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HYPERPLANES IN MATROIDS AND THE AXIOM OF CHOICE

MARIANNE MORILLON

ABSTRACT. We show that in set-theory without the axiom of choice **ZF**, the statement **sH**: “Every proper closed subset of a finitary matroid is the intersection of hyperplanes including it” implies **AC^{fin}**, the axiom of choice for (nonempty) finite sets. We also provide an equivalent of the statement **AC^{fin}** in terms of “graphic” matroids. Several open questions stay open in **ZF**, for example: does **sH** imply the Axiom of Choice?

1. INTRODUCTION

A *choice function* for a family $(A_i)_{i \in I}$ of nonempty sets is a family $(x_i)_{i \in I}$ such that for every $i \in I$, $x_i \in A_i$. The Axiom of Choice (**AC**) is the following statement: “*Every family of nonempty sets has a choice function.*” We work in set theory without the axiom of choice **ZF**. We shall also consider the more general set theory **ZFA** (see [8, p. 44-45]), a modified version of set theory, in which “atoms” (*i.e.* nonempty objects which are not sets) are allowed. Consider the statement **VB** (Vector Basis): “*Every vector space has a basis*” (see [7, Note 75 p. 271]). It is known that in **ZFA**, **VB** implies the Multiple Choice axiom **MC** ([7, form 67]), and that in **ZF**, **MC** is equivalent to **AC**, but it is an open question to know whether **VB** imply **AC** in **ZFA**. In this paper, we discuss various statements about “finitary matroids” (which can be seen as generalisations of vector spaces, see Section 2.3.3) and their links with **AC**. We show that the statement “*Every finitary matroid has a basis*” is equivalent to **AC** in **ZFA** (see Proposition 5). We then consider the three following consequences of **AC** involving hyperplanes in finitary matroids, possibly satisfying the “binary elimination property” (see Section 3.2):

sH: “*Every proper flat in a finitary matroid is the intersection of hyperplanes including it.*”

sH_{bep}: “*Every proper flat in a finitary matroid with the binary elimination property is the intersection of hyperplanes including it.*”

H: “*Every nonempty finitary matroid has an hyperplane.*”

It is known that **AC** \Rightarrow **sH** and of course **sH** \Rightarrow **H** and **sH** \Rightarrow **sH_{bep}**. In this paper, we shall prove that **sH_{bep}** implies the following axiom of choice for finite sets:

AC^{fin}: (form 62 of [7]) *Every nonempty family of finite nonempty sets has a choice function.*

It is known (see [7]) that **AC^{fin}** does not imply **AC** and that **AC^{fin}** is not provable in **ZF**. We do not know whether **H** implies **sH** or **sH_{bep}** or **AC^{fin}** nor do we know whether **H** or **sH** implies **AC** (see Figure 2 at the end of the paper). For every natural number $k \geq 2$ we consider the following consequence of **AC^{fin}**:

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\mathbf{AC}^k : “For every nonempty family $(A_i)_{i \in I}$ of finite sets with k -elements, $\prod_{i \in I} A_i$ is nonempty.”

We also denote by $\forall k \mathbf{AC}^k$ the following statement, which is form 61 of [7]:

For every natural number $k \geq 2$, for every nonempty family $(A_i)_{i \in I}$ of finite sets with k -elements, $\prod_{i \in I} A_i$ is nonempty.

In \mathbf{ZF} , for every natural number $n \geq 2$, $\mathbf{AC} \Rightarrow \mathbf{AC}^{\text{fin}} \Rightarrow \forall k \mathbf{AC}^k \Rightarrow \mathbf{AC}^n$, and it is known (see [7]) that in \mathbf{ZF} , none of these implications is reversible, and that \mathbf{AC}^n is not provable.

Using the natural structure of finitary matroid over a vector space (see Example 1), \mathbf{H} implies the following statement \mathbf{D} : “Given a commutative field \mathbb{K} and a non null vector space E over \mathbb{K} , there exists a non null linear form $f : E \rightarrow \mathbb{K}$ ”. For every commutative field \mathbb{K} , we denote by $\mathbf{D}_{\mathbb{K}}$ the previous statement restricted to vector spaces over \mathbb{K} : “For every non null \mathbb{K} -vector space E , the algebraic dual of E is non null.” In [10, Corollary 2], we proved that for every prime number p , the statement $\mathbf{D}_{\mathbb{F}_p}$ (where \mathbb{F}_p is the finite field $\mathbb{Z}/p\mathbb{Z}$) implies the statement $\mathbf{C}(p)$: “For every family $(A_i)_{i \in I}$ of nonempty finite sets, there exists a family $(B_i)_{i \in I}$ such that for every $i \in I$, $B_i \subseteq A_i$ and p does not divide the cardinal of B_i ”. Denoting by $\forall p \mathbf{C}(p)$ the statement $\forall p \in \mathbb{P} \mathbf{C}(p)$ where \mathbb{P} is the set of prime natural numbers, then $\forall p \mathbf{C}(p)$ implies (and thus is equivalent to) the statement $\forall k \mathbf{AC}^k$ (see [10, Remarks 3 and 4]). It follows that $\mathbf{sH} \Rightarrow \mathbf{H} \Rightarrow \mathbf{D} \Rightarrow \forall k \mathbf{AC}^k$. However, we do not know whether \mathbf{D} implies \mathbf{H} . Notice that in \mathbf{ZFA} , \mathbf{D} does not imply \mathbf{AC}^{fin} , since the statement $\forall p \mathbf{MC}(p)$ (see [7, form 218]) implies the Ingleton statement \mathbf{I} (the ultrametric counterpart of the Hahn-Banach statement, see [11]) which implies \mathbf{D} , but $\forall p \mathbf{MC}(p)$ does not imply \mathbf{AC}^{fin} (see Figure 2 at the end of the paper).

The paper is organized as follows. In Section 2 we review in set theory \mathbf{ZF} some definitions and results about operators on finite or infinite sets in the sense of Higgs ([3]) and Klee ([9]): finitary operators, matroidal operators with particular emphasis on circuits and hyperplanes. We introduce the three notions of “circuit-accessibility”, “hyperplane-accessibility” and “symmetric circuits”. In Section 3, we formulate an equivalent of \mathbf{AC} in terms of hyperplanes in a certain (non finitary) matroid, and we prove that the statement \mathbf{sH} restricted to certain binary matroids implies \mathbf{AC}^{fin} . Finally, in the last section, we prove that \mathbf{AC}^{fin} is equivalent to various statements about “graphic” matroids. We end with several questions about finitary matroids and \mathbf{AC} .

2. OPERATORS AND THE AXIOM OF CHOICE

2.1. Operators on a set.

2.1.1. *Operators and their circuits.* An operator on a set X (see [9, p. 138]) is a mapping $\phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ which is *isotonic* (for every subsets A, B of X , $(A \subseteq B \Rightarrow \phi(A) \subseteq \phi(B))$) and *enlarging* (for every subset A of X , $A \subseteq \phi(A)$). Given an operator ϕ on a set X , a subset D of X is said to be ϕ -*dependent* if there exists $x \in D$ such that $x \in \phi(D \setminus \{x\})$. A subset I of X is said to be ϕ -*independent* if I is not ϕ -dependent *i.e.* if for every $x \in I$, $x \notin \phi(I \setminus \{x\})$. Minimal ϕ -dependent subsets of X are called ϕ -*circuits*. A *loop* of the operator ϕ on X is an element x of X such that $\{x\}$ is a ϕ -circuit *i.e.* $\{x\}$ is ϕ -dependent *i.e.* $x \in \phi(\emptyset)$. Two distinct elements x, y of X are *parallel* if $\{x, y\}$ is a ϕ -circuit.

Remark 1. Given an operator ϕ on a set X :

- (1) the collection \mathcal{I}_ϕ of ϕ -independent subsets of X contains \emptyset and is *initial*: for all subsets A, B of X , if $A \subseteq B$ and $B \in \mathcal{I}_\phi$, then $A \in \mathcal{I}_\phi$;
- (2) the collection \mathcal{D}_ϕ of ϕ -dependent subsets of X does not contain \emptyset and is *final*: for every subsets A, B of X , if $A \subseteq B$ and $A \in \mathcal{D}_\phi$, then $B \in \mathcal{D}_\phi$.
- (3) The collection \mathcal{C}_ϕ of ϕ -circuits is an *antichain* of nonempty sets: no member of \mathcal{C}_ϕ includes another one.

2.1.2. *Finitary operators.* A operator ϕ on X is said to be *finitary* if for every subset Y of X and every $x \in \phi(Y)$, there exists a finite subset F of Y satisfying $x \in \phi(F)$. If the operator ϕ is finitary, then every ϕ -dependent set includes a (finite) ϕ -circuit.

Definition 1. Given two *finitary* operators ϕ_1 and ϕ_2 on sets X_1 and X_2 and given a bijection $f : X_1 \rightarrow X_2$, the following statements are equivalent:

- (1) for every subset I of X_1 , I is ϕ_1 -independent if and only if $f[I]$ is ϕ_2 -independent
- (2) for every subset C of X_1 , C is a ϕ_1 -circuit if and only if $f[C]$ is a ϕ_2 -circuit.

Every bijection $f : X_1 \rightarrow X_2$ satisfying one of the two previous statements is called an *isomorphism* of finitary operators.

2.1.3. *Hyperplanes of an operator.* A subset A of X is said to be ϕ -*spanning* if $\phi(A) = X$. Subsets of X which are both ϕ -independent and ϕ -spanning are called *bases* of the operator ϕ (or ϕ -bases). Maximal non-spanning subsets of X are called ϕ -*hyperplanes*. Subsets of X which are fixed points of ϕ are called *flats* or *closed subsets* of the operator ϕ .

Remark 2. Given an operator ϕ on a set X , for every nonempty family $(F_i)_{i \in I}$ of ϕ -closed subsets of X , $\bigcap_{i \in I} F_i$ is ϕ -closed, and thus, the *poset* \mathcal{L}_ϕ of ϕ -closed subsets of X endowed with the inclusion relation is a complete lattice (but it is not an induced sub-lattice of the lattice $(\mathcal{P}(X), \subseteq)$ in general).

2.2. Minors of an operator.

2.2.1. *Suboperators.* Given an operator ϕ on a set X , and a subset Y of X , the mapping $\phi_Y : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$ such that for every subset Z of Y , $\phi_Y(Z) = \phi(Z) \cap Y$ is an operator on Y , called the *suboperator* induced by ϕ on Y , or *restriction operator* of ϕ to Y (see [13, p. 263]). If the operator ϕ on X is finitary, then the suboperator ϕ_Y is also finitary.

Remark 3. Given an operator ϕ on a set X , and a subset Y of X , then:

- (1) The ϕ_Y -dependent subsets of Y are the ϕ -dependent sets that are included in Y ;
- (2) The ϕ_Y -independent subsets of Y are the ϕ -independent sets that are included in Y .
- (3) The ϕ_Y -circuits are the ϕ -circuits that are included in Y .

2.2.2. *Quotient operators.* Given an operator ϕ on a set X , and a subset Y of X , the mapping $\phi^Y : \mathcal{P}(Y) \rightarrow \mathcal{P}(Y)$ associating to every subset A of Y the set $Y \cap \phi(A \cup (X \setminus Y))$ is an operator on Y . The operator ϕ^Y on Y is called the *quotient operator* ϕ^Y , or the *contraction operator* ϕ^Y (see [13, p. 263]). If the operator ϕ on X is finitary, then the operator ϕ^Y is also finitary.

Proposition 1. *Given an operator ϕ on a set X and a proper flat F of ϕ , then:*

- (1) ϕ -flats including F are subsets $F \cup Z$ where Z is a flat of the quotient operator $\phi^{X \setminus F}$ on $X \setminus F$.
- (2) ϕ -hyperplanes including F are subsets $F \cup Z$ where Z is a hyperplane of the operator $\phi^{X \setminus F}$.

Proof. (1) Given a subset Z of $X \setminus F$, the following sentences are equivalent: $F \cup Z$ is a ϕ -flat; $\phi(F \cup Z) \subseteq F \cup Z$; $\phi(F \cup Z) \setminus F \subseteq Z$; $\phi^{X \setminus F}(Z) \subseteq Z$; Z is a $\phi^{X \setminus F}$ -flat subset of $X \setminus F$.

(2) Given a subset Z of $X \setminus F$, the following sentences are equivalent: $F \cup Z$ is a ϕ -hyperplane; $(F \cup Z)$ is a proper ϕ -flat but for every $x \in X \setminus (F \cup Z)$, $\phi((F \cup Z) \cup \{x\}) = X$; Z is a proper $\phi^{X \setminus F}$ -flat but for every $x \in X \setminus (F \cup Z)$, $\phi^{X \setminus F}(Z \cup \{x\}) = X \setminus F$; the subset Z of $X \setminus F$ is a $\phi^{X \setminus F}$ -hyperplane. \square

Remark 4. Proposition 1 implies that given a class \mathcal{O} of operators which is closed by quotient operators, if every $\phi \in \mathcal{O}$ has an hyperplane, then for every $\phi \in \mathcal{O}$, every proper flat of ϕ is included in a ϕ -hyperplane.

Definition 2. Given an operator ϕ on a set X , a *minor* of an operator ϕ on a set X is an operator ψ on a subset Y of X such that there exists a sequence of operators $(\phi_i)_{0 \leq i \leq n}$ such that $\phi_0 = \phi$, $\phi_n = \psi$ and for each $i \in \{1, \dots, n\}$, ϕ_i is a suboperator or a quotient operator of ϕ_{i-1} .

2.3. Finitary matroidal operators.

2.3.1. *Idempotency properties.* A *closure operator* on X is an operator ϕ on X which is *idempotent* (see [9, p. 140]): for every subset A of X , $\phi(\phi(A)) = \phi(A)$.

If the operator ϕ on X is idempotent, then for every subset Y of X , the operators ϕ_Y and ϕ^Y are also idempotent.

Proposition 2. *Given an idempotent operator ϕ on a set X , a subset H of X is a ϕ -hyperplane iff H is a maximal proper ϕ -closed subset of X .*

Proof. Given an operator ϕ on a set X , for every ϕ -hyperplane H , then either $\phi(H) = H$, and thus H is a maximal proper ϕ -closed subset of X , or $\phi(H)$ is spanning (else $H \subsetneq \phi(H) \subseteq \phi(\phi(H)) \subsetneq X$ and H would not be a ϕ -hyperplane since $\phi(H)$ would be a non spanning subset of X strictly including H). It follows that if ϕ is idempotent, then every ϕ -hyperplane is a maximal proper ϕ -closed subset of X (else, $\phi(H)$ would be spanning *i.e.* $X = \phi(\phi(H)) = \phi(H)$ by idempotency, and thus H would be spanning). Reciprocally, if H is a maximal proper ϕ -closed subset of X , then for every $x \in X \setminus H$, $\phi(H \cup \{x\})$ is closed and thus $\phi(H \cup \{x\}) = X$ whence H is a ϕ -hyperplane. \square

Definition 3. An operator ϕ on X is *circuit-accessible* if for every subset Y of X and every $x \in \phi(Y) \setminus Y$, there exists a ϕ -circuit C such that $x \in C \subseteq Y \cup \{x\}$.

Remark 5. Every finitary idempotent operator is circuit-accessible.

Proof. Let ϕ be a finitary idempotent operator on a set X . Given some subset A of X , and some $x \in \phi(A) \setminus A$, let I be a minimal finite subset of A such that $x \in \phi(I)$. Then I is independent, else there exists $y \in I$ such that $y \in \phi(I \setminus \{y\})$, whence, denoting by G the set $I \setminus \{y\}$, $x \in \phi(G \cup \{y\})$ and thus, by idempotency of ϕ and since $y \in \phi(G)$, $x \in \phi(G)$ which contradicts the minimality of I . Since $I \cup \{x\}$ is finite and dependent, there exists a ϕ -circuit C such that $C \subseteq I \cup \{x\}$. Since I is independent, $x \in C$ and finally, $x \in C \subseteq A \cup \{x\}$. It follows that ϕ is circuit-accessible. \square

2.3.2. *Exchange properties.* An operator ϕ on a set X is said to satisfy the *exchange property* (see property (E) in [9, p. 140]) if for every subsets Y, Z of X and every $x \in X$, if $x \in \phi(Y \cup Z)$ and $x \notin \phi(Y)$, then there exists $y \in Z$ such that $y \in \phi(((Y \cup Z) \setminus \{y\}) \cup \{x\})$.

Definition 4. Given an operator ϕ on a set X , a ϕ -circuit C is *symmetric* if for every $x \in C$, $x \in \phi(C \setminus \{x\})$.

Remark 6. If an operator ϕ on a set X satisfies the exchange property, then every ϕ -circuit is symmetric.

2.3.3. *Matroidal operators.* We say that an operator ϕ on a set X is *matroidal* if ϕ is idempotent and satisfies the exchange property.

Example 1 (The operator span_X associated to a vector space X). Given a vector space X over a commutative field \mathbb{K} , the operator span on X , associating to every subset Y of X the vector subspace generated by Y in X is a finitary matroidal operator on X . The span-independent subsets of X are the \mathbb{K} -linearly independent subsets of X ; the span-bases of X are the bases of the \mathbb{K} -vector space X ; the span-flats are the vector subspaces of X , and the span-hyperplanes of X are the kernels of non null linear forms $f : X \rightarrow \mathbb{K}$. The only loop of this operator is $\{0_X\}$.

Example 2 (The matroidal operator associated to a family of vectors). Given a \mathbb{K} -vector space X and a mapping $f : I \rightarrow X$, the mapping $\phi : \mathcal{P}(I) \rightarrow \mathcal{P}(I)$ associating to every subset J of I the set $\{i \in I : f(i) \in \text{span}(f[J])\}$ is a finitary matroidal operator. Loops of this operator are elements $i \in I$ such that $f(i) = 0_X$. Two elements i, j of I are parallel iff i, j are not loops and if $f(i)$ and $f(j)$ are colinear.

Given a (commutative) field \mathbb{F} , a finitary matroidal operator ϕ on a set X is said to be \mathbb{F} -*representable* if there exist a \mathbb{K} -vector space E and a mapping $f : I \rightarrow E$ such that the matroidal operator ϕ is isomorphic with the finitary matroidal operator associated to f .

Remark 7. There are many equivalent definitions for the notion of *matroid* on a finite set (see [15, Chapter 1] or [16, Chapter 2]). Given an infinite set X , the notion of finitary matroidal operator on X is equivalent to the notion of “transitive dependence relation” on X (see for example [17, p. 97], [1, Prop. 2.1 p. 253], [15, Chapter 20.5], [2, p. 2]). In **ZFC**, finitary matroids have bases, but infinite matroids do not have bases in general.

2.3.4. *Hyperplane-accessibility.*

Definition 5. An operator ϕ on a set X is *hyperplane-accessible* if every proper flat of ϕ is the intersection of the set of the ϕ -hyperplanes including it.

Given a commutative field \mathbb{K} , the statement $\mathbf{D}_{\mathbb{K}}$: “Every non null vector space has a non null linear form.” is equivalent to the statement “For every \mathbb{K} -vector space E , the finitary matroidal operator is hyperplane-accessible.”

2.4. Finitary operators and the Axiom of choice.

2.4.1. *Axiom of Choice and finitary operators.*

Proposition 3 ([14, p. 95] and [4]). **AC** is equivalent to each of the following statements:

- (1) AL'_3 : [14, p. 95] “For every finitary closure operator ϕ on a set X , for every collection \mathcal{F} of subsets of X which has finite character (i.e. for every subset Z of X , $Z \in \mathcal{F}$ iff for every finite subset Y of Z , $Y \in \mathcal{F}$), for every proper ϕ -flat F of X such that $F \in \mathcal{F}$, then there exists a maximal ϕ -flat G such that $F \subseteq G$ and $G \in \mathcal{F}$.”

- (2) AL_3'' : “For every finitary closure operator ϕ on a set X , for every proper ϕ -flat F of X and every $x \in X \setminus F$, then there exists a maximal ϕ -flat G such that $F \subseteq G$ and $x \notin G$.”
- (3) K (Krull): “Every proper ideal of commutative unitary ring has a maximal proper ideal.”

It follows that **AC** implies the statement **sH**: “Every finitary matroid is hyperplane-accessible.”

Proof. **AC** \Rightarrow AL_3' : The set $P := \{Z \in \mathcal{F} : F \subseteq Z \text{ and } \phi(Z) = Z\}$ endowed with the order induced by \subseteq is inductive (for every chain C of P , $\cup C \in P$) and thus, Zorn’s lemma implies a maximal element G of P . $AL_3' \Rightarrow AL_3''$: given a proper ϕ -flat F and $x \in X \setminus F$, the collection \mathcal{F} of subsets of X which do not contain x has the finite character, and thus AL_3' implies a maximal ϕ -flat including F and not containing x . $AL_3'' \Rightarrow K$: Given a proper ideal I of a commutative unitary ring A , consider the closure operator ϕ on A associating to each subset Z of A the ideal of A generated by Z . Then ϕ is finitary, and thus AL_3'' implies a maximal ϕ -closed subset M of A including I such that $1 \notin M$. $K \Rightarrow$ **AC**: this implication is due to Hodges (see [4]).

In the conditions of statement AL_3'' , if moreover ϕ satisfies the exchange property, then G is a ϕ -hyperplane, so the statement **sH** is the restriction of statement AL_3'' to finitary matroids. It follows that **AC** \Rightarrow $AL_3'' \Rightarrow$ **sH**. \square

2.4.2. Axiom of choice and finitary matroids.

Definition 6. An operator ϕ on a set X is said to satisfy the *interpolation property* (for bases) if for every ϕ -independent subset I of X and every ϕ -generating subset G of X such that $I \subseteq G$, there exists a ϕ -basis B such that $I \subseteq B \subseteq G$.

A *B-matroidal* operator on a set X (see [3, p. 217], [13, p. 264]) is a matroidal operator ϕ on X such that for every subset Y of X , the suboperator ϕ_Y satisfies the interpolation property. Of course, every suboperator of a B-matroidal operator is B-matroidal.

Proposition 4 ([3, p. 219]). *Every B-matroidal operator is hyperplane-accessible and circuit-accessible.*

Proof. Higgs defines a “C-matroid” as a matroidal operator which is both hyperplane-accessible and circuit-accessible. He proves that every B-matroid is a “C-matroid”. \square

Proposition 5. (1) **AC** is equivalent to each of the following statements:

FB_0 : “Every finitary matroid satisfies the interpolation property”

FB_1 : “Every finitary matroid is a B-matroid”

FB_2 : “Every finitary matroid has a basis”

FB_3 (form [1A] of [7]): “Given a vector space E , every generating subset of E includes a basis of E .”

FB_4 : “Every connected graph has a spanning tree.”

- (2) The statement **H**: “Every nonempty finitary matroid has an hyperplane.” is equivalent to the statement “Every proper flat of a finitary matroid is included in a hyperplane.”

Proof. (1) **AC** \Rightarrow FB_0 . Given a finitary matroidal operator ϕ on a set X , a ϕ -independent subset I of X and a ϕ -generating subset G of X such that $I \subseteq G$, consider the set \mathcal{J} of ϕ -independent subsets J such that $I \subseteq J \subseteq G$. Then the poset (\mathcal{J}, \subseteq) is inductive (every chain

$(J_t)_{t \in T}$ of this poset is dominated by $\bigcup_{t \in T} J_t$, so with Zorn's lemma, one gets a maximal element B of the poset (\mathcal{J}, \subseteq) , and B is a ϕ -basis such that $I \subseteq B \subseteq G$. $FB_0 \Rightarrow FB_1$ follows from the previous point and the fact that every submatroid of a finitary matroid is finitary. $FB_1 \Rightarrow FB_2$ is trivial. $FB_2 \Rightarrow FB_3$: Consider a vector space E and a generating subset G of E . The operator ϕ induced by span on G is finitary and matroidal, and thus FB_2 implies a ϕ -basis, which is a basis of the vector space E included in G . $FB_3 \Rightarrow FB_4$: See [6]. $FB_4 \Rightarrow \mathbf{AC}$: See [5].

(2) Given a finitary matroidal operator ϕ on a set X , and a proper flat F of ϕ , the statement **sH** applied to the finitary operator ϕ^F provides a hyperplane Z of ϕ^F , and then $F \cup Z$ is a ϕ -hyperplane using Proposition 1. \square

3. HYPERPLANES IN MATROIDS AND THE AXIOM OF CHOICE

3.1. The operator associated to an antichain of nonempty sets.

Proposition 6. *Every circuit-accessible operator ϕ on a set X such that ϕ -circuits are symmetric satisfies the exchange property.*

Proof. Assume that Y, Z are two subsets of X and that for some $x \in X$, $x \in \phi(Y \cup Z)$ but $x \notin \phi(Y)$. Since ϕ is circuit-accessible, let C be a ϕ -circuit such that $x \in C \subseteq (Y \cup Z) \cup \{x\}$. Since the circuit C is symmetric, $x \in \phi(C \setminus \{x\})$, and thus $C \setminus \{x\}$ meets Z (else $C \setminus \{x\} \subseteq Y$ so $\phi(C \setminus \{x\}) \subseteq \phi(Y)$ whence $x \in \phi(Y)$, which is contradictory!). Let $z \in (C \setminus \{x\}) \cap Z$; then, since the circuit C is symmetric, $z \in \phi(C \setminus \{z\}) \subseteq \phi(((Y \cup Z) \cup \{x\}) \setminus \{z\})$. \square

Lemma 1. *Given an antichain \mathcal{C} of nonempty subsets of a set X , denote by ϕ the operator on X associating to each subset Y of X the set $Y \cup B$ where B is the set of elements $x \in X$ such that there exists $C \in \mathcal{C}$ satisfying $x \in C \subseteq Y \cup \{x\}$.*

- (1) ϕ is an operator on X .
- (2) Each element of \mathcal{C} is a symmetric ϕ -circuit.
- (3) \mathcal{C} is the set of ϕ -circuits, and the operator ϕ on X is circuit-accessible.
- (4) The operator ϕ satisfies the exchange property.
- (5) If elements of \mathcal{C} are finite sets, then the operator ϕ is finitary.

Proof. (1) By definition of ϕ , the mapping ϕ is expansive; moreover ϕ is isotonic since if $Y_1 \subseteq Y_2 \subseteq X$, for every $x \in X$ and every $C \in \mathcal{C}$ such that $x \in C \subseteq Y_1 \cup \{x\}$, then $x \in C \subseteq Y_2 \cup \{x\}$, thus $\phi(Y_1) \subseteq \phi(Y_2)$.

(2) If $C \in \mathcal{C}$, then, by definition of ϕ , for every $x \in C$, $x \in \phi(C \setminus \{x\})$, thus C is ϕ -dependent; moreover, the set $I := C \setminus \{x\}$ is ϕ -independent, else let $y \in I$ such that $y \in \phi(I \setminus \{y\})$; then there would exist $C' \in \mathcal{C}$ such that $y \in C' \subseteq I \subsetneq C$ which is contradictory since \mathcal{C} is an antichain.

(3) Let C be a ϕ -circuit. Then there exists $x \in C$ such that $x \in \phi(C \setminus \{x\})$. By definition of ϕ , let $C' \in \mathcal{C}$ such that $x \in C' \subseteq (C \setminus \{x\}) \cup \{x\} = C$; using Point (2), C' is a ϕ -circuit, and since the set of ϕ -circuits is an antichain, $C' = C$, and thus $C \in \mathcal{C}$. Since \mathcal{C} is the set of ϕ -circuits, it follows by definition of ϕ that the operator ϕ is circuit-accessible.

(4) This follows from Proposition 6 using the fact that ϕ is circuit-accessible and has symmetric circuits.

(5) Trivial since ϕ is circuit-accessible. \square

3.2. Binary matroids. A family \mathcal{C} of subsets of a set X is said to satisfy the *binary elimination property* if for all distinct elements C_1, C_2 of \mathcal{C} , the symmetric difference $C_1\Delta C_2$ is a union of pairwise disjoint elements of \mathcal{C} .

Theorem 1 ([12, Th 9.1.2 p . 344]). *Given a matroidal operator ϕ on a finite set X and denoting by \mathcal{C} the set of ϕ -circuits, the following statements are equivalent:*

- (1) *The operator ϕ is representable over the two-element field \mathbb{F}_2*
- (2) *The symmetric difference of any set of circuits is either empty or contains a circuit*
- (3) *\mathcal{C} satisfies the binary elimination property*
- (4) *For all distinct circuits $C_1, C_2 \in \mathcal{C}$, $C_1\Delta C_2$ is a (finite) union of circuits*
- (5) *For all distinct circuits $C_1, C_2 \in \mathcal{C}$, $C_1\Delta C_2$ includes a circuit.*

The following corollary holds in **ZF** for *infinite* finitary matroids.

Corollary 1. *Given a finitary matroidal operator ϕ on a (non necessarily finite) set X and denoting by \mathcal{C} the set of ϕ -circuits, the following statements are equivalent:*

- (1) *ϕ is \mathbb{F}_2 -representable*
- (2) *Every finite submatroid of ϕ is \mathbb{F}_2 -representable*
- (3) *\mathcal{C} satisfies the binary elimination property*
- (4) *For all distinct ϕ -circuits $C_1, C_2 \in \mathcal{C}$, $C_1\Delta C_2$ is a (finite) union of circuits*
- (5) *For all distinct ϕ -circuits $C_1, C_2 \in \mathcal{C}$, $C_1\Delta C_2$ includes a circuit.*
- (6) *The symmetric difference of any set of ϕ -circuits is either empty or contains a circuit.*

Proof. (1) \Rightarrow (2) is easy and (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) are consequences of Theorem 1. We prove (6) \Rightarrow (1). We consider the vector space $\mathbb{F}_2^{(X)}$ and its canonical basis $(e_x)_{x \in X}$ where for every $x \in X$, $e_x : X \rightarrow \mathbb{F}_2$ is the indicator function of the singleton $\{x\}$. Let V be the vector subspace of $\mathbb{F}_2^{(X)}$ generated by the set $\{v_C := \sum_{x \in C} x : C \text{ } \phi\text{-circuit}\}$. Let Q be the quotient vector space $\mathbb{F}_2^{(X)}/V$ and let $f : X \rightarrow Q$ be the quotient mapping $x \mapsto e_x + V$. The (finitary) matroidal operator ψ associated to f is isomorphic with ϕ since ϕ and ψ have the same circuits: given a subset C of X , C is a ψ -circuit iff $\sum_{x \in C} (e_x + V) = 0_Q$ and for every proper subset I of C , $\sum_{x \in I} (e_x + V) \neq 0_Q$; equivalently, $\sum_{x \in C} e_x \in V$ and for every proper subset I of C , $\sum_{x \in I} e_x \notin V$; this means that there exist ϕ -circuits C_1, \dots, C_m such that $C = C_1\Delta \dots \Delta C_m$ and that no proper subset I of C is the symmetric difference of a nonempty sequence of ϕ -circuits; using (2) it means that C is a ϕ -circuit. \square

Definition 7. A finitary matroid is said to be *binary* if it satisfies one of the previous equivalent statements.

3.3. The matroidal operator associated to a family of pairwise disjoint nonempty sets.

Definition 8. Given an integer $n \geq 2$, a family \mathcal{C} of subsets of a set X is said to satisfy the *n -binary elimination property* if for all distinct elements C_1, C_2 of \mathcal{C} , the symmetric difference $C_1\Delta C_2$ is a union of at most n elements of \mathcal{C} .

Theorem 2. *Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, consider the set $X = \bigcup_{i \in I} A_i \cup \{O\}$ where O is some set such that $O \notin \bigcup_{i \in I} A_i$. For every $i \in I$, let $C_i^1 := A_i \cup \{O\}$, and for all distinct elements $i, j \in I$, let $C_{i,j}^2 = A_i \cup A_j$. Let $\mathcal{C} := \{C_i^1 : i \in I\} \cup \{C_{i,j}^2 : i, j \in I; i \neq j\}$.*

- (1) \mathcal{C} is an antichain of nonempty subsets of X
- (2) \mathcal{C} satisfies the 2-binary elimination property.
- (3) Let ϕ be the operator associated to the antichain \mathcal{C} . Then ϕ is finitary iff for every $i \in I$, the set A_i is finite.
- (4) The operator ϕ is idempotent (and thus matroidal).

Proof. Points (1), (2) and (3) are easy to check.

(4) Let Z be a subset of X . Let I_1 be the set of elements $i \in I$ such that $A_i \setminus Z$ has at least two elements. Let $I_2 = I \setminus I_1$. If $O \in Z$ then $\phi(Z) = Z \cup \bigcup_{i \in I_2} A_i$ and thus, $\phi(\phi(Z)) = \phi(Z)$. If $O \notin Z$ and if there exists $i_0 \in I_2$ such that $A_{i_0} \subseteq Z$, then $\phi(Z) = Z \cup \{O\} \cup \bigcup_{i \in I_2} A_i$ and thus, $\phi(\phi(Z)) = \phi(Z)$; if $O \notin Z$ and if for every $i \in I_2$, $A_i \setminus Z$ has exactly one element, then $\phi(Z) = Z$ and thus $\phi(\phi(Z)) = \phi(Z)$. \square

Definition 9. In the conditions of the previous theorem, we call ϕ the *matroidal operator associated to O and the family $(A_i)_{i \in I}$* .

Definition 10. Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, a *selector* for this family is a subset S of $\bigcup_{i \in I} A_i$ such that for every $i \in I$, $S \cap A_i$ has at most one element; the selector S is said to be *total* if for every $i \in I$, $S \cap A_i$ has exactly one element.

Theorem 3. *Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, consider the set $X = \bigcup_{i \in I} A_i \cup \{O\}$ where O is some set such that $O \notin \bigcup_{i \in I} A_i$. Let ϕ be the matroidal operator associated to O and the family $(A_i)_{i \in I}$.*

- (1) A subset L of X is ϕ -independent iff either ($O \in L$ and $\forall i \in I A_i \not\subseteq L$), or ($O \notin L$ and there exists at most one element $i_0 \in I$ such that $A_{i_0} \subseteq L$).
- (2) A subset G of X is ϕ -generating iff $S := (\bigcup_{i \in I} A_i) \setminus G$ is a selector for the family $(A_i)_{i \in I}$, which is not total if $O \notin G$.
- (3) A subset B of X is a ϕ -basis iff there exists a total selector S for the family $(A_i)_{i \in I}$ such that $B = ((\bigcup_{i \in I} A_i) \setminus S) \cup \{a\}$ where a is some element of $\{O\} \cup S$.
- (4) A proper subset F of X is a ϕ -flat iff ($O \in F$ or $\exists i_0 \in I A_{i_0} \subseteq F$) $\Rightarrow \forall i \in I A_i \setminus F$ is not a singleton
- (5) A subset H of X is a ϕ -hyperplane iff $H = (\bigcup_{i \in I} A_i) \setminus S$ where S is a total selector for the family $(A_i)_{i \in I}$, or $H = X \setminus \{x, y\}$ where $i_0 \in I$ and $x, y \in A_{i_0}$ with $x \neq y$.
- (6) The following statements are equivalent:
 - (a) The operator ϕ is hyperplane-accessible.
 - (b) Every family $(B_i)_{i \in I}$ such that for every $i \in I$, $\emptyset \subsetneq B_i \subseteq A_i$ has a total selector.
 - (c) The operator ϕ is B -matroidal.
 - (d) The operator ϕ satisfies the interpolation property for bases

Proof. Points (1), (2), (3), (4) and (5) are consequences of the definitions. We prove Point (6).

(a) \Rightarrow (b): Given a family $(B_i)_{i \in I}$ such that for every $i \in I$, $\emptyset \subsetneq B_i \subseteq A_i$, consider the proper ϕ -flat subset $F := \bigcup_{i \in I} (A_i \setminus B_i)$ of X ; since ϕ is hyperplane-accessible, let H be a ϕ -hyperplane such that $F \subseteq H$ and $O \notin H$; then $\bigcup_{i \in I} (A_i \setminus H)$ is a total selector for the family $(B_i)_{i \in I}$.

(b) \Rightarrow (c): Let Y be a subset of X . Let L be a ϕ -independent subset of Y and let G be a ϕ_Y -generating subset of Y such that $L \subseteq G$. Let $J := \{i \in I : Y \cap A_i \neq \emptyset\}$. Let $J_1 := \{i \in J : A_i \not\subseteq G\}$. Let $J_2 = \{i \in J : A_i \subseteq G \text{ and } A_i \not\subseteq L\}$. Let $J_3 = \{i \in J : A_i \subseteq L\}$: notice that $J = J_1 \cup J_2 \cup J_3$ and that J_1 , J_2 and J_3 are pairwise disjoint. For each $i \in J_1$,

let x_i be the element of $A_i \setminus G$. Using (b), consider a choice function $(x_i)_{i \in J_2}$ for the family $(A_i \setminus L)_{i \in J_2}$. If J_3 is nonempty, then J_3 has a unique element i_0 and let $x_{i_0} = O$ if $O \in Y$. If $O \in Y$, let $B := Y \setminus \{x_i : i \in J\}$, and if $O \notin Y$, let $B := Y \setminus \{x_i : i \in J_1 \cup J_2\}$. Then B is a ϕ_Y -basis such that $L \subseteq B \subseteq G$.

(c) \Rightarrow (d) follows from the definitions.

(d) \Rightarrow (a): Let F be a proper subset of X which is a ϕ -flat and let $x \in X \setminus F$. If $x = O$, then for every $i \in I$, $A_i \setminus F$ has at least one element (else O would belong to F), and thus F is ϕ -independent; then $G = \bigcup_{i \in I} A_i$ is ϕ -spanning and $F \subseteq G$: using the interpolation property, there exists a ϕ -basis B such that $F \subseteq B \subseteq G$; it follows that there exists a total selector S for $(A_i)_i$ and an element $i_0 \in I$ such that $B = A_{i_0} \cup (\bigcup_{i \neq i_0} A_i) \setminus S$; let $x_{i_0} \in A_{i_0} \setminus F$; then $H = B \setminus \{x_{i_0}\}$ is a ϕ -hyperplane including F such that $O \notin H$. If $x \neq O$, then let i_0 be the element of I such that $x \in A_{i_0}$. If $A_{i_0} \setminus F$ contains an element y distinct from x , then $H := X \setminus \{x, y\}$ is a ϕ -hyperplane including F and not containing x . If $A_{i_0} \setminus F = \{x\}$, then for every $i \in I \setminus \{i_0\}$, $A_i \setminus F \neq \emptyset$ and $O \notin F$ (else x would belong to F); using the independent set $L = F \setminus \{O\}$ and the generating set $G = \bigcup_i A_i$, consider a ϕ -basis B such that $L \subseteq B \subseteq G$; then B yields a selector S for the family $(A_i \setminus F)_{i \in I}$ (and thus $x \in S$). It follows that $H := (\bigcup_{i \in I} A_i) \setminus S$ is a ϕ -hyperplane including F . \square

Corollary 2. *AC is equivalent to the following statement: “For every nonempty family $(A_i)_{i \in I}$ of pairwise disjoint nonempty sets, and for every set O such that $O \notin \bigcup_{i \in I} A_i$, the matroidal operator associated to O and the family $(A_i)_{i \in I}$ has an hyperplane not containing O .”*

3.4. The axiom sH implies AC^{fin}. We denote by \mathbf{sH}_{bep} the axiom **sH** restricted to finitary matroids with the binary elimination property. For every natural number $n \geq 2$, we denote by \mathbf{sH}_{bep_n} the axiom **sH** restricted to finitary matroids with the n -binary elimination property. We denote by \mathbf{H}_{bep} (resp. \mathbf{H}_{bep_n}) the axiom **H** restricted to finitary matroids with the binary elimination property (resp. n -binary elimination property).

Remark 8. The matroidal operator associated to a family $(A_i)_{i \in I}$ of pairwise finite disjoint nonempty sets satisfies the 2-binary elimination property (and hence is binary).

Theorem 4. *In ZF, $\mathbf{sH} \Rightarrow \mathbf{sH}_{bep} \Rightarrow \mathbf{sH}_{bep_2} \Rightarrow \mathbf{AC}^{\text{fin}}$.*

Proof. Notice that \mathbf{AC}^{fin} is equivalent to the statement “For every nonempty family $(A_i)_{i \in I}$ of pairwise disjoint finite nonempty sets, $\prod_{i \in I} A_i$ is nonempty.”: given a family $(A_i)_{i \in I}$ of nonempty sets, consider the family $(A_i \times \{i\})_{i \in I}$. Given a nonempty family $(A_i)_{i \in I}$ of pairwise disjoint finite nonempty sets, consider the set $X = \bigcup_{i \in I} A_i \cup \{O\}$ where $O \notin \bigcup_{i \in I} A_i$, and consider the finitary matroidal operator ϕ on X associated to the family $(A_i)_{i \in I}$ (see Theorem 2). Since ϕ has no loops, $\phi(\emptyset) = \emptyset$, so \emptyset is a proper flat of ϕ and thus, \mathbf{sH}_{bep_2} implies a ϕ -hyperplane H not containing O . It follows from Theorem 3 that for each $i \in I$, $A_i \setminus H$ is a singleton $\{x_i\}$ where $(x_i)_{i \in I}$ is a choice function for the family $(A_i)_{i \in I}$. \square

Question 1. We have shown that $\mathbf{AC} \Rightarrow \mathbf{sH}_1 \Rightarrow \mathbf{sH} \Rightarrow \mathbf{sH}_{bep} \Rightarrow \mathbf{sH}_{bep_2} \Rightarrow \mathbf{AC}^{\text{fin}}$ and of course $\mathbf{sH} \Rightarrow \mathbf{H} \Rightarrow \mathbf{H}_{bep} \Rightarrow \mathbf{H}_{bep_2}$. Does \mathbf{sH}_{bep} imply **sH**? Does **H** imply \mathbf{AC}^{fin} ? Does **H** imply **sH**?

4. GRAPHIC MATROIDS AND THE FINITE AXIOM OF CHOICE

4.1. Strong and weak elimination properties.

Definition 11. A family \mathcal{C} of subsets of a set X is said to satisfy the *elimination property* if for all distinct elements $C_1, C_2 \in \mathcal{C}$, for every $x \in C_1 \cap C_2$, there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2$ and $x \notin C_3$. The family \mathcal{C} is said to satisfy the *strong elimination property* if for every elements $C_1, C_2 \in \mathcal{C}$, for every $x \in C_1 \cap C_2$ and every $y \in C_1 \setminus C_2$, then there exists $C_3 \in \mathcal{C}$ such that $y \in C_3 \subseteq C_1 \cup C_2$ and $x \notin C_3$.

Notice that the binary elimination property implies the strong elimination property, which in turn implies the elimination property.

Notation 1. For every finite set F , we denote by $|F|$ the cardinal of F .

The following result is classical:

Proposition 7 ([15], [2]). *Let \mathcal{C} be an antichain of nonempty finite subsets of a set X , and let ϕ be the (finitary) operator associated to \mathcal{C} . If \mathcal{C} satisfies the weak elimination property, then:*

- (1) \mathcal{C} satisfies the strong elimination property.
- (2) The operator ϕ is a closure operator.
- (3) The operator ϕ is matroidal.

Proof. (1) See [15, Theorem 2 p. 24] or [2, Lemme 4 p. 17].

(2) See [2, Théorème 8 p. 18]. We sketch the proof. Let A be a subset of X and let $x \in \phi(\phi(A))$. Let us show that $x \in \phi(A)$. Let $C \in \mathcal{C}$ such that $x \in C \subseteq \phi(A) \cup \{x\}$, and such that $C \cap (\phi(A) \setminus A)$ is minimal. If $(C \setminus \{x\}) \cap (\phi(A) \setminus A)$ is nonempty, let $y \in (C \setminus \{x\}) \cap (\phi(A) \setminus A)$; since $y \in \phi(A)$, let $C_1 \in \mathcal{C}$ such that $y \in C_1 \subseteq A \cup \{y\}$. Using the strong elimination property, let $C_2 \in \mathcal{C}$ such that $x \in C_2 \subseteq (C \cup C_1) \setminus \{y\}$: then $|C_2 \cap (\phi(A) \setminus A)| < |C \cap (\phi(A) \setminus A)|$, which contradicts the minimality of $C \cap (\phi(A) \setminus A)$. It follows that $(C \setminus \{x\}) \cap (\phi(A) \setminus A) = \emptyset$ and thus, $(C \setminus \{x\}) \subseteq A$ so $x \in \phi(A)$.

(3) Using Lemma 1, \mathcal{C} is the set of ϕ -circuits and ϕ satisfies the exchange property, whence the closure operator ϕ is a matroidal operator on X . \square

4.2. The binary matroid associated to a multigraph.

4.2.1. *Multigraphs.* A *multigraph* on a set V is given by a mapping $f : X \rightarrow [V]^1 \cup [V]^2$, where, for each natural number $n \geq 1$, $[V]^n$ is the set of n -element subsets of V . Elements of X such that $f(x) \in [V]^1$ are called *loops* of the multigraph.

Denoting by $(e_v)_{v \in V}$ the canonical basis of the vector space $\mathbb{F}_2^{(V)}$, the *incidence matrix* of the multigraph f is the mapping $\tilde{f} : X \rightarrow \mathbb{F}_2^{(V)}$ such that for every $x \in X$, $\tilde{f}(x)$ is $0_{\mathbb{F}_2^{(V)}}$ if $f(x) \in [V]^1$, and $\tilde{f}(x) = e_{v_1} + e_{v_2}$ if $f(x)$ is the two-element sets $\{v_1, v_2\}$. The *matroid associated to the multigraph f* is the (binary) matroidal operator on X associated to the incidence matrix \tilde{f} . Loops of this matroid correspond to loops of the multigraph. A matroidal operator which is isomorphic with the (binary hence finitary) matroid associated to a multigraph is said to be *graphic*.

4.2.2. *Simple graphs.* A simple graph on a set V is a binary relation R on V which is irreflexive (for every $x \in V$, $x \not R x$) and symmetric (for every $x, y \in V$, $x R y \Rightarrow y R x$). Elements of V are called the *vertices* of the graph, and pairs $\{x, y\}$ of vertices such that $x R y$ are the edges of the simple graph. A simple graph on a set V with set E of edges is also denoted by (V, E) . A (partial) subgraph of a simple graph G on a set X with set of edges E

is a simple graph (X', E') such that $X' \subseteq X$ and $E' \subseteq E$. Two graphs (V_1, E_1) and (V_2, E_2) are *isomorphic* when there exists a bijection $f : V_1 \rightarrow V_2$ which respects the edges.

Notation 2. Given some integer $n \geq 3$, we denote by C_n the simple graph on $\mathbb{Z}/n\mathbb{Z} = \{0, \dots, n-1\}$ with set of edges $E_n = \{\{i, i+n-1\} : i \in \mathbb{Z}/n\mathbb{Z}\}$, where $+_n$ is the additive law on $\mathbb{Z}/n\mathbb{Z}$;

Given some integer $n \geq 3$, a simple graph is a *n-cycle* if it is isomorphic with the simple graph C_n . Given a simple graph $G = (V, E)$, a *cycle of the graph G* is a (partial) subgraph of G which is isomorphic with a *n-cycle* for some natural number $n \geq 3$.

4.2.3. *Graphic matroids.* Given a set V and a multigraph $f : X \rightarrow [V]^1 \cup [V]^2$, if $E = f[X] \cap [V]^2$, then (V, E) is called the simple graph underlying the multigraph f . Reciprocally, every simple graph (V, E) underlies the multigraph $\text{id}_E : E \rightarrow E$ on V .

Proposition 8 ([12, Proposition 1.1.7]). *Let $G = (V, E)$ be a simple graph. Let \mathcal{C}_G be the set of (finite) subsets F of E such that F is the set of edges of a cycle of G . Then \mathcal{C}_G is the set of circuits of the (binary) matroidal operator \mathcal{M}_G associated to the multigraph $\text{id}_E : E \rightarrow E$.*

Proof. Let W be the \mathbb{F}_2 -vector space $\mathbb{F}_2^{(V)}$. For every $v \in V$ we denote by e_v the v -th vector of the canonical basis of W . We identify each edge $\{a, b\}$ of G with the vector $e_a + e_b$ of W . A subset F of E is a circuit of the matroid \mathcal{M}_G iff F is nonempty, $\sum_{e \in F} e = 0_W$ and for every nonempty proper subset G of F , $\sum_{e \in G} e \neq 0_W$; replacing each element $e = \{a, b\}$ of E by $e_a + e_b$, this means that $F \neq \emptyset$, every vertex of the subgraph $(\cup F, F)$ has even degree, but for every proper subset G of F , some vertex of the subgraph $(\cup G, G)$ has an odd degree; this means that F is a nonempty finite union of cycles of G , and that no proper subset of F is a cycle of G ; equivalently, F is a cycle of the graph G . \square

Remark 9. If $f : X \rightarrow [V]^1 \cup [V]^2$ is a multigraph on a set V , if \mathcal{M}_f is the matroid associated to the multigraph f , loops of \mathcal{M}_f are the singletons $\{x\}$ such that $x \in X$ and $f(x)$ is a singleton; circuits of cardinal two of \mathcal{M}_f are the pairs $\{x, y\}$ of distinct elements of X such that $f(x) = f(y)$. Given some natural number $n \geq 3$, then the *n-circuits* of \mathcal{M}_f are the *n-element subsets* $\{x_1, \dots, x_n\}$ of X such that $\{f(x_1), \dots, f(x_n)\}$ is the set of edges of a *n-cycle* of the underlying simple graph of f .

4.3. An equivalent of AC^{fin} in terms of graphic matroids.

Theorem 5. *The following statements are equivalent:*

- (1) AC^{fin}
- (2) *For every family $(A_i)_{i \in I}$ of pairwise disjoint nonempty finite sets with at least two elements, the (binary hence finitary) matroid associated to this family is graphic.*

Proof. (1) \Rightarrow (2) Let $(A_i)_{i \in I}$ be an infinite family of pairwise disjoint nonempty finite sets, such that for every $i \in I$, $n_i := |A_i| \geq 2$. Let \mathcal{M} be the matroid associated to the family $(A_i)_{i \in I}$: the underlying set of \mathcal{M} is $M := \bigcup_{i \in I} A_i \cup \{O\}$ where $O \notin \bigcup_{i \in I} A_i$. We consider a family $(V_i)_{i \in I}$ of pairwise disjoint linearly ordered finite sets such that for each $i \in I$, $|V_i| = n_i - 1$. We also consider two distinct elements a and b not belonging to $\bigcup_{i \in I} V_i$, and we define the set $V := \{a, b\} \cup \bigcup_{i \in I} V_i$. Since each V_i is linearly ordered, for each $i \in I$, we consider a graph G_i on $V_i \cup \{a, b\}$ which is a n_i -cycle and such that $\{a, b\}$ is an edge of this graph: we denote by E_i the set of edges of G_i which are not equal to the edge $\{a, b\}$ of G_i .

We consider the simple graph G on V which admits $E := \bigcup_{i \in I} E_i \cup \{a, b\}$ as set of edges (see Figure 1). Notice that every finite subgraph of G is planar. We denote by \mathcal{G} the matroid on E associated to the graph G . Using \mathbf{AC}^{fin} , we consider a family $(f_i)_{i \in I}$ such that for every $i \in I$, $f_i : E_i \rightarrow A_i$ is a bijection. It follows that $f := \bigcup_{i \in I} f_i$ is a bijection from $\bigcup_{i \in I} E_i$ to $\bigcup_{i \in I} A_i$ and we extend it into a bijection from E to M . Then the bijection f respects circuits of \mathcal{M}_G and \mathcal{M} thus the matroid \mathcal{M} is graphic.

(2) \Rightarrow (1) Let $(A_i)_{i \in I}$ be a family of pairwise disjoint nonempty finite sets with at least two

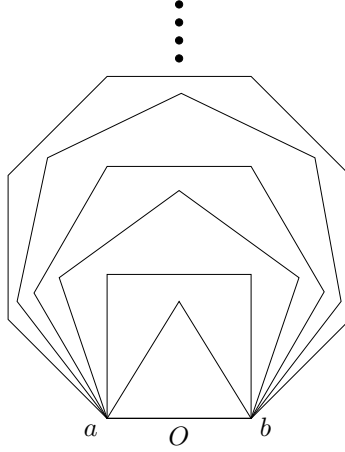


FIGURE 1. The graph G associated to the matroid \mathcal{M}

elements. Let \mathcal{M} be the finitary matroid on $\{O\} \cup \bigcup_{i \in I} A_i$ associated to this family. Let $G = (V, E)$ be a graph such that \mathcal{M} is the graphic matroid associated to G . Let a, b be the two extremities of the edge O of G . Then, for every $i \in I$, $A_i \cup \{O\}$ is the set of edges of a cycle of the graph G : let e_i be the the unique edge of A_i which is incident to the vertex a . Then $(e_i)_{i \in I}$ is a choice function for the family $(A_i)_{i \in I}$. \square

Consider the following well known consequences of \mathbf{AC} imply \mathbf{AC}^{fin} :

MG₁: “For every binary matroid \mathcal{M} , if every finite minor of \mathcal{M} is graphic then \mathcal{M} is graphic”.

MG₂: “For every binary matroid \mathcal{M} , if every finite submatroid of \mathcal{M} is graphic and planar then \mathcal{M} is graphic”.

MG₃: “For every binary matroid \mathcal{M} , if every finite minor of \mathcal{M} is graphic and planar then \mathcal{M} is graphic”.

Notice that both statements **MG₁** and **MG₂** imply **MG₃**. Moreover, every finite minor of the binary matroid used in the proof of Theorem 5 is graphic and planar, and thus, **MG₃** imply \mathbf{AC}^{fin} .

Question 2. Does \mathbf{AC}^{fin} or **sH** imply one of the statements **MG₁**, **MG₂** or **MG₃**?

Question 3. Is the following statement provable in **ZF**: “Every (infinite) graphic matroid is hyperplane-accessible.”

In the diagram in Figure 2, we add the statement **D_Q** which implies the statement $\mathbf{AC}^{\mathbb{Z}}$: “Every family of posets isomorphic with the linear order \mathbb{Z} has a nonempty product.” (see

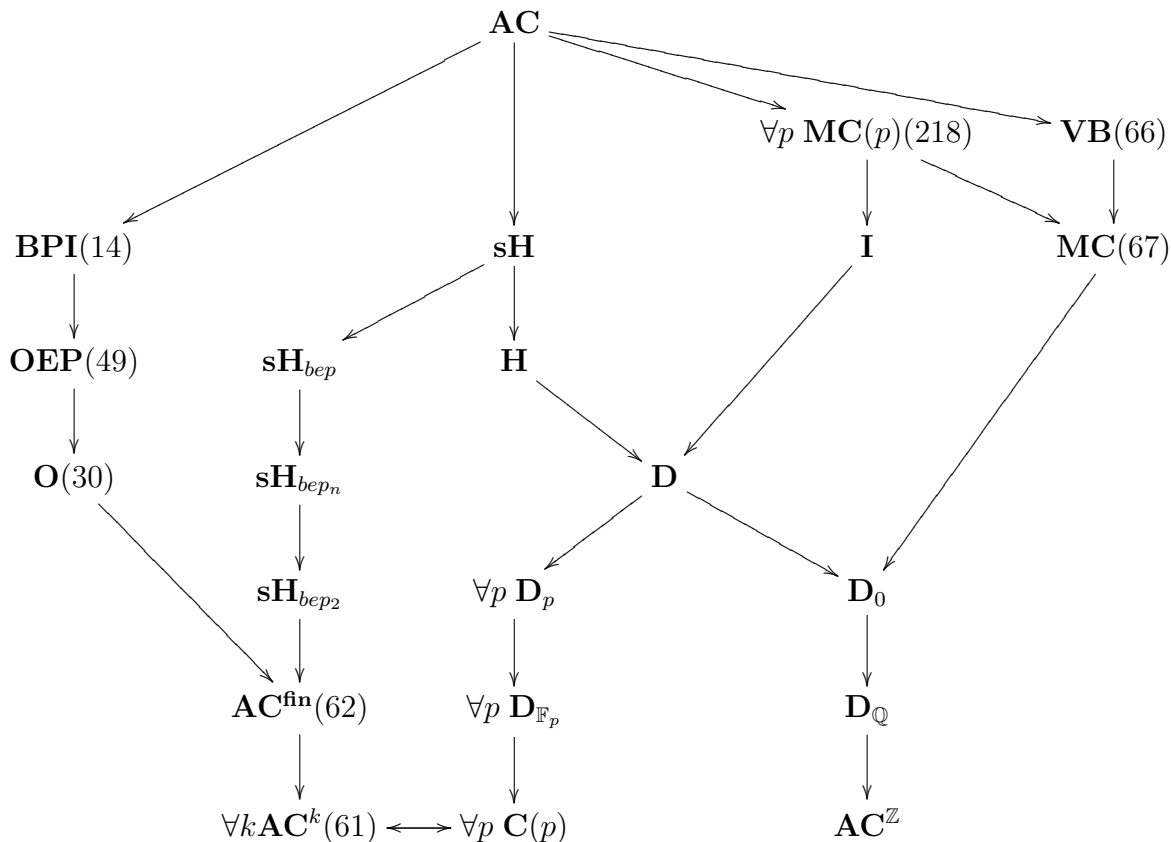


FIGURE 2. Summary diagram of the axioms

[10, Theorem 4]). We also add the statement \mathbf{D}_0 (*resp.* \mathbf{D}_p) which is \mathbf{D} restricted to vector spaces over a commutative field \mathbb{K} of characteristic 0 (*resp.* p).

Question 4. The statements \mathbf{BPI} (“Every non trivial Boolean algebra has a maximal ideal”), \mathbf{OEP} (“Every partial order on a set X can be extended into a linear order on X ”) and \mathbf{O} (“On every set X there exists a linear order”) (see forms 14, 49 and 30 of [7]) are well known consequences of \mathbf{AC} which are stronger than \mathbf{AC}^{fin} . Are there implications between one of them and \mathbf{H} or \mathbf{sH} or \mathbf{sH}_{bep} or $\mathbf{sH}_{\text{bep}_n}$ for some integer $n \geq 2$?

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