕ-closed, and thus, the *poset* $\mathcal{L}_\phi$ of ϕ-closed subsets of X endowed with the inclusion relation is a complete lattice (but it is not an induced sub-lattice of the lattice $(\mathcal{P}(X), \subseteq)$ in general).

2.2. **Minors of an operator.**

2.2.1. *Suboperators.* Given an operator ϕ on a set X, and a subset Y of X, the mapping $\phi_Y : \mathcal{P}(Y) \to \mathcal{P}(Y)$ such that for every subset Z of Y, $\phi_Y(Z) = \phi(Z) \cap Y$ is an operator on Y, called the *suboperator* induced by ϕ on Y, or *restriction operator* of ϕ to Y (see [13, p. 263]). If the operator ϕ on X is finitary, then the suboperator ϕ_Y is also finitary.

Remark 3. Given an operator ϕ on a set X, and a subset Y of X, then:

(1) The ϕ_Y-dependent subsets of Y are the ϕ-dependent sets that are included in Y;
(2) The ϕ_Y-independent subsets of Y are the ϕ-independent sets that are included in Y.
(3) The ϕ_Y-circuits are the ϕ-circuits that are included in Y.

2.2.2. *Quotient operators.* Given an operator ϕ on a set X, and a subset Y of X, the mapping $\phi^Y : \mathcal{P}(Y) \to \mathcal{P}(Y)$ associating to every subset A of Y the set $Y \cap \phi(A \cup (X \setminus Y))$ is an operator on Y. The operator ϕ^Y on Y is called the *quotient operator* ϕ^Y, or the *contraction* operator ϕ^Y (see [13, p. 263]). If the operator ϕ on X is finitary, then the operator ϕ^Y is also finitary.

Proposition 1. *Given an operator ϕ on a set X and a proper flat F of ϕ, then:*

(1) ϕ-flats including F are subsets $F \cup Z$ where Z is a flat of the quotient operator $\phi^{X \setminus F}$ on $X \setminus F$.
(2) ϕ-hyperplanes including F are subsets $F \cup Z$ where Z is a hyperplane of the operator $\phi^{X \setminus F}$.

Proof. (1) Given a subset Z of $X\backslash F$, the following sentences are equivalent: $F\cup Z$ is a ϕ-flat; $\phi(F\cup Z)\subseteq F\cup Z$; $\phi(F\cup Z)\backslash F\subseteq Z$; $\phi^{X\backslash F}(Z)\subseteq Z$; Z is a $\phi^{X\backslash F}$-flat subset of $X\backslash F$.
(2) Given a subset Z of $X\backslash F$, the following sentences are equivalent: $F\cup Z$ is a ϕ-hyperplane; $(F\cup Z)$ is a proper ϕ-flat but for every $x\in X\backslash(F\cup Z)$, $\phi((F\cup Z)\cup\{x\})=X$; Z is a proper $\phi^{X\backslash F}$-flat but for every $x\in X\backslash(F\cup Z)$, $\phi^{X\backslash F}(Z\cup\{x\})=X\backslash F$; the subset Z of $X\backslash F$ is a $\phi^{X\backslash F}$-hyperplane. □

Remark 4. Proposition 1 implies that given a class $\mathcal{O}$ of operators which is closed by quotient operators, if every $\phi\in\mathcal{O}$ has an hyperplane, then for every $\phi\in\mathcal{O}$, every proper flat of ϕ is included in a ϕ-hyperplane.

Definition 2. Given an operator ϕ on a set X, a *minor* of an operator ϕ on a set X is an operator ψ on a subset Y of X such that there exists a sequence of operators $(\phi_i)_{0\leq i\leq n}$ such that $\phi_0=\phi$, $\phi_n=\psi$ and for each $i\in\{1,\ldots,n\}$, ϕ_i is a suboperator or a quotient operator of ϕ_{i-1}.

2.3. **Finitary matroidal operators.**

2.3.1. *Idempotency properties.* A *closure operator* on X is an operator ϕ on X which is *idempotent* (see [9, p. 140]): for every subset A of X, $\phi(\phi(A))=\phi(A)$.

If the operator ϕ on X is idempotent, then for every subset Y of X, the operators ϕ_Y and ϕ^Y are also idempotent.

Proposition 2. *Given an idempotent operator ϕ on a set X, a subset H of X is a ϕ-hyperplane iff H is a maximal proper ϕ-closed subset of X.*

Proof. Given an operator ϕ on a set X, for every ϕ-hyperplane H, then either $\phi(H)=H$, and thus H is a maximal proper ϕ-closed subset of X, or $\phi(H)$ is spanning (else $H\subsetneq\phi(H)\subseteq\phi(\phi(H))\subsetneq X$ and H would not be a ϕ-hyperplane since $\phi(H)$ would be a non spanning subset of X strictly including H). It follows that if ϕ is idempotent, then every ϕ-hyperplane is a maximal proper ϕ-closed subset of X (else, $\phi(H)$ would be spanning *i.e.* $X=\phi(\phi(H))=\phi(H)$ by idempotency, and thus H would be spanning). Reciprocally, if H is a maximal proper ϕ-closed subset of X, then for every $x\in X\backslash H$, $\phi(H\cup\{x\})$ is closed and thus $\phi(H\cup\{x\})=X$ whence H is a ϕ-hyperplane. □

Definition 3. An operator ϕ on X is *circuit-accessible* if for every subset Y of X and every $x\in\phi(Y)\backslash Y$, there exists a ϕ-circuit C such that $x\in C\subseteq Y\cup\{x\}$.

Remark 5. Every finitary idempotent operator is circuit-accessible.

Proof. Let ϕ be a finitary idempotent operator on a set X. Given some subset A of X, and some $x\in\phi(A)\backslash A$, let I be a minimal finite subset of A such that $x\in\phi(I)$. Then I is independent, else there exists $y\in I$ such that $y\in\phi(I\backslash\{y\})$, whence, denoting by G the set $I\backslash\{y\}$, $x\in\phi(G\cup\{y\})$ and thus, by idempotency of ϕ and since $y\in\phi(G)$, $x\in\phi(G)$ which contradicts the minimality of I. Since $I\cup\{x\}$ is finite and dependent, there exists a ϕ-circuit C such that $C\subseteq I\cup\{x\}$. Since I is independent, $x\in C$ and finally, $x\in C\subseteq A\cup\{x\}$. It follows that ϕ is circuit-accessible. □

2.3.2. *Exchange properties.* An operator ϕ on a set X is said to satisfy the *exchange property* (see property (E) in [9, p. 140]) if for every subsets Y, Z of X and every $x\in X$, if $x\in\phi(Y\cup Z)$ and $x\notin\phi(Y)$, then there exists $y\in Z$ such that $y\in\phi(((Y\cup Z)\backslash\{y\})\cup\{x\})$.

Definition 4. Given an operator ϕ on a set X, a ϕ-circuit C is *symmetric* if for every $x \in C$, $x \in \phi(C\backslash\{x\})$.

Remark 6. If an operator ϕ on a set X satisfies the exchange property, then every ϕ-circuit is symmetric.

2.3.3. *Matroidal operators.* We say that an operator ϕ on a set X is *matroidal* if ϕ is idempotent and satisfies the exchange property.

Example 1 (The operator span_X associated to a vector space X). Given a vector space X over a commutative field $\mathbb{K}$, the operator span on X, associating to every subset Y of X the vector subspace generated by Y in X is a finitary matroidal operator on X. The span-independent subsets of X are the $\mathbb{K}$-linearly independent subsets of X; the span-bases of X are the bases of the $\mathbb{K}$-vector space X; the span-flats are the vector subspaces of X, and the span-hyperplanes of X are the kernels of non null linear forms $f : X \to \mathbb{K}$. The only loop of this operator is $\{0_X\}$.

Example 2 (The matroidal operator associated to a family of vectors). Given a $\mathbb{K}$-vector space X and a mapping $f : I \to X$, the mapping $\phi : \mathcal{P}(I) \to \mathcal{P}(I)$ associating to every subset J of I the set $\{i \in I : f(i) \in \text{span}(f[J])\}$ is a finitary matroidal operator. Loops of this operator are elements $i \in I$ such that $f(i) = 0_X$. Two elements i, j of I are parallel iff i, j are not loops and if $f(i)$ and $f(j)$ are colinear.

Given a (commutative) field $\mathbb{F}$, a finitary matroidal operator ϕ on a set X is said to be $\mathbb{F}$-*representable* if there exist a $\mathbb{K}$-vector space E and a mapping $f : I \to E$ such that the matroidal operator ϕ is isomorphic with the finitary matroidal operator associated to f.

Remark 7. There are many equivalent definitions for the notion of *matroid* on a finite set (see [15, Chapter 1] or [16, Chapter 2]). Given an infinite set X, the notion of finitary matroidal operator on X is equivalent to the notion of "transitive dependence relation" on X (see for example [17, p. 97], [1, Prop. 2.1 p. 253], [15, Chapter 20.5], [2, p. 2]). In **ZFC**, finitary matroids have bases, but infinite matroids do not haves bases in general.

2.3.4. *Hyperplane-accessibility.*

Definition 5. An operator ϕ on a set X is *hyperplane-accessible* if every proper flat of ϕ is the intersection of the set of the ϕ-hyperplanes including it.

Given a commutative field $\mathbb{K}$, the statement $\mathbf{D}_{\mathbb{K}}$: *"Every non null vector space has a non null linear form."* is equivalent to the statement *"For every $\mathbb{K}$-vector space E, the finitary matroidal operator is hyperplane-accessible."*

2.4. **Finitary operators and the Axiom of choice.**

2.4.1. *Axiom of Choice and finitary operators.*

Proposition 3 ([14, p. 95] and [4]). **AC** *is equivalent to each of the following statements:*

(1) AL_3'*:* [14, p. 95] *"For every finitary closure operator ϕ on a set X, for every collection $\mathcal{F}$ of subsets of X which has finite character (*i.e. *for every subset Z of X, $Z \in \mathcal{F}$ iff for every finite subset Y of Z, $Y \in \mathcal{F}$), for every proper ϕ-flat F of X such that $F \in \mathcal{F}$, then there exists a maximal ϕ-flat G such that $F \subseteq G$ and $G \in \mathcal{F}$."*

(2) AL_3'': *"For every finitary closure operator ϕ on a set X, for every proper ϕ-flat F of X and every $x \in X \backslash F$, then there exists a maximal ϕ-flat G such that $F \subseteq G$ and $x \notin G$."*
(3) K *(Krull): "Every proper ideal of commutative unitary ring has a maximal proper ideal."*

It follows that **AC** *implies the statement* **sH***:* "Every finitary matroid is hyperplane-accessible."

Proof. **AC** $\Rightarrow AL_3'$: The set $P := \{Z \in \mathcal{F} : F \subseteq Z$ and $\phi(Z) = Z\}$ endowed with the order induced by $\subseteq$ is inductive (for every chain C of P, $\cup C \in P$) and thus, Zorn's lemma implies a maximal element G of P. $AL_3' \Rightarrow AL_3''$: given a proper ϕ-flat F and $x \in X \backslash F$, the collection $\mathcal{F}$ of subsets of X which do not contain x has the finite character, and thus AL_3' implies a maximal ϕ-flat including F and not containing x. $AL_3'' \Rightarrow K$: Given a proper ideal I of a commutative unitary ring A, consider the closure operator ϕ on A associating to each subset Z of A the ideal of A generated by Z. Then ϕ is finitary, and thus AL_3'' implies a maximal ϕ-closed subset M of A including I such that $1 \notin M$. $K \Rightarrow$ **AC**: this implication is due to Hodges (see [4]).
In the conditions of statement AL_3'', if moreover ϕ satisfies the exchange property, then G is a ϕ-hyperplane, so the statement **sH** is the restriction of statement AL_3'' to finitary matroids. It follows that **AC** $\Rightarrow AL_3'' \Rightarrow$ **sH**. □

2.4.2. *Axiom of choice and finitary matroids.*

Definition 6. An operator ϕ on a set X is said to satisfy the *interpolation property (for bases)* if for every ϕ-independent subset I of X and every ϕ-generating subset G of X such that $I \subseteq G$, there exists a ϕ-basis B such that $I \subseteq B \subseteq G$.

A *B-matroidal* operator on a set X (see [3, p. 217], [13, p. 264]) is a matroidal operator ϕ on X such that for every subset Y of X, the suboperator ϕ_Y satisfies the interpolation property. Of course, every suboperator of a B-matroidal operator is B-matroidal.

Proposition 4 ([3, p. 219])**.** *Every B-matroidal operator is hyperplane-accessible and circuit-accessible.*

Proof. Higgs defines a "C-matroid" as a matroidal operator which is both hyperplane-accessible and circuit-accessible. He proves that every B-matroid is a "C-matroid". □

Proposition 5. (1) **AC** *is equivalent to each of the following statements:*

FB_0*:* "Every finitary matroid satisfies the interpolation property"

FB_1*:* "Every finitary matroid is a B-matroid"

FB_2*:* "Every finitary matroid has a basis"

FB_3 *(form [1A] of* [7]*):* "Given a vector space E, every generating subset of E includes a basis of E."

FB_4 "Every connected graph has a spanning tree."

(2) *The statement* **H***:* "Every nonempty finitary matroid has an hyperplane." *is equivalent to the statement "Every proper flat of a finitary matroid is included in a hyperplane."*

Proof. (1) **AC** $\Rightarrow FB_0$. Given a finitary matroidal operator ϕ on a set X, a ϕ-independent subset I of X and a ϕ-generating subset G of X such that $I \subseteq G$, consider the set $\mathcal{J}$ of ϕ-independent subsets J such that $I \subseteq J \subseteq G$. Then the *poset* $(\mathcal{J}, \subseteq)$ is inductive (every chain

$(J_t)_{t\in T}$ of this poset is dominated by $\bigcup_{t\in T} J_t$), so with Zorn's lemma, one gets a maximal element B of the *poset* $(\mathcal{J}, \subseteq)$, and B is a ϕ-basis such that $I \subseteq B \subseteq G$. $FB_0 \Rightarrow FB_1$ follows from the previous point and the fact that every submatroid of a finitary matroid is finitary. $FB_1 \Rightarrow FB_2$ is trivial. $FB_2 \Rightarrow FB_3$: Consider a vector space E and a generating subset G of E. The operator ϕ induced by span on G is finitary and matroidal, and thus FB_2 implies a ϕ-basis, which is a basis of the vector space E included in G. $FB_3 \Rightarrow FB_4$: See [6]. $FB_4 \Rightarrow$ **AC**: See [5].
(2) Given a finitary matroidal operator ϕ on a set X, and a proper flat F of ϕ, the statement **sH** applied to the finitary operator ϕ^F provides a hyperplane Z of ϕ^F, and then $F \cup Z$ is a ϕ-hyperplane using Proposition 1. □

3. Hyperplanes in matroids and the axiom of choice

3.1. The operator associated to an antichain of nonempty sets.

Proposition 6. *Every circuit-accessible operator ϕ on a set X such that ϕ-circuits are symmetric satisfies the exchange property.*

Proof. Assume that Y, Z are two subsets of X and that for some $x \in X$, $x \in \phi(Y \cup Z)$ but $x \notin \phi(Y)$. Since ϕ is circuit-accessible, let C be a ϕ-circuit such that $x \in C \subseteq (Y \cup Z) \cup \{x\}$. Since the circuit C is symmetric, $x \in \phi(C\backslash\{x\})$, and thus $C\backslash\{x\}$ meets Z (else $C\backslash\{x\} \subseteq Y$ so $\phi(C\backslash\{x\}) \subseteq \phi(Y)$ whence $x \in \phi(Y)$, which is contradictory!). Let $z \in (C\backslash\{x\}) \cap Z$; then, since the circuit C is symmetric, $z \in \phi(C\backslash\{z\}) \subseteq \phi(((Y \cup Z) \cup \{x\})\backslash\{z\})$. □

Lemma 1. *Given an antichain $\mathcal{C}$ of nonempty subsets of a set X, denote by ϕ the operator on X associating to each subset Y of X the set $Y \cup B$ where B is the set of elements $x \in X$ such that there exists $C \in \mathcal{C}$ satisfying $x \in C \subseteq Y \cup \{x\}$.*

(1) ϕ is an operator on X.
(2) Each element of $\mathcal{C}$ is a symmetric ϕ-circuit.
(3) $\mathcal{C}$ is the set of ϕ-circuits, and the operator ϕ on X is circuit-accessible.
(4) The operator ϕ satisfies the exchange property.
(5) If elements of $\mathcal{C}$ are finite sets, then the operator ϕ is finitary.

Proof. (1) By definition of ϕ, the mapping ϕ is expansive; moreover ϕ is isotonic since if $Y_1 \subseteq Y_2 \subseteq X$, for every $x \in X$ and every $C \in \mathcal{C}$ such that $x \in C \subseteq Y_1 \cup \{x\}$, then $x \in C \subseteq Y_2 \cup \{x\}$, thus $\phi(Y_1) \subseteq \phi(Y_2)$.
(2) If $C \in \mathcal{C}$, then, by definition of ϕ, for every $x \in \mathcal{C}$, $x \in \phi(C\backslash\{x\})$, thus C is ϕ-dependent; moreover, the set $I := C\backslash\{x\}$ is ϕ-independent, else let $y \in I$ such that $y \in \phi(I\backslash\{y\})$; then there would exist $C' \in \mathcal{C}$ such that $y \in C' \subseteq I \subsetneq C$ which is contradictory since $\mathcal{C}$ is an antichain.
(3) Let C be a ϕ-circuit. Then there exists $x \in C$ such that $x \in \phi(C\backslash\{x\})$. By definition of ϕ, let $C' \in \mathcal{C}$ such that $x \in C' \subseteq (C\backslash\{x\}) \cup \{x\} = C$; using Point (2), C' is a ϕ-circuit, and since the set of ϕ-circuits is an antichain, $C' = C$, and thus $C \in \mathcal{C}$. Since $\mathcal{C}$ is the set of ϕ-circuits, it follows by definition of ϕ that the operator ϕ is circuit-accessible .
(4) This follows from Proposition 6 using the fact that ϕ is circuit-accessible and has symmetric circuits.
(5) Trivial since ϕ is circuit-accessible. □

3.2. **Binary matroids.** A family $\mathcal{C}$ of subsets of a set X is said to satisfy the *binary elimination property* if for all distinct elements C_1, C_2 of $\mathcal{C}$, the symmetric difference $C_1 \Delta C_2$ is a union of pairwise disjoint elements of $\mathcal{C}$.

Theorem 1 ([12, Th 9.1.2 p . 344])**.** *Given a matroidal operator ϕ on a* finite *set X and denoting by $\mathcal{C}$ the set of ϕ-circuits, the following statements are equivalent:*

(1) The operator ϕ is representable over the two-element field $\mathbb{F}_2$
(2) The symmetric difference of any set of circuits is either empty or contains a circuit
(3) $\mathcal{C}$ satisfies the binary elimination property
(4) For all distinct circuits C_1, $C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ is a (finite) union of circuits
(5) For all distinct circuits C_1, $C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ includes a circuit.

The following corollary holds in **ZF** for *infinite* finitary matroids.

Corollary 1. *Given a finitary matroidal operator ϕ on a (non necessarily finite) set X and denoting by $\mathcal{C}$ the set of ϕ-circuits, the following statements are equivalent:*

(1) ϕ is $\mathbb{F}_2$-representable
(2) Every finite submatroid of ϕ is $\mathbb{F}_2$-representable
(3) $\mathcal{C}$ satisfies the binary elimination property
(4) For all distinct ϕ-circuits C_1, $C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ is a (finite) union of circuits
(5) For all distinct ϕ-circuits C_1, $C_2 \in \mathcal{C}$, $C_1 \Delta C_2$ includes a circuit.
(6) The symmetric difference of any set of ϕ-circuits is either empty or contains a circuit.

Proof. (1) $\Rightarrow$ (2) is easy and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are consequences of Theorem 1. We prove (6) $\Rightarrow$ (1). We consider the vector space ${\mathbb{F}_2}^{(X)}$ and its canonical basis $(e_x)_{x \in X}$ where for every $x \in X$, $e_x : X \to \mathbb{F}_2$ is the indicator function of the singleton $\{x\}$. Let V be the vector subspace of ${\mathbb{F}_2}^{(X)}$ generated by the set $\{v_C := \sum_{x \in C} x : C\ \phi\text{-circuit}\}$. Let Q be the quotient vector space ${\mathbb{F}_2}^{(X)}/V$ and let $f : X \to Q$ be the quotient mapping $x \mapsto e_x + V$. The (finitary) matroidal operator ψ associated to f is isomorphic with ϕ since ϕ and ψ have the same circuits: given a subset C of X, C is a ψ-circuit iff $\sum_{x \in C}(e_x + V) = 0_Q$ and for every proper subset I of C, $\sum_{x \in I}(e_x + V) \neq 0_Q$; equivalently, $\sum_{x \in C} e_x \in V$ and for every proper subset I of C, $\sum_{x \in I} e_x \notin V$; this means that there exist ϕ-circuits $C_1, \dots, C_m$ such that $C = C_1 \Delta \dots \Delta C_m$ and that no proper subset I of C is the symmetric difference of a nonempty sequence of ϕ-circuits; using (2) it means that C is a ϕ-circuit. $\square$

Definition 7. A finitary matroid is said to be *binary* if it satisfies one of the previous equivalent statements.

3.3. **The matroidal operator associated to a a family of pairwise disjoint nonempty sets.**

Definition 8. Given an integer $n \geq 2$, a family $\mathcal{C}$ of subsets of a set X is said to satisfy the *n-binary elimination property* if for all distinct elements C_1, C_2 of $\mathcal{C}$, the symmetric difference $C_1 \Delta C_2$ is a union of at most n elements of $\mathcal{C}$.

Theorem 2. *Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, consider the set $X = \bigcup_{i \in I} A_i \cup \{O\}$ where O is some set such that $O \notin \bigcup_{i \in I} A_i$. For every $i \in I$, let $C_i^1 := A_i \cup \{O\}$, and for all distinct elements $i, j \in I$, let $C_{i,j}^2 = A_i \cup A_j$. Let $\mathcal{C} := \{C_i^1 : i \in I\} \cup \{C_{i,j}^2 : i, j \in I; i \neq j\}$.*

(1) $\mathcal{C}$ is an antichain of nonempty subsets of X
(2) $\mathcal{C}$ satisfies the 2-binary elimination property.
(3) Let ϕ be the operator associated to the antichain $\mathcal{C}$. Then ϕ is finitary iff for every $i \in I$, the set A_i is finite.
(4) The operator ϕ is idempotent (and thus matroidal).

Proof. Points (1), (2) and (3) are easy to check.
(4) Let Z be a subset of X. Let I_1 be the set of elements $i \in I$ such that $A_i \backslash Z$ has at least two elements. Let $I_2 = I \backslash I_1$. If $O \in Z$ then $\phi(Z) = Z \cup \bigcup_{i \in I_2} A_i$ and thus, $\phi(\phi(Z)) = \phi(Z)$. If $O \notin Z$ and if there exists $i_0 \in I_2$ such that $A_{i_0} \subseteq Z$, then $\phi(Z) = Z \cup \{O\} \cup \bigcup_{i \in I_2} A_i$ and thus, $\phi(\phi(Z)) = \phi(Z)$; if $O \notin Z$ and if for every $i \in I_2$, $A_i \backslash Z$ has exactly one element, then $\phi(Z) = Z$ and thus $\phi(\phi(Z)) = \phi(Z)$. □

Definition 9. In the conditions of the previous theorem, we call ϕ the *matroidal operator associated to O and the family $(A_i)_{i \in I}$.*

Definition 10. Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, a *selector* for this family is a subset S of $\bigcup_{i \in I} A_i$ such that for every $i \in I$, $S \cap A_i$ has at most one element; the selector S is said to be *total* if for every $i \in I$, $S \cap A_i$ has exactly one element.

Theorem 3. *Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, consider the set $X = \bigcup_{i \in I} A_i \cup \{O\}$ where O is some set such that $O \notin \bigcup_{i \in I} A_i$. Let ϕ be the matroidal operator associated to O and the family $(A_i)_{i \in I}$.*
(1) A subset L of X is ϕ-independent iff either ($O \in L$ and $\forall i \in I A_i \nsubseteq L$), or ($O \notin L$ and there exists at most one element $i_0 \in I$ such that $A_{i_0} \subseteq L$).
(2) A subset G of X is ϕ-generating iff $S := (\bigcup_{i \in I} A_i) \backslash G$ is a selector for the family $(A_i)_{i \in I}$, which is not total if $O \notin G$.
(3) A subset B of X is a ϕ-basis iff there exists a total selector S for the family $(A_i)_{i \in I}$ such that $B = ((\bigcup_{i \in I} A_i) \backslash S) \cup \{a\}$ where a is some element of $\{O\} \cup S$.
(4) A proper subset F of X is a ϕ-flat iff ($O \in F$ or $\exists i_0 \in I A_{i_0} \subseteq F$) $\Rightarrow$ $\forall i \in I$ $A_i \backslash F$ is not a singleton
(5) A subset H of X is a ϕ-hyperplane iff $H = (\bigcup_{i \in I} A_i) \backslash S$ where S is a total selector for the family $(A_i)_{i \in I}$, or $H = X \backslash \{x, y\}$ where $i_0 \in I$ and $x, y \in A_{i_0}$ with $x \neq y$.
(6) The following statements are equivalent:
  (a) The operator ϕ is hyperplane-accessible.
  (b) Every family $(B_i)_{i \in I}$ such that for every $i \in I$, $\varnothing \subsetneq B_i \subseteq A_i$ has a total selector.
  (c) The operator ϕ is B-matroidal.
  (d) The operator ϕ satisfies the interpolation property for bases

Proof. Points (1), (2), (3), (4) and (5) are consequences of the definitions. We prove Point (6).
(a) $\Rightarrow$ (b): Given a family $(B_i)_{i \in I}$ such that for every $i \in I$, $\varnothing \subsetneq B_i \subseteq A_i$, consider the proper ϕ-flat subset $F := \bigcup_{i \in I}(A_i \backslash B_i)$ of X; since ϕ is hyperplane-accessible, let H be a ϕ-hyperplane such that $F \subseteq H$ and $O \notin H$; then $\bigcup_{i \in I}(A_i \backslash H)$ is a total selector for the family $(B_i)_{i \in I}$.
(b) $\Rightarrow$ (c): Let Y be a subset of X. Let L be a ϕ-independent subset of Y and let G be a ϕ_Y-generating subset of Y such that $L \subseteq G$. Let $J := \{i \in I : Y \cap A_i \neq \varnothing\}$. Let $J_1 := \{i \in J : A_i \nsubseteq G\}$. Let $J_2 = \{i \in J : A_i \subseteq G$ and $A_i \nsubseteq L\}$. Let $J_3 = \{i \in J : A_i \subseteq L\}$: notice that $J = J_1 \cup J_2 \cup J_3$ and that J_1, J_2 and J_3 are pairwise disjoint. For each $i \in J_1$,

let x_i be the element of $A_i \backslash G$. Using (b), consider a choice function $(x_i)_{i\in J_2}$ for the family $(A_i \backslash L)_{i\in J_2}$. If J_3 is nonempty, then J_3 has a unique element i_0 and let $x_{i_0} = O$ if $O \in Y$. If $O \in Y$, let $B := Y \backslash \{x_i : i \in J\}$, and if $O \notin Y$, let $B := Y \backslash \{x_i : i \in J_1 \cup J_2\}$. Then B is a ϕ_Y-basis such that $L \subseteq B \subseteq G$.
(c) $\Rightarrow$ (d) follows from the definitions.
(d) $\Rightarrow$ (a): Let F be a proper subset of X which is a ϕ-flat and let $x \in X \backslash F$. If $x = O$, then for every $i \in I$, $A_i \backslash F$ has at least one element (else O would belong to F), and thus F is ϕ-independent; then $G = \bigcup_{i\in I} A_i$ is ϕ-spanning and $F \subseteq G$: using the interpolation property, there exists a ϕ-basis B such that $F \subseteq B \subseteq G$; it follows that there exists a total selector S for $(A_i)_i$ and en element $i_0 \in I$ such that $B = A_{i_0} \cup (\bigcup_{i\neq i_0} A_i) \backslash S$; let $x_{i_0} \in A_{i_0} \backslash F$; then $H = B \backslash \{x_{i_0}\}$ is a ϕ-hyperplane including F such that $O \notin H$. If $x \neq O$, then let i_0 be the element of I such that $x \in A_{i_0}$. If $A_{i_0} \backslash F$ contains an element y distinct from x, then $H := X \backslash \{x, y\}$ is a ϕ-hyperplane including F and not containing x. If $A_{i_0} \backslash F = \{x\}$, then for every $i \in I \backslash \{i_0\}$, $A_i \backslash F \neq \varnothing$ and $O \notin F$ (else x would belong to F); using the independent set $L = F \backslash \{O\}$ and the generating set $G = \bigcup_i A_i$, consider a ϕ-basis B such that $L \subseteq B \subseteq G$; then B yields a selector S for the family $(A_i \backslash F)_{i\in I}$ (and thus $x \in S$). It follows that $H := (\bigcup_{i\in I} A_i) \backslash S$ is a ϕ-hyperplane including F. □

Corollary 2. AC *is equivalent to the following statement:* "For every nonempty family $(A_i)_{i\in I}$ of pairwise disjoint nonempty sets, and for every set O such that $O \notin \bigcup_{i\in I} A_i$, the matroidal operator associated to O and the family $(A_i)_{i\in I}$ has an hyperplane not containing O."

3.4. **The axiom sH implies $\mathbf{AC^{fin}}$.** We denote by $\mathbf{sH}_{bep}$ the axiom **sH** restricted to finitary matroids with the binary elimination property. For every natural number $n \geq 2$, we denote by $\mathbf{sH}_{bep_n}$ the axiom **sH** restricted to finitary matroids with the n-binary elimination property. We denote by $\mathbf{H}_{bep}$ (*resp.* $\mathbf{H}_{bep_n}$) the axiom **H** restricted to finitary matroids with the binary elimination property (*resp.* n-binary elimination property).

Remark 8. The matroidal operator associated to a family $(A_i)_{i\in I}$ of pairwise finite disjoint nonempty sets satisfies the 2-binary elimination property (and hence is binary).

Theorem 4. *In* **ZF**, $\mathbf{sH} \Rightarrow \mathbf{sH}_{bep} \Rightarrow \mathbf{sH}_{bep_2} \Rightarrow \mathbf{AC^{fin}}$.

Proof. Notice that $\mathbf{AC^{fin}}$ is equivalent to the statement "For every nonempty family $(A_i)_{i\in I}$ of *pairwise disjoint* finite nonempty sets, $\prod_{i\in I} A_i$ is nonempty.": given a family $(A_i)_{i\in I}$ of nonempy sets, consider the family $(A_i \times \{i\})_{i\in I}$. Given a nonempty family $(A_i)_{i\in I}$ of pairwise disjoint finite nonempty sets, consider the set $X = \bigcup_{i\in I} A_i \cup \{O\}$ where $O \notin \bigcup_{i\in I} A_i$, and consider the finitary matroidal operator ϕ on X associated to the family $(A_i)_{i\in I}$ (see Theorem 2). Since ϕ has no loops, $\phi(\varnothing) = \varnothing$, so $\varnothing$ is a proper flat of ϕ and thus, $\mathbf{sH}_{bep_2}$ implies a ϕ-hyperplane H not containing O. It follows from Theorem 3 that for each $i \in I$, $A_i \backslash H$ is a singleton $\{x_i\}$ where $(x_i)_{i\in I}$ is a choice function for the family $(A_i)_{i\in I}$. □

Question 1. We have shown that $\mathbf{AC} \Rightarrow \mathbf{sH}_1 \Rightarrow \mathbf{sH} \Rightarrow \mathbf{sH}_{bep} \Rightarrow \mathbf{sH}_{bep_2} \Rightarrow \mathbf{AC}^{fin}$ and of course $\mathbf{sH} \Rightarrow \mathbf{H} \Rightarrow \mathbf{H}_{bep} \Rightarrow \mathbf{H}_{bep_2}$. Does $\mathbf{sH}_{bep}$ imply **sH**? Does **H** imply $\mathbf{AC^{fin}}$? Does **H** imply **sH**?

4. Graphic matroids and the finite axiom of choice

4.1. **Strong and weak elimination properties.**

Definition 11. A family $\mathcal{C}$ of subsets of a set X is said to satisfy the *elimination property* if for all distinct elements C_1, $C_2 \in \mathcal{C}$, for every $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2$ and $x \notin C_3$. The family $\mathcal{C}$ is said to satisfy the *strong elimination property* if for every elements C_1, $C_2 \in \mathcal{C}$, for every $x \in C_1 \cap C_2$ and every $y \in C_1 \backslash C_2$, then there exists $C_3 \in \mathcal{C}$ such that $y \in C_3 \subseteq C_1 \cup C_2$ and $x \notin C_3$.

Notice that the binary elimination property implies the strong elimination property, which in turn implies the elimination property.

Notation 1. For every finite set F, we denote by $|F|$ the cardinal of F.

The following result is classical:

Proposition 7 ([15], [2])**.** *Let $\mathcal{C}$ be an antichain of nonempty finite subsets of a set X, and let ϕ be the (finitary) operator associated to $\mathcal{C}$. If $\mathcal{C}$ satisfies the weak elimination property, then:*

(1) $\mathcal{C}$ satisfies the strong elimination property.
(2) The operator ϕ is a closure operator.
(3) The operator ϕ is matroidal.

Proof. (1) See [15, Theorem 2 p. 24] or [2, Lemme 4 p. 17].
(2) See [2, Théorème 8 p. 18]. We sketch the proof. Let A be a subset of X and let $x \in \phi(\phi(A))$. Let us show that $x \in \phi(A)$. Let $C \in \mathcal{C}$ such that $x \in C \subseteq \phi(A) \cup \{x\}$, and such that $C \cap (\phi(A) \backslash A)$ is minimal. If $(C \backslash \{x\}) \cap (\phi(A) \backslash A)$ is nonempty, let $y \in (C \backslash \{x\}) \cap (\phi(A) \backslash A)$; since $y \in \phi(A)$, let $C_1 \in \mathcal{C}$ such that $y \in C_1 \subseteq A \cup \{y\}$. Using the strong elimination property, let $C_2 \in \mathcal{C}$ such that $x \in C_2 \subseteq (C \cup C_1) \backslash \{y\}$: then $|C_2 \cap (\phi(A) \backslash A)| < |C \cap (\phi(A) \backslash A)|$, which contradicts the minimality of $C \cap (\phi(A) \backslash A)$. It follows that $(C \backslash \{x\}) \cap (\phi(A) \backslash A) = \varnothing$ and thus, $(C \backslash \{x\}) \subseteq A$ so $x \in \phi(A)$.
(3) Using Lemma 1, $\mathcal{C}$ is the set of ϕ-circuits and ϕ satisfies the exchange property, whence the closure operator ϕ is a matroidal operator on X. $\square$

4.2. **The binary matroid associated to a multigraph.**

4.2.1. *Multigraphs.* A *multigraph* on a set V is given by a mapping $f : X \to [V]^1 \cup [V]^2$, where, for each natural number $n \geq 1$, $[V]^n$ is the set of n-element subsets of V. Elements of X such that $f(x) \in [V]^1$ are called *loops* of the multigraph.

Denoting by $(e_v)_{v \in V}$ the canonical basis of the vector space $\mathbb{F}_2^{(V)}$, the *incidence matrix* of the multigraph f is the mapping $\tilde{f} : X \to \mathbb{F}_2^{(V)}$ such that for every $x \in X$, $\tilde{f}(x)$ is $0_{\mathbb{F}_2^{(V)}}$ if $f(x) \in [V]^1$, and $\tilde{f}(x) = e_{v_1} + e_{v_2}$ if $f(x)$ is the two-element sets $\{v_1, v_2\}$. The *matroid associated to the multigraph* f is the (binary) matroidal operator on X associated to the incidence matrix $\tilde{f}$. Loops of this matroid correspond to loops of the multigraph. A matroidal operator which is isomorphic with the (binary hence finitary) matroid associated to a multigraph is said to be *graphic*.

4.2.2. *Simple graphs.* A simple graph on a set V is a binary relation R on V which is irreflexive (for every $x \in V$, $x \not R x$) and symmetric (for every $x, y \in V$, $xRy \Rightarrow yRx$). Elements of V are called the *vertices* of the graph, and pairs $\{x, y\}$ of vertices such that xRy are the edges of the simple graph. A simple graph on a set V with set E of edges is also denoted by (V, E). A (partial) subgraph of a simple graph G on a set X with set of edges E

is a simple graph (X', E') such that $X' \subseteq X$ and $E' \subseteq E$. Two graphs (V_1, E_1) and (V_2, E_2) are *isomorphic* when there exists a bijection $f : V_1 \to V_2$ which respects the edges.

Notation 2. Given some integer $n \geq 3$, we denote by C_n the simple graph on $\mathbb{Z}/n\mathbb{Z} = \{0, \ldots, n-1\}$ with set of edges $E_n = \{\{i, i +_n 1\} : i \in \mathbb{Z}/n\mathbb{Z}\}$, where $+_n$ is the additive law on $\mathbb{Z}/n\mathbb{Z}$;

Given some integer $n \geq 3$, a simple graph is a *n-cycle* if it is isomorphic with the simple graph C_n. Given a simple graph $G = (V, E)$, a *cycle of the graph* G is a (partial) subgraph of G which is isomorphic with a n-cycle for some natural number $n \geq 3$.

4.2.3. *Graphic matroids.* Given a set V and a multigraph $f : X \to [V]^1 \cup [V]^2$, if $E = f[X] \cap [V]^2$, then (V, E) is called the simple graph underlying the multigraph f. Reciprocally, every simple graph (V, E) underlies the multigraph $\mathrm{id}_E : E \to E$ on V.

Proposition 8 ([12, Proposition 1.1.7])**.** *Let $G = (V, E)$ be a simple graph. Let $\mathcal{C}_G$ be the set of (finite) subsets F of E such that F is the set of edges of a cycle of G. Then $\mathcal{C}_G$ is the set of circuits of the (binary) matroidal operator $\mathcal{M}_G$ associated to the multigraph $\mathrm{id}_E : E \to E$.*

Proof. Let W be the $\mathbb{F}_2$-vector space $\mathbb{F}_2^{(V)}$. For every $v \in V$ we denote by e_v the v-th vector of the canonical basis of W. We identify each edge $\{a, b\}$ of G with the vector $e_a + e_b$ of W. A subset F of E is a circuit of the matroid $\mathcal{M}_G$ iff F is nonempty, $\sum_{e \in F} e = 0_W$ and for every nonempty proper subset G of F, $\sum_{e \in G} e \neq 0_W$; replacing each element $e = \{a, b\}$ of E by $e_a + e_b$, this means that $F \neq \varnothing$, every vertex of the subgraph $(\cup F, F)$ has even degree, but for every proper subset G of F, some vertex of the subgraph $(\cup G, G)$ has an odd degree; this means that F is a nonempty finite union of cycles of G, and that no proper subset of F is a cycle of G; equivalently, F is a cycle of the graph G. □

Remark 9. If $f : X \to [V]^1 \cup [V]^2$ is a multigraph on a set V, if $\mathcal{M}_f$ is the matroid associated to the multigraph f, loops of $\mathcal{M}_f$ are the singletons $\{x\}$ such that $x \in X$ and $f(x)$ is a singleton; circuits of cardinal two of $\mathcal{M}_f$ are the pairs $\{x, y\}$ of distinct elements of X such that $f(x) = f(y)$. Given some natural number $n \geq 3$, then the n-circuits of $\mathcal{M}_f$ are the n-element subsets $\{x_1, \ldots, x_n\}$ of X such that $\{f(x_1), \ldots, f(x_n)\}$ is the set of edges of a n-cycle of the underlying simple graph of f.

4.3. An equivalent of $\mathbf{AC}^{\mathbf{fin}}$ in terms of graphic matroids.

Theorem 5. *The following statements are equivalent:*

(1) $\mathbf{AC}^{\mathbf{fin}}$
(2) For every family $(A_i)_{i \in I}$ of pairwise disjoint nonempty finite sets with at least two elements, the (binary hence finitary) matroid associated to this family is graphic.

Proof. (1) $\Rightarrow$ (2) Let $(A_i)_{i \in I}$ be an infinite family of pairwise disjoint nonempty finite sets, such that for every $i \in I$, $n_i := |A_i| \geq 2$. Let $\mathcal{M}$ be the matroid associated to the family $(A_i)_{i \in I}$: the underlying set of $\mathcal{M}$ is $M := \bigcup_{i \in I} A_i \cup \{O\}$ where $O \notin \bigcup_{i \in I} A_i$. We consider a family $(V_i)_{i \in I}$ of pairwise disjoint linearly ordered finite sets such that for each $i \in I$, $|V_i| = n_i - 1$. We also consider two distinct elements a and b not belonging to $\bigcup_{i \in I} V_i$, and we define the set $V := \{a, b\} \cup \bigcup_{i \in I} V_i$. Since each V_i is linearly ordered, for each $i \in I$, we consider a graph G_i on $V_i \cup \{a, b\}$ which is a n_i-cycle and such that $\{a, b\}$ is an edge of this graph: we denote by E_i the set of edges of G_i which are not equal to the edge $\{a, b\}$ of G_i.

We consider the simple graph G on V which admits $E := \bigcup_{i\in I} E_i \cup \{a,b\}$ as set of edges (see Figure 1). Notice that every finite subgraph of G is planar. We denote by $\mathcal{G}$ the matroid on E associated to the graph G. Using $\mathbf{AC^{fin}}$, we consider a family $(f_i)_{i\in I}$ such that for every $i \in I$, $f_i : E_i \to A_i$ is a bijection. It follows that $f := \bigcup_{i\in I} f_i$ is a bijection from $\bigcup_{i\in I} E_i$ to $\bigcup_{i\in I} A_i$ and we extend it into a bijection from E to M. Then the bijection f respects circuits of $\mathcal{M}_G$ and $\mathcal{M}$ thus t the matroid $\mathcal{M}$ is graphic.

(2) $\Rightarrow$ (1) Let $(A_i)_{i\in I}$ be a family of pairwise disjoint nonempty finite sets with at least two

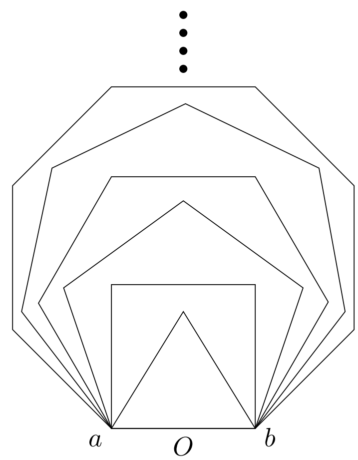

FIGURE 1. The graph G associated to the matroid $\mathcal{M}$

elements. Let $\mathcal{M}$ be the finitary matroid on $\{O\} \cup \bigcup_{i\in I} A_i$ associated to this family. Let $G = (V,E)$ be a graph such that $\mathcal{M}$ is the graphic matroid associated to G. Let a, b be the two extremities of the edge O of G. Then, for every $i \in I$, $A_i \cup \{O\}$ is the set of edges of a cycle of the graph G: let e_i be the the unique edge of A_i which is incident to the vertex a. Then $(e_i)_{i\in I}$ is a choice function for the family $(A_i)_{i\in I}$. $\square$

Consider the following well known consequences of $\mathbf{AC}$ imply $\mathbf{AC^{fin}}$:

$\mathbf{MG}_1$: "For every binary matroid $\mathcal{M}$, if every finite minor of $\mathcal{M}$ is graphic then $\mathcal{M}$ is graphic".

$\mathbf{MG}_2$: "For every binary matroid $\mathcal{M}$, if every finite submatroid of $\mathcal{M}$ is graphic and planar then $\mathcal{M}$ is graphic".

$\mathbf{MG}_3$: "For every binary matroid $\mathcal{M}$, if every finite minor of $\mathcal{M}$ is graphic and planar then $\mathcal{M}$ is graphic".

Notice that both statements $\mathbf{MG}_1$ and $\mathbf{MG}_2$ imply $\mathbf{MG}_3$. Moreover, every finite minor of the binary matroid used in the proof of Theorem 5 is graphic and planar, and thus, $\mathbf{MG}_3$ imply $\mathbf{AC^{fin}}$.

Question 2. Does $\mathbf{AC^{fin}}$ or $\mathbf{sH}$ imply one of the statements $\mathbf{MG}_1$, $\mathbf{MG}_2$ or $\mathbf{MG}_3$?

Question 3. Is the following statement provable in $\mathbf{ZF}$: *"Every (infinite) graphic matroid is hyperplane-accessible."*

In the diagram in Figure 2, we add the statement $\mathbf{D}_{\mathbb{Q}}$ which implies the statement $\mathbf{AC}^{\mathbb{Z}}$: *"Every family of posets isomorphic with the linear order $\mathbb{Z}$ has a nonempty product."* (see

$$
\begin{tikzcd}
 & & \mathbf{AC} \arrow[ddll] \arrow[dd] \arrow[drr] \arrow[drrr] & & & \\
 & & & & \forall p\ \mathbf{MC}(p)(218) \arrow[d] \arrow[dr] & \mathbf{VB}(66) \arrow[d] \\
\mathbf{BPI}(14) \arrow[d] & & \mathbf{sH} \arrow[dl] \arrow[d] & & \mathbf{I} \arrow[ddl] & \mathbf{MC}(67) \arrow[dddl] \\
\mathbf{OEP}(49) \arrow[d] & \mathbf{sH}_{bep} \arrow[d] & \mathbf{H} \arrow[dr] & & & \\
\mathbf{O}(30) \arrow[ddr] & \mathbf{sH}_{bep_n} \arrow[d] & & \mathbf{D} \arrow[dl] \arrow[dr] & & \\
 & \mathbf{sH}_{bep_2} \arrow[d] & \forall p\ \mathbf{D}_p \arrow[d] & & \mathbf{D}_0 \arrow[d] & \\
 & \mathbf{AC^{fin}}(62) \arrow[d] & \forall p\ \mathbf{D}_{\mathbb{F}_p} \arrow[d] & & \mathbf{D}_{\mathbb{Q}} \arrow[d] & \\
 & \forall k \mathbf{AC}^k(61) \arrow[r, leftrightarrow] & \forall p\ \mathbf{C}(p) & & \mathbf{AC}^{\mathbb{Z}} &
\end{tikzcd}
$$

FIGURE 2. Summary diagram of the axioms

[10, Theorem 4]). We also add the statement $\mathbf{D}_0$ (*resp.* $\mathbf{D}_p$) which is $\mathbf{D}$ restricted to vector spaces over a commutative field $\mathbb{K}$ of characteristic 0 (*resp.* p).

Question 4. The statements **BPI** ("Every non trivial Boolean algebra has a maximal ideal"), **OEP** ("Every partial order on a set X can be extended into a linear order on X") and **O** ("On every set X there exists a linear order") (see forms 14, 49 and 30 of [7]) are well known consequences of **AC** which are stronger than $\mathbf{AC^{fin}}$. Are there implications between one of them and **H** or **sH** or $\mathbf{sH}_{bep}$ or $\mathbf{sH}_{bep_n}$ for some integer $n \geq 2$?

References

[1] P. M. Cohn. *Universal algebra*, volume 6 of *Mathematics and its Applications*. D. Reidel Publishing Co., Dordrecht-Boston, Mass., second edition, 1981.

[2] J.-C. Fournier. *Introduction à la notion de matroïde*, volume 3 of *Publications Mathématiques d'Orsay 79 [Mathematical Publications of Orsay 79]*. Université de Paris-Sud, Département de Mathématique, Orsay, 1979. Géométrie combinatoire.

[3] D. A. Higgs. Matroids and duality. *Colloq. Math.*, 20:215–220, 1969.

[4] W. Hodges. Krull implies Zorn. *J. London Math. Soc. (2)*, 19(2):285–287, 1979.

[5] H. Höft and P. Howard. A graph theoretic equivalent to the axiom of choice. *Z. Math. Logik Grundlagen Math.*, 19:191, 1973.

[6] P. Howard. Bases, spanning sets, and the axiom of choice. *MLQ Math. Log. Q.*, 53(3):247–254, 2007.

[7] P. Howard and J. E. Rubin. *Consequences of the Axiom of Choice*, volume 59. American Mathematical Society, Providence, RI, 1998.

[8] T. J. Jech. *The Axiom of Choice*. North-Holland Publishing Co., Amsterdam, 1973.
[9] V. Klee. The greedy algorithm for finitary and cofinitary matroids. In *Combinatorics (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968)*, pages 137–152, 1971.
[10] M. Morillon. Linear forms and axioms of choice. *Comment. Math. Univ. Carolin.*, 50(3):421–431, 2009.
[11] M. Morillon. Multiple choices imply the Ingleton and Krein-Milman axioms. *J. Symb. Log.*, 85(1):439–455, 2020.
[12] J. Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.
[13] J. G. Oxley. Infinite matroids. *Proc. London Math. Soc. (3)*, 37(2):259–272, 1978.
[14] H. Rubin and J. E. Rubin. *Equivalents of the axiom of choice*. North-Holland Publishing Co., Amsterdam-London, 1970. Studies in Logic and the Foundations of Mathematics.
[15] D. J. A. Welsh. *Matroid theory*. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976. L. M. S. Monographs, No. 8.
[16] N. White, editor. *Theory of matroids*, volume 26 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1986.
[17] O. Zariski and P. Samuel. *Commutative algebra. Vol. 1*. Springer-Verlag, New York-Heidelberg-Berlin, 1975. With the cooperation of I. S. Cohen, Corrected reprinting of the 1958 edition, Graduate Texts in Mathematics, No. 28.

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