Error analysis for the finite element approximation of the Brinkmann-Darcy-Forchheimer model for porous media with mixed boundary conditions

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Abstract

This paper deals with the finite element approximation of the Darcy-Brinkman-Forchheimer equation, involving a porous media with spatially-varying porosity, with mixed boundary condition such as inhomogeneous Dirichlet and traction boundary conditions. We first prove that the considered problem has a unique solution if the source terms are small enough. The convergence of a Taylor-Hood finite element approximation using a finite element interpolation of the porosity is then proved under similar smallness assumptions. Some optimal error estimates are next obtained when assuming the solution to the Darcy-Brinkman-Forchheimer model are smooth enough. We end this paper by providing a fixed-point method to solve iteratively the discrete non-linear problems and with some numerical experiments to make more precise the smallness assumptions on the source terms and to illustrate the theoretical convergence results.

Keywords:
Darcy-Brinkman-Forchheimer model, Mixed boundary conditions, Finite element, Porous media.

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1. Introduction

The Darcy-Brinkman-Forchheimer (DBF) model for porous media is obtained from the incompressible Navier-Stokes equation in a porous media through volume averaging. The later are then completed with closure models for the unknown terms arising in the volume averaged equations. We refer to [45, 15, 36, 47, 48] for the physical modeling of fluid flows in porous media based on volume averaging of the Navier-Stokes equations and the drivation of the DBF model.

Regarding the mathematical study of the steady-state DBF equation, Kaloni & Guo [26] studied the case where the convective non-linear terms vanishes and show that this problem has a solution for inhomogeneous Dirichlet boundary conditions which is unique for small enough source terms. The existence and uniqueness of solution to the full non-linear problem with inhomogeneous

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Dirichlet boundary condition can be found in [41, 40]. It is worth noting that the results from [26, 41, 40] have been obtained with similar techniques. The latter are based on the study of a finite dimensional approximation of the non-linear variational problem whose well-posedness comes from [17, p. 597, Lemma IX.3.1] (see also [44, p. 164, Lemma 1.4]) and next proving that the limit satisfy the variational formulation of the DBF model. The existence of solution for homogeneous Dirichlet boundary conditions when considering a generalized Forcheimer term have been obtained in [46] using the Leray-Schauder theorem and the uniqueness again holds for source terms that are small enough. We emphasize that the need of small data to ensure well-posedness of some non-linear PDE is quite classical and is also required when dealing with incompressible Navier-Stokes equation with homogeneous Dirichlet conditions (see e.g. [44, 17]) or fluid-porous media interface problems [13, 19].

In this paper, we are interested in the finite element approximation of the Darcy-Brinkmann-Forchheimer equation with mixed boundary conditions since such setting is involved in many physical applications such as those from [45, 1] (see also [41, p. 39, Section 2.4] and [40, Section 3]). As a result, we first need to study the existence and uniqueness of solution to the DBF problem in this setting. Since the model we are interested in is derived from Navier-Stokes equations, we recall below some works dealing with the existence and uniqueness of solution to incompressible steady-state Navier-Stokes equations in the case of mixed boundary conditions. The regularity of the solutions for three dimensional Lipschitz domain with homogeneous mixed boundary conditions involving Dirichlet on some part of the boundary and, on the other part, a vanishing normal trace together with either zero tangential part of the normal stress tensor or the curl of the velocity can be found in [14]. In the case of polyhedral domains with more general inhomogenous mixed boundary conditions and non-divergence-free velocity fields, one can found existence, uniqueness and regularity results in [28] (see also [27] for similar results for Stokes flows). Finally, the existence and uniqueness of solution to Navier-Stokes equations with mixed inhomogeneous conditions have been considered in [35] where the inhomogeneous Dirichlet boundary conditions have been handled thanks to the introduction of an additional variable. The latter thus yields a non-linear saddle-point problem very similar to the standard weak-formulation of the incompressible Navier-Stokes equation but with a continuous bilinear form b that is different from the usual one since it does not only involves the divergence of the velocity but also a surface term taking into account the inhomogeneous Dirichlet condition. Regarding the previously mentioned results, even if the steady state incompressible Navier-Stokes system and the Darcy-Brinkman-Forchheimer model have a similar structure, we cannot deduce from them the existence and uniqueness of a solution to the DBF problem with mixed boundary condition. A first part of the present paper is then going to be dedicated to proved existence and uniqueness of solution to the DBF problem with mixed boundary conditions.

Regarding the convergence of a finite element approximation to the DBF problem, there is actually a large literature on the analysis of mixed finite element method applied to the (generalized) Darcy-Forchheimer model (see e.g.

[33, 24, 39, 25, 37]). Nevertheless, compared to the Darcy-Forchheimer model, the DBF equation also involves a nonlinear convective term and a Laplacian of the velocity and seems, to the best of our knowledge, to have been less studied. We can still refer to [34] where the convergence of finite difference method on a staggered grid applied to the unsteady DBF coupled to a solute transport equation have been obtained. For the steady-state DBF, the case of inhomogeneous Dirichlet boundary condition have been studied in [41] where optimal error estimates have been obtained for smooth enough solution. We would like to emphasize that all the aforementioned convergence results are obtained either in the case of a homogeneous porous media or without considering a discrete version of the porosity field in the discrete problem. Note however that having a finite element approximation of the porosity has its own interest since the latter is usually used in the numerical computations. Another advantage of having a finite element formulation with discrete parameters is when one deals with so-called topology optimization problems (see e.g. [4, 42, 5, 38]) or parametric optimization problem [6, Chapter VI]. Indeed, using discretization of these parameters are nedded to define discrete optimization problem and to prove the convergence of discrete optimal solution toward the continuous optimal solutions.

In this paper, we are interested in proving the existence and uniqueness of solution to the Darcy-Brinkman-Forchheimer model with a spatially-varying porosity as well as the convergence of a finite element approximation involving a discrete porosity. The paper is thus organized as follow: we begin to prove an existence and uniqueness result for a DBF model with mixed boundary conditions. A finite element method using a discrete porosity is then investigated for which we prove the convergence as well as optimal error estimates for smooth enough solutions. We study next the convergence of a fixed point iteration used to solve the discrete non-linear problem. Since all our results are obtained by assuming that the source term are small enough, we end this paper with some numerical simulations to get more precise estimates on how small these source term have to be in order for the finite element method to be convergent.

2. Steady-state Darcy-Brinkman-Forchheimer model for porous media

We consider a viscous flow inside a porous medium embedded inside a computational domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ which is assumed to be a bounded open set with Lipschitz boundary with outward unitary normal $\vec{n}$. We assume the porous media has a spatially varying porosity $\varepsilon : x \in \Omega \mapsto \varepsilon(x) \in (0, 1]$ and that it is modeled by the Darcy-Brinkman-Forchheimer equation. The latter can be found in e.g. [20, 29, 46] and in dimensionless form reads as follows

$$\left\{ \begin{array}{rcll} -\mathrm{div}\left(2\mathrm{Re}^{-1}\varepsilon S(\vec{u}) - \varepsilon\vec{u}\otimes\vec{u}\right) + \varepsilon\nabla p + \alpha(\varepsilon)\vec{u} + \beta(\varepsilon)\vec{u}|\vec{u}| & = & \varepsilon\vec{f}, & \text{in } \Omega, \\ \mathrm{div}\,(\varepsilon\vec{u}) & = & 0, & \text{in } \Omega, \end{array} \right. \tag{1}$$

where Re is the Reynolds number and $\vec{f}$ is an external force field (e.g. gravity). The tensor $S(\vec{u})$ is the symmetric part of the Jacobian matrix of the velocity field $\vec{u}$. We emphasize that one always has $\alpha(1) = \beta(1) = 0$ and thus the standard Navier-Stokes equation is recovered for $\varepsilon = 1$.

Remark 1. *Using the formula*

$$\operatorname{div}(\vec{u} \otimes \vec{v}) = \operatorname{div}(\vec{u})\vec{v} + (\vec{u} \cdot \nabla)\, \vec{v},$$

together with the incompressibility condition $\operatorname{div}(\varepsilon \vec{u}) = 0$*, the non-linear term can be written as*

$$\operatorname{div}(\varepsilon \vec{u} \otimes \vec{u}) = \operatorname{div}(\varepsilon \vec{u})\vec{u} + \varepsilon\, (\vec{u} \cdot \nabla)\, \vec{u} = \varepsilon\, (\vec{u} \cdot \nabla)\, \vec{u}.$$

We consider the following set of boundary conditions

$$\begin{cases} \vec{u} = 0 & \text{on } \Gamma_w, \\ \vec{u} = \vec{u}_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ \varepsilon \left(2\text{Re}^{-1} S(\vec{u}) - p\right) \vec{n} = 0 & \text{on } \Gamma_{\text{out}}, \end{cases} \tag{2}$$

where we assumed that $\partial\Omega = \Gamma = \Gamma_w \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ where each part correspond respectively to the walls, the inlet and the outlet. We also assume that

$$\Gamma_{\text{out}} \cap \Gamma_{\text{in}} = \emptyset.$$

In this paper, we assumed that the porosity, the Darcy and Forchheimer terms satisfy the next set of assumptions

$$\begin{aligned} &\varepsilon \in L^\infty(\Omega) \cap W^{1,4}(\Omega) \text{ and } 0 < \varepsilon_0 \le \varepsilon(x) \le 1 \text{ a.e. in } \Omega \\ &s \in [\varepsilon_0, 1[\mapsto \alpha(s) \in \mathbb{R}^+ \text{ and } s \in [\varepsilon_0, 1[\mapsto \beta(s) \in \mathbb{R}^+ \text{ are differentiable.} \\ &s \in [\varepsilon_0, 1[\mapsto \alpha'(s) \in \mathbb{R}^+ \text{ and } s \in [\varepsilon_0, 1[\mapsto \beta'(s) \in \mathbb{R}^+ \text{ are bounded.} \end{aligned} \tag{3}$$

Assumptions (3) are satisfied by many example of Darcy and Forchheimer coefficients one may find in the literature (see e.g. [3, 45, 15, 40]).

2.1. Well-posedness of the Darcy-Brinkman-Forchheimer problem

We are working with boundary condition that are defined on parts of $\partial\Omega$ and we then need some special functional spaces to handle them properly. For any bounded open set $\mathcal{O}$ with Lipschitz boundary, we note $\Gamma_c \subset \partial\mathcal{O}$ a part of the boundary. The trace space $H^{1/2}_{00}(\Gamma_c)$ is defined in [31, Theorem 11.7] (see also e.g. [30, 32] where some properties are recalled) and can be obtained as the completion of smooth function with compact support in Γ_c with respect to the norm

$$\|\mu\|^2_{H^{1/2}(\Gamma_c)} := \|\mu\|^2_{L^2(\Gamma_c)} + \int_{\Gamma_c} \int_{\Gamma_c} \frac{|\mu(x) - \mu(y)|^2}{|x - y|^d}\, dx\, dy.$$

Denoting by $E_0\mu$ the extension by 0 outside Γ_c, we have that any $\mu \in H^{1/2}_{00}(\Gamma_c)$ satisfy $E_0\mu \in H^{1/2}(\partial\mathcal{O})$ with

$$\|E_0(\mu)\|_{H^{1/2}(\partial\mathcal{O})} \le C\, \|\mu\|_{H^{1/2}_{00}(\Gamma_c)},$$

for a generic constant $C > 0$. As a result, we have the equivalent more usable definition of this trace space

$$H_{00}^{1/2}(\Gamma_{\mathrm{c}}) := \left\{ \mu \in H^{1/2}(\Gamma_{\mathrm{c}}) \mid E_0\mu \in H^{1/2}(\partial\mathcal{O}) \right\}.$$

Moreover, the linear application $E_0 : \mu \in H_{00}^{1/2}(\Gamma_{\mathrm{c}}) \mapsto E_0(\mu) \in H^{1/2}(\partial\mathcal{O})$ is continuous. We finally emphasize that if $H_{00}^{1/2}(\Gamma_{\mathrm{c}})$ is endowed with the next norm

$$\|\mu\|^2_{H_{00}^{1/2}(\Gamma_{\mathrm{c}})} := \|\mu\|^2_{H^{1/2}(\Gamma_{\mathrm{c}})} + \int_{\Gamma_{\mathrm{c}}} \frac{|\mu(s)|^2}{\operatorname{dist}(s, \partial\Gamma_{\mathrm{c}})}\, ds,$$

then it is a Banach space.

We are now in position to give the weak formulation to Problem (1,2). In order to take into account the inhomogeneous Dirichlet boundary condition, we introduce the following Hilbert spaces

$$\mathbf{X}_1 := \left\{ \vec{v} \in H^1(\Omega)^d \mid \vec{v}|_{\Gamma_w = 0} \right\}, \ \mathbf{X} := \left\{ \vec{v} \in H^1(\Omega)^d \mid \vec{v}|_{\Gamma_w \cup \Gamma_{\mathrm{in}}} = 0 \right\}.$$

Using Korn inequality, we get that

$$\|\vec{u}\|_{\mathbf{X}} := \|S(\vec{u})\|_{L^2(\Omega)} := \sqrt{\int_\Omega S(\vec{u}) \mathbin{:} S(\vec{u})\, dx},$$

is a norm on either $\mathbf{X}_1$ or $\mathbf{X}$ where $A \mathbin{:} B = \operatorname{trace}(AB)$. We also denote by $C_K > 0$ the constant such that

$$\|\vec{u}\|_{L^2(\Omega)} \leq C_K \|\vec{u}\|_{\mathbf{X}},$$

and emphasize that C_K only depend on Ω. Thanks to the assumptions on Γ_{out} and Γ_{in}, any $\vec{u} \in \mathbf{X}_1$ satisfy $\vec{u}|_{\Gamma_{\mathrm{in}}} \in H_{00}^{1/2}(\Gamma_{\mathrm{in}})^d$.

Using Lemma 19, we get that Problem (1,2) is equivalent the following variational formulation

$$\begin{array}{l} \textit{Find } (\vec{u}, p) \in \mathbf{X}_1 \times L^2(\Omega) \textit{ such that} \\ \vec{u}|_{\Gamma_{\mathrm{in}}} = \vec{u}_{\mathrm{in}} \text{ and} \\ \left\{ \begin{array}{rll} a(\varepsilon; \vec{u}, \vec{v}) + c(\varepsilon; \vec{u}, \vec{u}, \vec{v}) + b(\varepsilon; \vec{v}, p) & = \left\langle \vec{F}, \vec{v} \right\rangle_{\mathbf{X}' \times \mathbf{X}}, & \forall \vec{v} \in \mathbf{X}, \\ b(\varepsilon; \vec{u}, q) & = 0, & \forall q \in L^2(\Omega), \end{array} \right. \end{array} \tag{4}$$

where $\vec{F} = \varepsilon \vec{f}$ and

$$\begin{aligned} a(\varepsilon; \vec{u}, \vec{v}) &= 2\mathrm{Re}^{-1} \int_\Omega \varepsilon S(\vec{u}) \mathbin{:} S(\vec{v}) + \alpha(\varepsilon) \vec{u} \cdot \vec{v}\, dx, \\ c(\varepsilon; \vec{u}, \vec{v}, \vec{w}) &= \int_\Omega \varepsilon (\vec{u} \cdot \nabla)\vec{v} \cdot \vec{w} + \beta(\varepsilon)|\vec{u}|\vec{v} \cdot \vec{w}\, dx, \\ b(\varepsilon; \vec{u}, q) &= -\int_\Omega q \operatorname{div}(\varepsilon \vec{u})\, dx. \end{aligned}$$

To deal with the inhomogeneous Dirichlet condition, we introduce a divergence-free extension of $\vec{u}_{\rm in}$. Using Lemma 18, we can write $\vec{u} = \vec{w} + \vec{V}$ where $\vec{w} \in \mathbf{X}$. satisfy the variational formulation

$$\begin{aligned}
&\textit{Find } (\vec{w}, p) \in \mathbf{X} \times L^2(\Omega) \textit{ such that}\\
&\begin{cases} a(\varepsilon; \vec{w}, \vec{v}) + b(\varepsilon; \vec{v}, p) &= \langle G(\varepsilon; \vec{w}), \vec{v} \rangle_{\mathbf{X}' \times \mathbf{X}}, \quad \forall \vec{v} \in \mathbf{X}, \\ \qquad\qquad b(\varepsilon; \vec{w}, q) &= 0, \qquad\qquad\qquad\qquad \forall q \in L^2(\Omega), \end{cases}
\end{aligned} \tag{5}$$

where the non-linear term is defined as

$$\langle G(\varepsilon; \vec{w}), \vec{v} \rangle_{\mathbf{X}' \times \mathbf{X}} = \left\langle \vec{F}, \vec{v} \right\rangle_{\mathbf{X}' \times \mathbf{X}} - c(\varepsilon; \vec{w} + \vec{V}, \vec{w} + \vec{V}, \vec{v}) - a(\varepsilon; \vec{V}, \vec{v}). \tag{6}$$

We are going to study the well-posedness to Problem (5) with a fixed-point approach. Therefore, we begin to study the linear problem

$$\begin{aligned}
&\textit{Find } (\vec{w}, p) \in \mathbf{X} \times L^2(\Omega) \textit{ such that}\\
&\begin{cases} a(\varepsilon; \vec{w}, \vec{v}) + b(\varepsilon; \vec{v}, p) &= \left\langle \vec{F}, \vec{v} \right\rangle_{\mathbf{X}' \times \mathbf{X}}, \quad \forall \vec{v} \in \mathbf{X}, \\ \qquad\qquad b(\varepsilon; \vec{w}, q) &= 0, \qquad\qquad\qquad \forall q \in L^2(\Omega), \end{cases}
\end{aligned} \tag{7}$$

where $\vec{F} \in \mathbf{X}'$ is some source term. Problem (7) is a standard linear saddle-point problem whose well-posedness has been studied in e.g. [12, 18, 16]. Since the bilinear form $a(\varepsilon; \cdot, \cdot)$ is continuous and coercive, namely

$$\begin{aligned}
&|a(\varepsilon; \vec{u}, \vec{v})| \leq \left(C_K^2 \, \|\alpha(\varepsilon)\|_{L^\infty(\Omega)} + 2\mathrm{Re}^{-1} \right) \|\vec{u}\|_{\mathbf{X}} \, \|\vec{v}\|_{\mathbf{X}}, \\
&a(\varepsilon; \vec{u}, \vec{u}) \geq 2\mathrm{Re}^{-1} \varepsilon_0 \, \|\vec{u}\|_{\mathbf{X}}^2,
\end{aligned} \tag{8}$$

we only need to get the so-called inf-sup condition on the bilinear form b in order to get the well-posedness to Problem (7) (see e.g. [12, II.1, Proposition 1.3], [16, p. 474, Theorem A.56] or [18, p. 59, Theorem 4.1]). The latter is obtained in the next lemma.

Lemma 2. *Let $\beta > 0$ be the inf-sup constant when $\varepsilon(x) = 1$ for all $x \in \Omega$. Assume that $\varepsilon \in L^\infty(\Omega) \cap W^{1,r(d)}(\Omega)$, with $r(2) > 2$ and $r(3) = 3$ and that*

$$\forall x \in \Omega, \ 0 < \varepsilon_0 \leq \varepsilon(x) \leq 1.$$

Then there exists a constant $\beta > 0$ such that

$$\inf_{q \in L^2(\Omega) \setminus \{0\}} \sup_{\vec{u} \in \boldsymbol{X}} \frac{b(\varepsilon; \vec{u}, q)}{\|\vec{u}\|_{\boldsymbol{X}} \, \|q\|_{L^2(\Omega)}} \geq \beta,$$

where there is a generic constant $C > 0$ such that $\beta(\varepsilon) > 0$ is given by

$$\beta(\varepsilon) = \beta \frac{1}{C \left(\varepsilon_0^{-1} + \varepsilon_0^{-2} \, \|\nabla \varepsilon\|_{L^3(\Omega)} \right)}.$$

Proof. Adapting techniques from [8, p. 6, Eq. (2.13)] to the boundary conditions considered in this paper, the following inf-sup condition can be obtained

$$\inf_{q\in L^2(\Omega)\setminus\{0\}} \sup_{\vec{u}\in\mathbf{X}} \frac{b(1;\vec{u},q)}{\|\vec{u}\|_{\mathbf{X}}\,\|q\|_{L^2(\Omega)}} \geq \beta > 0.$$

From [18, p. 58, Lemma 4.1], the inf-sup condition is equivalent to the statement: for any $q \in L^2(\Omega)$ there exists a $\vec{v} = \vec{v}(q) \in \mathbf{X}$ such that

$$b(1;\vec{v},q) \geq C_1 \|q\|^2_{L^2(\Omega)} \text{ and } \|\vec{v}\|_{\mathbf{X}} \leq C_2 \|q\|_{L^2(\Omega)}, \tag{9}$$

where, in that case, $\beta = C_1/C_2$. Since the application $\vec{u} \in \mathbf{X} \mapsto \varepsilon\vec{u} \in \mathbf{X}$ is an isomorphism (see [7, p. 3, Lemma 2.1]), we have some $\vec{u} \in \mathbf{X}$ such that $\vec{v} = \varepsilon\vec{u}$ and (9) becomes

$$b(1;\varepsilon\vec{u},q) = b(\varepsilon;\vec{u},q) \geq C_1 \|q\|^2_{L^2(\Omega)} \text{ and } \|\varepsilon\vec{u}\|_{\mathbf{X}} \leq C_2 \|q\|_{L^2(\Omega)}.$$

To conclude, note that

$$\begin{aligned}
\|\vec{u}\|_{\mathbf{X}} &= \left\|\frac{\varepsilon\vec{u}}{\varepsilon}\right\|_{\mathbf{X}} \leq C\left(\varepsilon_0^{-1} + \varepsilon_0^{-2}\|\nabla\varepsilon\|_{L^3(\Omega)}\right)\|\varepsilon\vec{u}\|_{\mathbf{X}} \\
&\leq C_2\|q\|_{L^2(\Omega)}\,C\left(\varepsilon_0^{-1} + \varepsilon_0^{-2}\|\nabla\varepsilon\|_{L^3(\Omega)}\right)\|\varepsilon\vec{u}\|_{\mathbf{X}},
\end{aligned}$$

from which we get that for any $q \in L^2(\Omega)$, there exists a $\vec{u} = \vec{u}(q) \in \mathbf{X}$ so that

$$b(\varepsilon;\vec{u},q) \geq C_1 \|q\|^2_{L^2(\Omega)} \text{ and } \|\vec{u}\|_{\mathbf{X}} \leq \widetilde{C_2}\|q\|_{L^2(\Omega)}.$$

The desired inf-sup condition then follows with

$$\beta(\varepsilon) = \frac{C_1}{\widetilde{C_2}} = \beta\frac{1}{C\left(\varepsilon_0^{-1} + \varepsilon_0^{-2}\|\nabla\varepsilon\|_{L^3(\Omega)}\right)}.$$

From (8) and Lemma 2, we can apply [12, II.1, Proposition 1.3] (see also [16, p. 474, Theorem A.56] or [18, p. 59, Theorem 4.1]) to get the existence and uniqueness of solution to (5).

Theorem 3. *Problem (7) has a unique solution $(\vec{w},p) \in \boldsymbol{X}\times L^2(\Omega)$ that satisfies*

$$\begin{aligned}
\|\vec{w}\|_{\mathrm{X}} &\leq \frac{1}{\alpha_0}\left\|\vec{F}\right\|_{\boldsymbol{X}'}, \\
\|p\|_{L^2(\Omega)} &\leq \frac{1}{\beta(\varepsilon)}\left(1 + \frac{\|a\|}{\alpha_0}\right)\left\|\vec{F}\right\|_{\boldsymbol{X}'},
\end{aligned}$$

where α_0 (respectively $\|a\|$) is the coercivity (respectively the continuity) constant of $a(\varepsilon;\cdot,\cdot)$.

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbf{X}$. From the continuous Sobolev' embedding $H^1(\Omega) \subset L^4(\Omega)$ together with Hölder inequality, we get that

$$|c(\varepsilon; \vec{u}, \vec{v}, \vec{w})| \leq C \left(1 + \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right) \|\vec{u}\|_{\mathbf{X}} \|\vec{v}\|_{\mathbf{X}} \|\vec{w}\|_{\mathbf{X}} = C_{\mathrm{NL}} \|\vec{u}\|_{\mathbf{X}} \|\vec{v}\|_{\mathbf{X}} \|\vec{w}\|_{\mathbf{X}}, \tag{10}$$

where C is a positive constant that depends only on Ω. The following result gives some properties of the non-linear term (6) which are needed to prove the well-posedness of (7).

Lemma 4. *The nonlinear function $G(\varepsilon; \cdot) : \boldsymbol{X} \to \boldsymbol{X}'$ defined in (6) satisfies the following estimates*

$$\begin{aligned} \|G(\varepsilon; \vec{w})\|_{\boldsymbol{X}'} \quad \leq \quad & \left\|\vec{F}\right\|_{\boldsymbol{X}'} + 2C_{\mathrm{NL}} M(\varepsilon)^2 \|\vec{u}_{\mathrm{in}}\|^2_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d} + \|a\| M(\varepsilon) \|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d} \\ & + 2C_{\mathrm{NL}} \|\vec{w}\|^2_{\boldsymbol{X}}, \end{aligned}$$

$$\|G(\varepsilon; \vec{w}_1) - G(\varepsilon; \vec{w}_2)\|_{\boldsymbol{X}'} \leq C_{\mathrm{L}} \left(\|\vec{w}_1\|_{\boldsymbol{X}} + \|\vec{w}_2\|_{\boldsymbol{X}} + M(\varepsilon) \|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d}\right) \|\vec{w}_1 - \vec{w}_2\|_{\boldsymbol{X}},$$

where $C_{\mathrm{L}} = C(\Omega) \max\left\{1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\}$ for some generic constant $C(\Omega) > 0$.

Proof. Using (10) with Lemma 18, we get that

$$\begin{aligned} \left|\langle G(\varepsilon; \vec{w}), \vec{v}\rangle_{\mathbf{X}' \times \mathbf{X}}\right| \quad \leq \quad & \left\|\vec{F}\right\|_{\mathbf{X}'} \|\vec{v}\|_{\mathbf{X}} + \|a\| \left\|\vec{V}\right\|_{\mathbf{X}} \|\vec{v}\|_{\mathbf{X}} + C_{\mathrm{NL}} \|\vec{v}\|_{\mathbf{X}} \left\|\vec{w} + \vec{V}\right\|^2_{\mathbf{X}} \\ \leq \quad & \|\vec{v}\|_{\mathbf{X}} \left(\left\|\vec{F}\right\|_{\mathbf{X}'} + 2C_{\mathrm{NL}} M(\varepsilon)^2 \|\vec{u}_{\Gamma_{\mathrm{in}}}\|^2_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d}\right) \\ + \quad & \|\vec{v}\|_{\mathbf{X}} \left(\|a\| M(\varepsilon) \|\vec{u}_{\Gamma_{\mathrm{in}}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d} + 2C_{\mathrm{NL}} \|\vec{w}\|^2_{\mathbf{X}}\right). \end{aligned}$$

Taking the supremum over $\vec{v} \in \mathbf{X}$ with $\|\vec{v}\|_{\mathbf{X}} \leq 1$ then gives the first estimate. For the second estimate, we start by noting that

$$\begin{aligned} < G(\varepsilon; \vec{w}_1) \quad - \quad & G(\varepsilon; \vec{w}_2), \vec{v} >_{\mathbf{X}' \times \mathbf{X}} = c(\varepsilon; \vec{w}_2 + \vec{V}, \vec{w}_2 + \vec{V}, \vec{v}) - c(\varepsilon; \vec{w}_1 + \vec{V}, \vec{w}_1 + \vec{V}, \vec{v}) \\ = \quad & \int_\Omega \varepsilon \left(((\vec{w}_2 + \vec{V}) \cdot \nabla)(\vec{w}_2 + \vec{V}) - ((\vec{w}_1 + \vec{V}) \cdot \nabla)(\vec{w}_1 + \vec{V})\right) \cdot \vec{v} \, dx \\ + \quad & \int_\Omega \beta(\varepsilon) \left(\left|\vec{w}_2 + \vec{V}\right| (\vec{w}_2 + \vec{V}) - \left|\vec{w}_1 + \vec{V}\right| (\vec{w}_1 + \vec{V})\right) \cdot \vec{v} \, dx. \end{aligned}$$

For two vector fields $\vec{a}, \vec{b}$ we have the following bounds

$$\begin{aligned} \left|\vec{a}|\vec{a}| - \vec{b}|\vec{b}|\right| = \left|(\vec{a} - \vec{b})|\vec{a}| - \vec{b}(|\vec{b}| - |\vec{a}|)\right| \leq \left|\vec{a} - \vec{b}\right| \left(|\vec{a}| + |\vec{b}|\right), \\ \left|\vec{a} \cdot \nabla\vec{a} - \vec{b} \cdot \nabla\vec{b}\right| \leq \left|\vec{a} - \vec{b}\right| |\nabla\vec{a}| + \left|\vec{b}\right| \left|\nabla(\vec{a} - \vec{b})\right|. \end{aligned}$$

Gathering the previous estimates, one can get that

$$\begin{aligned} |< G(\varepsilon; \vec{w}_1) - G(\varepsilon; \vec{w}_2), \vec{v} >_{\mathbf{X}' \times \mathbf{X}}| \quad \leq \quad & C(\Omega) \max\left\{1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\} \\ \times \quad & \left(\|\vec{w}_1\|_{\mathbf{X}} + \|\vec{w}_2\|_{\mathbf{X}} + M(\varepsilon) \|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d}\right) \\ \times \quad & \|\vec{w}_1 - \vec{w}_2\|_{\mathbf{X}}, \end{aligned}$$

and the proof is thus finished.

We are now in position to prove the existence and uniqueness of solution to Problem (5).

Theorem 5. *Assume that ε satisfies the standing assumptions (3). Then one has a $\eta > 0$ such that if*

$$\left\|\vec{F}\right\|_{\boldsymbol{X}'} + \|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d} \leq \eta,$$

there exists a unique $(\vec{w}, p) \in \boldsymbol{X} \times L^2(\Omega)$ satisfying the weak formulation (5) and the estimate

$$\|\vec{w}\|_{\boldsymbol{X}} + \|p\|_{L^2(\Omega)} \leq C_{\text{stab}} \left(\left\|\vec{F}\right\|_{\boldsymbol{X}'} + \|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d} \right),$$

where $C_{\text{stab}} > 0$ is a generic constant that does not depend on ε.

Proof. Let $\mathcal{S} : \vec{F} \in \mathbf{X}' \mapsto (\vec{w}, p) \in \mathbf{X} \times L^2(\Omega)$ where $(\vec{w}, p) \in \mathbf{X} \times L^2(\Omega)$ is the unique solution to the linear Problem (7) which is ensured by Theorem 3. The non-linear variational problem (5) is then equivalent to the following fixed point equation

$$(\vec{w}, p) = \mathcal{T}(\vec{w}, p), \tag{11}$$

where

$$\mathcal{T} = \mathcal{S}\mathcal{G} \text{ with } \mathcal{G}(\vec{w}, p) = G(\varepsilon; \vec{w}).$$

Let $\mathbf{B}_R = \left\{ (\vec{v}, p) \in \mathbf{X} \times L^2(\Omega) \| \ \|\vec{v}\|_{\mathbf{X}} + \|p\|_{L^2(\Omega)} \leq R \right\}$. We set

$$R = R_0 \left(\left\|\vec{F}\right\|_{\mathbf{X}'} + \|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d} \right).$$

Note that $\mathcal{T} : \mathbf{X} \times L^2(\Omega) \mapsto \mathbf{X} \times L^2(\Omega)$ and we thus only need to show that $\mathcal{T} : \mathbf{B}_R \to \mathbf{B}_R$ for some $R > 0$ and that $\mathcal{T}$ is a contraction mapping. Thanks to Lemma 4 and Theorem 3, these two properties holds if $\left\|\vec{F}\right\|_{\mathbf{X}'}$, $\|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d}$ and R_0 are small enough. The Banach fixed point theorem then gives the existence and uniqueness of $(\vec{w}, p) \in \mathbf{B}_R$ satisfying (11).

We end this section by noting that Theorem 5 gives the existence and uniqueness of a solution to (4) since $\vec{u} = \vec{w} + \vec{V}$ where $\vec{V} \in \mathbf{X}_1$ is the divergence-free lifting defined in Lemma 18.

3. Finite element approximation of the Darcy-Brinkman-Forchheimer model

We consider a quasi-uniform family of triangulations (see [16, p. 76, Definition 1.140]) $\{\mathcal{T}_h\}_{h>0}$ of Ω whose elements are triangle ($d = 2$) or tetrahedrons

($d = 3$) denoted by K. We emphasize that $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$. The parameter h_K is the diameter of the circle or sphere inscribed in K and we define

$$h = \max_{K \in \mathcal{T}_h} h_K.$$

We consider the Taylor-Hood finite element technique [43] (see also [18, p. 176, Chapter II, Section 4.2]) which consist in looking for piecewise-polynomial approximations $(\vec{w}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h$ of $(\vec{w}, p) \in \mathbf{X} \times L^2(\Omega)$ with

$$\begin{aligned} \mathbf{X}_h &= \{\vec{v}_h \in \mathbf{X} \mid \forall K \in \mathcal{T}_h,\ \vec{v}_h|_K \in \mathbb{P}_2(K)\}, \\ \mathbf{M}_h &= \left\{q_h \in \mathcal{C}^0(\overline{\Omega}) \mid \forall K \in \mathcal{T}_h,\ q_h|_K \in \mathbb{P}_1(K)\right\}. \end{aligned}$$

It is worth noting that we choose this finite element combination since our numerical simulations are done with it, but the convergence results in this section also hold for other finite element spaces.

We now consider some $\varepsilon_h \in \mathbf{M}_h$ that approximates ε in the following sense

$$\begin{aligned} &\forall x \in \Omega,\ \varepsilon_0 \le \varepsilon_h(x) \le 1, \\ &\|\varepsilon_h - \varepsilon\|_{L^\infty(\Omega)} \le Ch \|\varepsilon\|_{W^{1,\infty}(\Omega)}, \\ &\|\nabla\varepsilon_h - \nabla\varepsilon\|_{L^r(\Omega)} \le Ch^l \|\varepsilon\|_{W^{l+1,r}(\Omega)},\ r \ge 4. \end{aligned} \tag{12}$$

Note that these assumptions are satisfied if $\varepsilon \in W^{1,\infty}(\Omega) \cap W^{l+1,r}(\Omega)$ for $l > 0$ and if one takes $\varepsilon_h = \mathcal{I}_h \varepsilon$ where $\mathcal{I}_h$ is the global interpolation operator (see [16, Corollary 1.109 and Corollary 1.110]).

This section is now devoted to finite element discretization of (4) which actually amounts to consider the finite element discretization of (5) since the solution of these two problems are related thanks to $\vec{u} = \vec{w} + \vec{V}$. As a result, if $\vec{w}_h$ denotes the velocity associated to (5), the finite element approximation of the solution to Problem (4) is going to be $\vec{u}_h = \vec{w}_h + \mathcal{I}_{\mathbf{X}_h} \vec{V}$ where $\mathcal{I}_{\mathbf{X}_h} : \mathbf{X} \mapsto \mathbf{X}_h$ is the finite element interpolate operator. The discrete problem associated to (5) reads

$$\textit{Find } (\vec{w}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h \textit{ such that}$$
$$\left\{ \begin{aligned} a(\varepsilon_h; \vec{w}_h, \vec{v}_h) + b(\varepsilon_h; \vec{v}_h, p_h) &= \langle G(\varepsilon_h; \vec{w}_h), \vec{v}_h \rangle_{\mathbf{X}' \times \mathbf{X}}, && \forall \vec{v}_h \in \mathbf{X}_h, \\ b(\varepsilon_h; \vec{w}_h, q_h) &= 0, && \forall q_h \in \mathbf{M}_h, \end{aligned} \right. \tag{13}$$

where the non-linear term is defined as in (6). We are going to study the existence and uniqueness of discrete solution to (13) as we did for the continuous problem, using a fixed-point approach. As a result, we are going to study first the discretization of (7) which is

$$\textit{Find } (\vec{w}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h \textit{ such that}$$
$$\left\{ \begin{aligned} a(\varepsilon_h; \vec{w}_h, \vec{v}_h) + b(\varepsilon_h; \vec{v}_h, p_h) &= \left\langle \vec{F}, \vec{v}_h \right\rangle_{\mathbf{X}' \times \mathbf{X}}, && \forall \vec{v}_h \in \mathbf{X}_h, \\ b(\varepsilon_h; \vec{w}_h, q_h) &= 0, && \forall q_h \in \mathbf{M}_h. \end{aligned} \right. \tag{14}$$

Note that (14) is again a saddle-point problem with a coercive bilinear form a (see (8)) and we thus need a (discrete) inf-sup condition in order to get the well-posedness of (14). This is done in the next subsection.

3.1. Discrete inf-sup conditions

This section is devoted to prove that the bilinear forms $b(\varepsilon;\cdot,\cdot)$ and $b(\varepsilon_h;\cdot,\cdot)$ both satisfies inf-sup conditions. The latter together with the coercivity of the bilinear forms $a(\varepsilon_h;\cdot,\cdot)$ and $a(\varepsilon;\cdot,\cdot)$ are necessary to prove that the linear discrete problems (14), either with ε or ε_h, are well-posed.

Lemma 6 (Discrete inf-sup with ε). *Assume that ε is regular enough so that (12) holds and that at least one edge ($d = 2$) or a face ($d = 3$) of an element of $\mathcal{T}_h$ is contained in Γ_{out} (see [9, Assumption 3.1]). Then the following inf-sup condition holds:*

$$\inf_{q_h\in \boldsymbol{M}_h\setminus\{0\}} \sup_{\vec{u}_h\in \boldsymbol{X}_h} \frac{b(\varepsilon;\vec{u}_h,q_h)}{\|\vec{u}_h\|_{\boldsymbol{X}}\|q_h\|_{L^2(\Omega)}} \geq \beta(h,\varepsilon),$$

$$\beta(h,\varepsilon) = \frac{\min\left\{\frac{1}{2},\frac{\mu c_0}{4|\Omega|}\right\} - C_1 C(\Omega)h}{\max\left\{\left(\varepsilon_0^{-1}+\varepsilon_0^{-2}\|\nabla\varepsilon\|_{L^3(\Omega)}\right),\mu C_2\right\}},$$

where $c_0, C_1, C_2, C(\Omega)$ are generic positive constants and only μ defined in (18) depends on ε.

Proof. The proof is adapted from [7, p. 18, Proposition 3.7] and [9, p. 14, Lemma 3.2]. Let $q_h \in \mathbf{M}_h$ written as

$$q_h = \widetilde{q}_h + \overline{q},\ \overline{q} = \frac{1}{|\Omega|}\int_\Omega q_h(x)\,dx.$$

Since $\widetilde{q}_h \in L^2_0(\Omega)$, the continuous inf-sup condition from [18, p. 24, Corollary 2.4] gives the existence of $\vec{v} \in H^1_0(\Omega)$ such that

$$\operatorname{div}(\vec{\widetilde{v}}) = -\widetilde{q}_h \text{ and } \left\|\vec{\widetilde{v}}\right\|_{\mathbf{X}} \leq C_1\|\widetilde{q}_h\|_{L^2(\Omega)}. \tag{15}$$

Since $\overline{q} \in \mathbb{R}$, we have

$$\forall \vec{v}\in\mathbf{X},\ b(\varepsilon;\vec{v},\overline{q}) = -\overline{q}\int_{\Gamma_{\text{out}}} \varepsilon\vec{v}\cdot\vec{n}\,d\sigma.$$

Let $\varphi \in \mathcal{C}_c^\infty(\overline{\Omega})$ be a smooth function such that $\int_{\Gamma_{\text{out}}}\varepsilon\varphi\,d\sigma = c_0 > 0$. The Cauchy-Schwartz inequality together with the continuity of the trace operator give

$$\int_{\Gamma_{\text{out}}}\mathcal{I}_{\mathbf{X}_h}(\varepsilon\varphi)\,d\sigma \geq c_0 - C(\Omega)\|\mathcal{I}_{\mathbf{X}_h}(\varepsilon\varphi)-\varepsilon\varphi\|_{\mathbf{X}} \geq c_0 - C(\Omega)h^l\|\varepsilon\varphi\|_{W^{l+1,2}(\Omega)} \geq \frac{c_0}{2},$$

where we used [16, p. 61, Corollary (1.110)] and we assumed that h is small enough to get the two last lower bounds. We now consider a regular extension $\vec{n}_*$ of the unit normal vector $\vec{n}$ inside Ω and set

$$\vec{\overline{v}} = -\overline{q}\mathcal{I}_{\mathbf{X}_h}(\varepsilon\varphi\vec{n}_*).$$

For h small enough, the previous estimate then gives that

$$b(\varepsilon;\vec{\overline{v}},\overline{q}) \geq \frac{c_0}{2|\Omega|}\left\|\overline{q}\right\|^2_{L^2(\Omega)} \text{ and } \left\|\vec{\overline{v}}\right\|_{\mathbf{X}} \leq C_2\left\|\overline{q}\right\|_{L^2(\Omega)}. \tag{16}$$

Now setting

$$\vec{u}_h = \mathcal{I}_{\mathbf{X}_h}\left(\varepsilon^{-1}\vec{\overline{v}}\right) + \mu\vec{\overline{v}}. \tag{17}$$

and using (15) and (16), we infer

$$\begin{aligned}
b(\varepsilon;\vec{u}_h,q_h) &= b\left(\varepsilon;\varepsilon^{-1}\vec{\overline{v}},q_h\right) + b\left(\varepsilon;\mathcal{I}_{\mathbf{X}_h}\left(\varepsilon^{-1}\vec{\overline{v}}\right) - \varepsilon^{-1}\vec{\overline{v}},q_h\right)\\
&\geq b\left(\varepsilon;\mathcal{I}_{\mathbf{X}_h}\left(\varepsilon^{-1}\vec{\overline{v}}\right) - \varepsilon^{-1}\vec{\overline{v}},q_h\right) + \left\|\widetilde{q}_h\right\|^2_{L^2(\Omega)} + \mu\frac{c_0}{2|\Omega|}\left\|\overline{q}\right\|^2_{L^2(\Omega)}\\
&- \mu C(\Omega)\left(C_2(1+\left\|\nabla\varepsilon\right\|_{L^3(\Omega)})\right)\left\|\overline{q}\right\|_{L^2(\Omega)}\left\|\widetilde{q}_h\right\|_{L^2(\Omega)}.\\
&\geq b\left(\varepsilon;\mathcal{I}_{\mathbf{X}_h}\left(\varepsilon^{-1}\vec{\overline{v}}\right) - \varepsilon^{-1}\vec{\overline{v}},q_h\right) + \frac{1}{2}\left\|\widetilde{q}_h\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left(\mu\frac{c_0}{|\Omega|} - \delta^2\right)\left\|\overline{q}\right\|^2_{L^2(\Omega)},
\end{aligned}$$

where the last lower bound has been obtain thanks to Young inequality $ab \leq a^2/(2\delta) + b^2\delta/2$ applied with

$$a = \left\|\widetilde{q}_h\right\|_{L^2(\Omega)},\ b = \left\|\overline{q}\right\|_{L^2(\Omega)},\ \delta = \mu C(\Omega)\left(C_2(1+\left\|\nabla\varepsilon\right\|_{L^3(\Omega)})\right).$$

If we now chose μ as

$$\mu = \frac{c_0}{2|\Omega|\left(C(\Omega)\left(C_2(1+\left\|\nabla\varepsilon\right\|_{L^3(\Omega)})\right)\right)^2}, \tag{18}$$

we end up with

$$\begin{aligned}
b(\varepsilon;\vec{u}_h,q_h) &\geq b\left(\varepsilon;\mathcal{I}_{\mathbf{X}_h}\left(\varepsilon^{-1}\vec{\overline{v}}\right) - \varepsilon^{-1}\vec{\overline{v}},q_h\right) + \frac{1}{2}\left\|\widetilde{q}_h\right\|^2_{L^2(\Omega)} + \frac{\mu c_0}{4|\Omega|}\left\|\overline{q}\right\|^2_{L^2(\Omega)}\\
&\geq b\left(\varepsilon;\mathcal{I}_{\mathbf{X}_h}\left(\varepsilon^{-1}\vec{\overline{v}}\right) - \varepsilon^{-1}\vec{\overline{v}},q_h\right) + \min\left\{\frac{1}{2},\frac{\mu c_0}{4|\Omega|}\right\}\left\|q_h\right\|^2_{L^2(\Omega)}
\end{aligned} \tag{19}$$

Using now [7, Proof of Proposition 3.7, (ii)], we have

$$\begin{aligned}
b\left(\varepsilon;\mathcal{I}_{\mathbf{X}_h}\left(\varepsilon^{-1}\vec{\overline{v}}\right) - \varepsilon^{-1}\vec{\overline{v}},q_h\right) &\leq Ch\left\|\vec{\overline{v}}\right\|_{\mathbf{X}}\left\|q_h\right\|_{L^2(\Omega)}\\
&\leq C_1Ch\left\|\widetilde{q}_h\right\|_{L^2(\Omega)}\left\|q_h\right\|_{L^2(\Omega)}\\
&\leq C_1Ch\left\|q_h\right\|^2_{L^2(\Omega)},
\end{aligned} \tag{20}$$

where we used (15) for the last upper bound. From (17), (15) and (16), we get

$$\begin{aligned}
\left\|\vec{u}_h\right\|_{\mathbf{X}} &\leq \left\|\mathcal{I}_{\mathbf{X}_h}\left(\varepsilon^{-1}\vec{\overline{v}}\right)\right\|_{\mathbf{X}} + \mu\left\|\vec{\overline{v}}\right\|_{\mathbf{X}} \leq C(\Omega)\left\|\varepsilon^{-1}\vec{\overline{v}}\right\|_{\mathbf{X}} + \mu C_2\left\|\overline{q}\right\|_{L^2(\Omega)}\\
&\leq \max\left\{\left(\varepsilon_0^{-1} + \varepsilon_0^{-2}\left\|\nabla\varepsilon\right\|_{L^3(\Omega)}\right),\mu C_2\right\}\left\|q_h\right\|_{L^2(\Omega)}.
\end{aligned} \tag{21}$$

The desired inf-sup condition is finally proved by gathering (21), (19) and (20).

We emphasize that we are interested in solving the discrete problem (13) which involve the discretization of ε. Therefore, we extend Lemma 6 for the bilinear form $b(\varepsilon_h;\cdot,\cdot)$.

Lemma 7 (Discrete inf-sup with ε_h). *Assume that the assumptions of Lemma 6 hold. Then one has the following inf-sup condition*

$$\inf_{q_h\in \boldsymbol{M}_h\setminus\{0\}} \sup_{\vec{u}_h\in \boldsymbol{X}_h} \frac{b(\varepsilon_h;\vec{u}_h,q_h)}{\|\vec{u}_h\|_{\boldsymbol{X}}\|q_h\|_{L^2(\Omega)}} \geq \beta_2(h,\varepsilon),$$
$$\beta_2(h,\varepsilon) = \beta(h,\varepsilon) - h^l \frac{C(\Omega)\,\|\varepsilon\|_{W^{s,r}(\Omega)}}{\max\left\{\left(\varepsilon_0^{-1} + \varepsilon_0^{-2}\,\|\nabla\varepsilon\|_{L^3(\Omega)}\right), \mu C_2\right\}},$$

where $\beta(h,\varepsilon)$ is defined in Lemma 6, μ in (18) and $C_2>0$ is a generic constant.

Proof. We have that

$$b(\varepsilon_h;\vec{u}_h,q_h) = b(\varepsilon;\vec{u}_h,q_h) + \int_\Omega \operatorname{div}\left((\varepsilon-\varepsilon_h)\vec{u}_h\right) q_h\,dx.$$

From [7, p. 13; Lemma 3.1], we have the estimate

$$\forall \vec{w}\in H^1(\Omega),\ \|(\varepsilon-\varepsilon_h)\,\vec{w}\|_{H^1(\Omega)} \leq Ch^l\,\|\varepsilon\|_{W^{l+1,r}(\Omega)}\,\|\vec{w}\|_{H^1(\Omega)},$$

which gives

$$b(\varepsilon_h;\vec{u}_h,q_h) \geq b(\varepsilon;\vec{u}_h,q_h) - Ch^l\,\|\varepsilon\|_{W^{s,r}(\Omega)}\,\|\vec{u}_h\|_{\mathbf{X}}.$$

Using now q_h as in the proof of Lemma 6, $\vec{u}_h$ given by (17) and the estimate (21), we obtain

$$b(\varepsilon_h;\vec{u}_h,q_h) \geq \left(\min\left\{\frac{1}{2},\frac{\mu c_0}{4|\Omega|}\right\} - C_1C(\Omega)h - Ch^l\,\|\varepsilon\|_{W^{l+1,r}(\Omega)}\right)\|q_h\|^2_{L^2(\Omega)},$$

and the desired inf-sup condition then follows from the estimate (21) on $\vec{u}_h$.

We end this section by noting that both inf-sup constants satisfy $\beta(h,\varepsilon)\geq\beta_*(\varepsilon)$ and $\beta_2(h,\varepsilon)\geq\beta_*(\varepsilon)$ with

$$\beta_*(\varepsilon) = \frac{1}{8|\Omega|}\frac{\min\{2|\Omega|,\mu c_0\}}{\max\left\{\left(\varepsilon_0^{-1}+\varepsilon_0^{-2}\,\|\nabla\varepsilon\|_{L^3(\Omega)}\right),\mu C_2\right\}},$$

if the mesh-size h is assumed to be small enough.

3.2. Convergence of the finite element method

We now have all the necessary tools to show that $(\vec{w}_h,p_h)$ satisfying (13) converge toward $(\vec{w},p)$ which is the solution to the variational formulation (5). First we are going to prove that (13) has a unique solution under assumptions similar to those giving the well-posedness of the continuous problem (5) (see Theorem 5), namely for source terms that are small enough.

Theorem 8. *Assume that ε satisfies (3). Then one has a $\eta > 0$ and a $h_{\min} > 0$ such that if*

$$\left\|\vec{F}\right\|_{\boldsymbol{X}'} + \|\vec{u}_{\rm in}\|_{H_{00}^{1/2}(\Gamma_{\rm in})^d} \leq \eta, \ \ h \leq h_{\min},$$

there exists a unique $(\vec{w}_h, p_h) \in \boldsymbol{X} \times L^2(\Omega)$ satisfying the weak formulation (13) and the estimate

$$\|\vec{w}_h\|_{\boldsymbol{X}} + \|p_h\|_{L^2(\Omega)} \leq C_{\rm stab}\left(\left\|\vec{F}\right\|_{\boldsymbol{X}'} + \|\vec{u}_{\rm in}\|_{H_{00}^{1/2}(\Gamma_{\rm in})^d}\right),$$

where $C_{\rm stab} > 0$ is a generic constant that does not depend on ε.

Proof. We proceed as in the proof of Theorem 5 and we are thus going to write (13) as a fixed point equation on $\mathbf{X}_h \times \mathbf{M}_h$. Owning to [18, p. 59, Theorem 4.1], the coercivity of the bilinear form a (8) and Lemma 7, Problem (14) has a unique solution. We introduce the operator $\mathcal{S}_h : \vec{F} \in \mathbf{X}' \mapsto (\vec{w}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h$ where $(\vec{w}_h, p_h)$ is the unique solution to (14). The non-linear discrete problem (13) can then be written as the next fixed-point equation

$$(\vec{w}_h, p_h) = \mathcal{S}_h\mathcal{G}_h(\vec{w}_h, p_h) \text{ where } \mathcal{G}_h(\vec{w}_h, p_h) = G(\varepsilon_h; \vec{w}_h),$$

where $\mathcal{S}_h\mathcal{G}_h : \mathbf{X}_h \times \mathbf{M}_h \mapsto \mathbf{X}_h \times \mathbf{M}_h$. Since the properties of $G(\varepsilon; \cdot) : \mathbf{X} \to \mathbf{X}'$ proved in Lemma 4 are also valid for $G(\varepsilon_h; \cdot) : \mathbf{X}_h \to \mathbf{X}'$, the proof of the present theorem can be done exactly as the proof of Theorem 5.

We are now in position to prove the convergence of the finite element approximation.

Theorem 9. *Assume that ε satisfies (3) and that*

$$\left\|\vec{F}\right\|_{\boldsymbol{X}'} + \|\vec{u}_{\rm in}\|_{H_{00}^{1/2}(\Gamma_{\rm in})^d} \leq \eta,$$

where $\eta > 0$ is given in Theorem 8. Let $(\vec{w}_h, p_h)$ be the unique solution to (13) and $(\vec{w}, p)$ be the unique solution to (5). We then have the following convergence

$$\lim_{h \to 0}\left(\|\vec{w}_h - \vec{w}\|_{\boldsymbol{X}} + \|p_h - p\|_{L^2(\Omega)}\right) = 0.$$

Proof. Thanks to Theorem 8, (13) has a unique solution $(\vec{w}_h, p_h)$ for h small enough. It also satisfies

$$\|\vec{w}_h\|_{\mathbf{X}} + \|p_h\|_{L^2(\Omega)} \leq C_{\rm stab}\left(\left\|\vec{F}\right\|_{\mathbf{X}'} + \|\vec{u}_{\rm in}\|_{H_{00}^{1/2}(\Gamma_{\rm in})^d}\right),$$

where $C_{\rm stab} > 0$ does not depend on the mesh-size. As a result, there exists $(\vec{w}, p) \in \mathbf{X} \times L^2(\Omega)$ and subsequences such that

$$\begin{aligned}&(\vec{w}_{h_l}, p_{h_l}) \rightharpoonup (\vec{w}, p) \text{ weakly in } H^1(\Omega)^d \times L^2(\Omega),\\ &\vec{w}_{h_l} \to \vec{w} \text{ strongly in } L^4(\Omega),\end{aligned}$$

with $h_l \to 0$ as $l \to +\infty$.

(Step 1) Identification of the weak limit: Now, let $(\vec{v}, q) \in \mathcal{C}^\infty(\Omega)^d \times \mathcal{C}^\infty(\Omega)$. Then there exists a sequence $(\vec{v}_h, q_h)_h \subset \mathbf{X}_h \times \mathbf{M}_h$ such that

$$\lim_{h \to 0} \|\vec{v}_h - \vec{v}\|_{\mathbf{X}} + \|q_h - q\|_{L^2(\Omega)} = 0.$$

From (12), we have that $\varepsilon_h \to \varepsilon$ strongly in $L^\infty(\Omega)$ and $\nabla\varepsilon_h \to \nabla\varepsilon$ strongly in $L^r(\Omega)$ for some $r \geq 4$. We then get that

$$\begin{aligned}
&\lim_{l \to +\infty} a(\varepsilon_{h_l}; \vec{w}_{h_l}, \vec{w}_{h_l}) = a(\varepsilon_{h_l}; \vec{w}_{h_l}, \vec{w}_{h_l}),\\
&\lim_{l \to +\infty} \langle G(\varepsilon_{h_l}; \vec{w}_{h_l}), \vec{v}_{h_l} \rangle_{\mathbf{X}' \times \mathbf{X}} = \langle G(\varepsilon; \vec{w}), \vec{v} \rangle_{\mathbf{X}' \times \mathbf{X}},\\
&\lim_{l \to +\infty} b(\varepsilon_{h_l}; \vec{w}_{h_l}, q_{h_l}) = b(\varepsilon; \vec{w}, q),
\end{aligned}$$

where for the last limit, we used

$$b(\varepsilon_h; \vec{w}_h, q_h) = -\int_\Omega q_h \nabla\varepsilon_h \cdot \vec{w}_h + q_h \varepsilon_h \mathrm{div}(\vec{w}_h)\, dx.$$

We then get that the limit $(\vec{w}, p) \in \mathbf{X} \times L^2(\Omega)$ satisfies (5) for all $(\vec{v}, q) \in \mathcal{C}^\infty(\Omega)^d \times \mathcal{C}^\infty(\Omega)$. The density of smooth function in $\mathbf{X}$ and in $L^2(\Omega)$ ensure that $(\vec{w}, p) \in \mathbf{X} \times L^2(\Omega)$ satisfy (5) for all $(\vec{v}, q) \in \mathbf{X} \times L^2(\Omega)$. We have thus show that $(\vec{w}_h, p_h)$ weakly converges toward $(\vec{w}, p)$ which is the solution to (5). The uniqueness of the solution to Problem (5) shows that the whole sequence weakly converges toward $(\vec{w}, p)$.

(Step 2) Strong convergence of $\vec{w}_h$ toward $\vec{w}$ in X: We now show that we have strong convergence in $\mathbf{X} \times L^2(\Omega)$. We work below with a subsequence still denoted with index l keeping in mind that the uniqueness of the limit is going to ensure that the whole sequence is going to converge. Note first that $\varepsilon_{h_l} \to \varepsilon$ strong in $L^\infty(\Omega)$, $\vec{w}_{h_l} \to \vec{w}$ strongly in $L^4(\Omega)$ and that $(\vec{w}_{h_l}, p_{h_l}) \rightharpoonup (\vec{w}, p)$ weakly in $H^1(\Omega)^d \times L^2(\Omega)$ ensure that

$$\begin{aligned}
&\lim_{l \to +\infty} \int_\Omega \varepsilon_{h_l} (\vec{w}_{h_l} \cdot \nabla)\vec{w}_{h_l} \cdot \vec{w}_{h_l} + \beta(\varepsilon_{h_l}) |\vec{w}_{h_l}| \vec{w}_{h_l} \cdot \vec{w}_{h_l}\, dx\\
= &\int_\Omega \varepsilon (\vec{w} \cdot \nabla)\vec{w} \cdot \vec{w} + \beta(\varepsilon)|\vec{w}|\vec{w} \cdot \vec{w}\, dx.
\end{aligned}$$

Now taking $\vec{v}_h = \vec{w}_h$ in the weak formulation (13) and recalling that the non-linear term is defined in (6), we obtain

$$\begin{aligned}
2\mathrm{Re}^{-1} \left\| \sqrt{\varepsilon_{h_l}} S(\vec{w}_{h_l}) \right\|^2_{L^2(\Omega)} &= \langle G(\varepsilon_{h_l}; \vec{w}_{h_l}), \vec{w}_{h_l} \rangle_{\mathbf{X}' \times \mathbf{X}} - \int_\Omega \alpha(\varepsilon_n) |\vec{w}_{h_l}|^2\, dx\\
&\to_{l \to +\infty} \langle G(\varepsilon; \vec{w}), \vec{w} \rangle_{\mathbf{X}' \times \mathbf{X}} - \int_\Omega \alpha(\varepsilon) |\vec{w}|^2\, dx\\
&= 2\mathrm{Re}^{-1} \left\| \sqrt{\varepsilon} S(\vec{w}) \right\|^2_{L^2(\Omega)},
\end{aligned}$$

where we used that $(\vec{w}, p) \in \mathbf{X} \times L^2(\Omega)$ satisfy (5). The weak H^1-convergence together with the previous norm convergence then yield

$$\lim_{n \to +\infty} \sqrt{\varepsilon_{h_l}} S(\vec{w}_{h_l}) = \sqrt{\varepsilon} S(\vec{w}_{h_l}), \text{ strongly in } L^2(\Omega).$$

Since $\varepsilon_h \to \varepsilon$ strongly in L^∞, we finally get that $\vec{w}_h \to \vec{w}$ strongly in $H^1(\Omega)$.

(Step 3) Strong convergence of p_h toward p in $L^2(\Omega)$: Let $\mathbf{B} \subset \mathbf{X}_h$ be the unit ball of $\mathbf{X}_h$. To get strong convergence on the pressure, we start by using the inf-sup condition from Lemma 7 which yields

$$\begin{aligned} \|p_{h_l} - p\|_{L^2(\Omega)} &\leq \frac{1}{\beta_*(\varepsilon_{h_l})} \sup_{\vec{v}_{h_l} \in \mathbf{B}_{h_l}} b(\varepsilon_{h_l}; \vec{v}_{h_l}, p_{h_l} - p) \\ &= \frac{1}{\beta_*(\varepsilon_{h_l})} \sup_{\vec{v}_{h_l} \in \mathbf{B}_{h_l}} b(\varepsilon_{h_l}; \vec{v}_{h_l}, p_{h_l}) - b(\varepsilon; \vec{v}_{h_l}, p) + \int_\Omega p \operatorname{div}(\vec{v}_{h_l}(\varepsilon_{h_l} - \varepsilon))\, dx. \end{aligned} \tag{22}$$

Since $\mathbf{X}_h \subset \mathbf{X}$, one can take $\vec{v} = \vec{v}_h \in \mathbf{X}_h$ in the weak formulation (5). Since $(\vec{w}_h, p_h)$ satisfies (13) and $(\vec{w}, p)$ satisfies (5), we have

$$\begin{aligned} |b(\varepsilon_{h_l}; \vec{v}_{h_l}, p_{h_l}) - b(\varepsilon; \vec{v}_{h_l}, p)| &\leq \left|\langle G(\varepsilon_{h_l}; \vec{w}_{h_l}) - G(\varepsilon; \vec{w}), \vec{v}_{h_l}\rangle_{\mathbf{X}' \times \mathbf{X}}\right| \\ &+ |a(\varepsilon; \vec{w}, \vec{v}_{h_l}) - a(\varepsilon_{h_l}; \vec{w}_{h_l}, \vec{v}_{h_l})| \\ &\leq \left|\langle G(\varepsilon_{h_l}; \vec{w}_{h_l}) - G(\varepsilon; \vec{w}_{h_l}), \vec{v}_{h_l}\rangle_{\mathbf{X}' \times \mathbf{X}}\right| \\ &+ \left|\langle G(\varepsilon; \vec{w}_{h_l}) - G(\varepsilon; \vec{w}), \vec{v}_{h_l}\rangle_{\mathbf{X}' \times \mathbf{X}}\right| \\ &+ |a(\varepsilon; \vec{w}, \vec{v}_{h_l}) - a(\varepsilon_{h_l}; \vec{w}_{h_l}, \vec{v}_{h_l})|\,. \end{aligned} \tag{23}$$

Since $\vec{w}_{h_l} \to \vec{w}$ in H^1-strong, the sequence $(\vec{w}_{h_l})_l \subset \mathbf{X}$ is uniformly bounded, Lemma 4 gives that

$$\begin{aligned} \lim_{l \to +\infty} \sup_{\vec{v}_{h_l} \in \mathbf{B}_{h_l}} \left|\langle G(\varepsilon; \vec{w}_{h_l}) - G(\varepsilon; \vec{w}), \vec{v}_{h_l}\rangle_{\mathbf{X}' \times \mathbf{X}}\right| &\leq \lim_{l \to +\infty} \|G(\varepsilon; \vec{w}_{h_l}) - G(\varepsilon; \vec{w})\|_{\mathbf{X}'} \\ &= 0. \end{aligned} \tag{24}$$

Using that $\varepsilon_{h_l} \to \varepsilon$ in $L^\infty-$strong, that $\vec{w}_{h_l} \to \vec{w}$ in H^1-strong and some easy computations, one gets

$$\lim_{l \to +\infty} \sup_{\vec{v}_{h_l} \in \mathbf{B}_{h_l}} |a(\varepsilon; \vec{w}, \vec{v}_{h_l}) - a(\varepsilon_{h_l}; \vec{w}_{h_l}, \vec{v}_{h_l})| = 0. \tag{25}$$

Some computations similar to those done in the proof of Lemma 4 together with the strong L^∞ convergence of ε_{h_l} toward ε also give

$$\lim_{l \to +\infty} \sup_{\vec{v}_{h_l} \in \mathbf{B}_{h_l}} \left|\langle G(\varepsilon_{h_l}; \vec{w}_{h_l}) - G(\varepsilon; \vec{w}_{h_l}), \vec{v}_{h_l}\rangle_{\mathbf{X}' \times \mathbf{X}}\right| = 0. \tag{26}$$

Gathering (24,25,25) with (23) show that

$$\lim_{l \to +\infty} \sup_{\vec{v}_{h_l} \in \mathbf{B}_{h_l}} b(\varepsilon_{h_l}; \vec{v}_{h_l}, p_{h_l}) - b(\varepsilon; \vec{v}_{h_l}, p) = 0. \tag{27}$$

We now estimate the last term of (22) which reads

$$\begin{aligned}
\sup_{\vec{v}_{h_l}\in\mathbf{B}_{h_l}} \left|\int_\Omega p\mathrm{div}((\varepsilon_{h_l}-\varepsilon)\vec{v}_{h_l})\,dx\right| &\le \sup_{\vec{v}_{h_l}\in\mathbf{B}_{h_l}} \|\nabla\varepsilon_{h_l}-\nabla\varepsilon\|_{L^4(\Omega)}\|\vec{v}_{h_l}\|_{L^4(\Omega)^d}\|p\|_{L^2(\Omega)} \\
&+ \sup_{\vec{v}_{h_l}\in\mathbf{B}_{h_l}} \Big(\|\varepsilon_{h_l}-\varepsilon\|_{L^\infty(\Omega)}\|\vec{v}_{h_l}\|_{L^2(\Omega)^d}\Big)\|p\|_{L^2(\Omega)} \\
&\le C(\Omega)\|p\|_{L^2(\Omega)}\Big(\|\nabla\varepsilon_{h_l}-\nabla\varepsilon\|_{L^4(\Omega)}+\|\varepsilon_{h_l}-\varepsilon\|_{L^\infty(\Omega)}\Big) \\
&\to 0 \text{ as } l\to+\infty, \qquad (28)
\end{aligned}$$

where (12) was used to get the last limit. Using the fact that the inf-sup constant satisfy $\lim_{l\to+\infty}\beta_*(\varepsilon_{h_l}) = \beta_*(\varepsilon)$ (see Lemma 7), we can finally use (22) together with (28,27) as well as the uniqueness of the limit, to obtain that $p_h \to p$ strongly in $L^2(\Omega)$ as h goes to 0.

Theorem 9 gives the convergence of $(\vec{w}_h, p_h)$ solution to (13) toward $(\vec{w}, p)$ satisfying (5). Since the solution to the weak formulation of the Darcy-Brinkman-Forchheimer model (4) is related to (5) through $\vec{u} = \vec{w} + \vec{V}$, its finite element approximation is given as $\vec{u}_h = \vec{w}_h + \mathcal{I}_{\mathbf{X}_h}\vec{V}$. We have then also proved that the finite element approximation $(\vec{u}_h, p_h)$ strongly converge toward $(\vec{u}, p)$ which is the unique solution to (4).

Remark 10. *Assumptions (12) on the regularity of ε can be weakened by assuming that one has $(\varepsilon_h)_h \in \boldsymbol{M}_h$ which converges strongly toward ε in $L^\infty(\Omega)\cap W^{1,4}(\Omega)$.*

3.3. Optimal error estimates

Theorem 9 does not give any detail about the order of convergence of the finite element method but holds for any weak solution to the Darcy-Brinkman-Forchheimer problem without additional regularity. In this section, we prove optimal error estimate for the finite element approximation of the linear and non-linear problems (5) and (7), respectively.

Convergence estimate for the linear problem

We consider here the linear problem (7) whose finite element discretization is (14) when the given porosity is also discretized. We emphasize that Problem (14) falls into the class of discrete saddle-point problem such as those studied in [12, p. 65, II.2.6]. The existence and uniqueness of $(\vec{w}_h, p_h)$ satisfying (14) is ensured by the inf-sup condition from Lemma 7 and the coercivity and continuity of the bilinear form a_h (see (8)). We also have the following convergence result.

Theorem 11. *Assume that $\alpha :]0,1] \to \mathbb{R}^+$ is Lipchitz continuous. Let $(\vec{w}, p)$ be the unique solution to (7). Assume that h is small enough so that the inf-sup* *condition from Lemma 7 holds and let $(\vec{w}_h, p_h)$ be the unique solution to (14). We then have*

$$\begin{aligned}\|\vec{w}_h - \vec{w}\|_{\boldsymbol{X}} + \|p_h - p\|_{L^2(\Omega)} &\leq C\left(\inf_{\vec{v}_h\in \boldsymbol{X}_h} \|\vec{w} - \vec{v}_h\|_{\boldsymbol{X}} + \inf_{q_h\in \boldsymbol{M}_h} \|p - q_h\|_{L^2(\Omega)}\right) \\ &+ C\max\left\{\|\varepsilon_h - \varepsilon\|_{L^\infty(\Omega)}, \|\varepsilon - \varepsilon_h\|_{W^{1,4}(\Omega)}\right\}.\end{aligned}$$

Proof. We apply [12, p. 67, Proposition 2.16] to get

$$\begin{aligned}\|\vec{w}_h - \vec{w}\|_{\mathbf{X}} &+ \|p_h - p\|_{L^2(\Omega)} \leq C\left(\inf_{\vec{v}_h\in \mathbf{X}_h} \|\vec{w} - \vec{v}_h\|_{\mathbf{X}} + \inf_{q_h\in \mathbf{M}_h} \|p - q_h\|_{L^2(\Omega)}\right) \\ &+ \sup_{\vec{v}_h\in \mathbf{X}_h} \frac{|a(\varepsilon;\vec{w},\vec{v}_h) - a(\varepsilon_h;\vec{w},\vec{v}_h)|}{\|\vec{v}_h\|_{\mathbf{X}}} + \sup_{\vec{v}_h\in \mathbf{X}_h} \frac{|b(\varepsilon;\vec{v}_h,p) - b(\varepsilon_h;\vec{v}_h,p)|}{\|\vec{v}_h\|_{\mathbf{X}}} \\ &+ \sup_{q_h\in \mathbf{M}_h} \frac{|b(\varepsilon;\vec{w},q_h) - b(\varepsilon_h;\vec{w},q_h)|}{\|q_h\|_{L^2(\Omega)}}.\end{aligned}$$

Note that

$$\begin{aligned}a(\varepsilon;\vec{w},\vec{v}) - a(\varepsilon_h;\vec{w},\vec{v}) &= \int_\Omega (\varepsilon - \varepsilon_h)\, S(\vec{w}){:}S(\vec{v}) + (\alpha(\varepsilon) - \alpha(\varepsilon_h))\, \vec{w}\cdot\vec{v}\, dx, \\ b(\varepsilon;\vec{w},q) - b(\varepsilon_h;\vec{w},q) &= \int_\Omega q\mathrm{div}\left((\varepsilon_h - \varepsilon)\vec{w}\right)\, dx.\end{aligned}$$

The error estimate then follows easily thanks to the Lipschitz continuity of $s \in]0,1] \mapsto \alpha(s) \in \mathbb{R}^+$ and the Hölder inequality.

If $\varepsilon \in W^{1,\infty}(\Omega) \cap W^{l+1,r}(\Omega)$ for some $l > 0$ then the estimates (12) hold. Assuming also that the solution $(\vec{w}, p)$ to (7) are in $H^{s+1}(\Omega)^d \times H^s(\Omega)$ then the error estimate from Theorem 11 reads

$$\begin{aligned}\|\vec{w}_h - \vec{w}\|_{\mathbf{X}} + \|p_h - p\|_{L^2(\Omega)} &\leq Ch^s\left(\|\vec{w}\|_{H^{s+1}(\Omega)} + \|p\|_{H^s(\Omega)}\right) \\ &+ C\max\left\{h\,\|\varepsilon\|_{W^{1,\infty}(\Omega)}, h^l\,\|\varepsilon\|_{W^{l+1,r}(\Omega)}\right\},\end{aligned}$$

where we used [16, p. 61, Corollary 1.110] to get the dependence of the inf with respect to the meshsize.

Let us now consider $\vec{u} = \vec{w} + \vec{V}$ where $\vec{V}$ is the divergence-free lifting of the inhomogeneous Dirichlet boundary condition introduced in Lemma 18. It is worth noting that $(\vec{u}, p)$ satisfy the linear Darcy-Brinkman-Forchheimer problem with inhomogeneous Dirichlet boundary condition on Γ_{in}. The finite element discretization of $\vec{u}$ is then $\vec{u}_h = \vec{w}_h + \mathcal{I}_{\mathbf{X}_h}\vec{V}$ and one has the error estimate

$$\begin{aligned}\|\vec{u}_h - \vec{u}\|_{\mathbf{X}} + \|p_h - p\|_{L^2(\Omega)} &\leq Ch^s\left(\|\vec{w}\|_{H^{s+1}(\Omega)} + \|p\|_{H^s(\Omega)}\right) + \left\|\vec{V} - \mathcal{I}_{\mathbf{X}_h}\vec{V}\right\|_{\mathbf{X}} \\ &+ C\max\left\{h\,\|\varepsilon\|_{W^{1,\infty}(\Omega)}, h^l\,\|\varepsilon\|_{W^{l+1,r}(\Omega)}\right\},\end{aligned}$$

where we used that $(\vec{w}, p) \in H^{s+1}(\Omega)^d \times H^s(\Omega)$.

The discrete non-linear problem without using a discrete porosity

We consider now the discrete problem associated to (5) where the porosity is not discretized. The latter is very similar to (13) and reads

$$\begin{aligned}
&\textit{Find } (\vec{w}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h \textit{ such that}\\
&\begin{cases} a(\varepsilon; \vec{w}_h, \vec{v}_h) + b(\varepsilon; \vec{v}_h, p_h) &= \langle G(\varepsilon; \vec{w}_h), \vec{v}_h \rangle_{\mathbf{X}' \times \mathbf{X}}, \quad \forall \vec{v}_h \in \mathbf{X}_h, \\ b(\varepsilon; \vec{w}_h, q_h) &= 0, \qquad\qquad\qquad\qquad\quad \forall q_h \in \mathbf{M}_h. \end{cases}
\end{aligned} \tag{29}$$

We emphasize that the existence and uniqueness of solution to (29) can be proved with arguments similar to those used to get Theorem 8. We are now going to compute the effective order of convergence of $(\vec{w}_h, p_h)$ toward $(\vec{w}, p)$ which satisfy (5). This can be done using the results from [10] (see also [21, p. 14, Section 4.2], [22] and [9, Theorem 4.5]) and relies on several properties that we check below.

We recall that (5) is equivalent to

$$\mathcal{F}(\vec{w}, p) := (\vec{w}, p) - \mathcal{S}\mathcal{G}(\vec{w}, p) \text{ with } \mathcal{G}(\vec{w}, p) = G(\varepsilon; \vec{w}),$$

where $\mathcal{S} : \vec{F} \in \mathbf{X}' \mapsto (\vec{w}, p) \in \mathbf{X} \times L^2(\Omega)$ is the unique solution of (7). Now, let $\mathcal{S}_h : \vec{F} \in \mathbf{X}' \mapsto (\vec{w}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h$ be the operator associated to any right-hand side $\vec{F}$ the solution to the following linear discrete problem

$$\begin{aligned}
&\textit{Find } (\vec{w}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h \textit{ such that}\\
&\begin{cases} a(\varepsilon; \vec{w}_h, \vec{v}_h) + b(\varepsilon; \vec{v}_h, p_h) &= \left\langle \vec{F}, \vec{v}_h \right\rangle_{\mathbf{X}' \times \mathbf{X}}, \quad \forall \vec{v}_h \in \mathbf{X}_h, \\ b(\varepsilon; \vec{w}_h, q_h) &= 0, \qquad\qquad\qquad \forall q_h \in \mathbf{M}_h. \end{cases}
\end{aligned} \tag{30}$$

Then (29) is equivalent to the non-linear equation

$$\mathcal{F}_h(\vec{w}_h, p_h) := (\vec{w}_h, p_h) - \mathcal{S}_h\mathcal{G}(\vec{w}_h, p_h) \text{ with } \mathcal{G}(\vec{w}_h, p_h) = G(\varepsilon; \vec{w}_h). \tag{31}$$

Since (30) is a linear saddle-point problem where the bilinear form a is coercive and continuous (see (8) and the bilinear form b satisfy an inf-sup condition (see Lemma 6), we can apply [12, p. 54, Proposition 2.4] and get that the operator $\mathcal{S}_h$ verifies

$$\begin{aligned}
&\left\| \mathcal{S}_h \vec{F} \right\|_{\mathbf{X} \times L^2(\Omega)} \leq C \left\| \vec{F} \right\|_{\mathbf{X}'}, \\
&\left\| (\mathcal{S}_h - \mathcal{S}) \vec{F} \right\|_{\mathbf{X} \times L^2(\Omega)} \leq C \inf_{(\vec{v}_h, q_h) \in \mathbf{X}_h \times \mathbf{M}_h} \left\| \mathcal{S}_h \vec{F} - (\vec{v}_h, q_h) \right\|_{\mathbf{X} \times L^2(\Omega)}.
\end{aligned} \tag{32}$$

From (32) and the density of smooth function in $\mathbf{X} \times L^2(\Omega)$, we obtain

$$\lim_{h \to 0} \left\| (\mathcal{S}_h - \mathcal{S}) \vec{F} \right\|_{\mathbf{X} \times L^2(\Omega)} = 0. \tag{33}$$

We prove below that the differential $D_{\mathcal{F}}(\vec{w}, p)$ of $\mathcal{F}$ at $(\vec{w}, p)$ is an isomorphism of $\mathbf{X} \times L^2(\Omega)$.

Lemma 12. *Let $(\vec{w},p) \in \boldsymbol{X} \times L^2(\Omega)$ be the solution to $\mathcal{F}(\vec{w},p) = 0$. Then there exists $\eta > 0$ such that if*

$$\left\|\vec{F}\right\|_{\boldsymbol{X}} + \|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d} \leq \eta,$$

then $D_{\mathcal{F}}(\vec{w},p) : \boldsymbol{X} \times L^2(\Omega) \to \boldsymbol{X} \times L^2(\Omega)$ is an isomorphism with bounded inverse.

Proof. A computation gives

$$D_{\mathcal{F}}(\vec{w},p)[\delta\vec{w},\delta p] = (\delta\vec{w},\delta p) - \mathcal{S}D_{\mathcal{G}}(\vec{w},p)[\delta\vec{w},\delta p].$$

We recall that $\mathcal{G}(\vec{w},p) = G(\varepsilon;\vec{w})$ where the non-linear term $G(\varepsilon;\cdot)$ is defined in (6). Then

$$\begin{aligned} \langle D_{\mathcal{G}}(\vec{w},p)[\delta\vec{w},\delta p],\vec{v}\rangle_{\mathbf{X}'\times\mathbf{X}} \quad = \quad & -\int_\Omega \varepsilon\left(\left(\vec{w}+\vec{V}\right)\cdot\nabla\right)\delta\vec{w}\cdot\vec{v}\,dx \\ & -\int_\Omega \varepsilon\left((\delta\vec{w})\cdot\nabla\right)\left(\vec{w}+\vec{V}\right)\cdot\vec{v}\,dx \\ & -\int_\Omega \beta(\varepsilon)\left\{\left|\vec{w}+\vec{V}\right|\delta\vec{w}\cdot\vec{v}\right\}\,dx \\ & -\int_\Omega \beta(\varepsilon)\left\{\frac{\vec{w}+\vec{V}}{\left|\vec{w}+\vec{V}\right|}\cdot\delta\vec{w}((\vec{w}+\vec{V})\cdot\vec{v})\right\}\,dx. \end{aligned} \tag{34}$$

To study the invertibility of the operator $[\delta\vec{w},\delta p] \to D_{\mathcal{F}}(\vec{w},p)[\delta\vec{w},\delta p]$, we consider the equation $D_{\mathcal{F}}(\vec{w},p)[\delta\vec{w},\delta p] = \vec{F}$ which is equivalent to the linear saddle-point problem

$$\begin{aligned} & Find\ (\delta\vec{w},\delta p) \in \mathbf{X}\times L^2(\Omega)\ such\ that\ \forall\ \vec{v}\in\mathbf{X},\ q\in L^2(\Omega): \\ & \left\{\begin{array}{ll} a(\varepsilon;\delta\vec{w},\vec{v}) - \langle D_{\mathcal{G}}(\vec{w},p)[\delta\vec{w},\delta p],\vec{v}\rangle_{\mathbf{X}'\times\mathbf{X}} + b(\varepsilon;\vec{v},\delta p) & = \left\langle\vec{F},\vec{v}\right\rangle_{\mathbf{X}'\times\mathbf{X}}, \\ b(\varepsilon;\delta\vec{w},q) & = 0. \end{array}\right. \end{aligned} \tag{35}$$

Theorem 5 and Lemma 18 give that $(\vec{w},p) \in \mathbf{X}\times L^2(\Omega)$ and $\vec{V}$ satisfy the next estimate

$$\|\vec{w}\|_{\mathbf{X}} + \|p\|_{L^2(\Omega)} \leq C_{\mathrm{stab}}\left(\left\|\vec{F}\right\|_{\mathbf{X}'} + \|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d}\right),\ \left\|\vec{V}\right\|_{\mathbf{X}} \leq M(\varepsilon)\,\|\vec{u_{\mathrm{in}}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d}.$$

As a result, there exists a constant $C > 0$ such that

$$\left|\langle D_{\mathcal{G}}(\vec{w},p)[\delta\vec{w},\delta p],\delta\vec{w}\rangle_{\mathbf{X}'\times\mathbf{X}}\right| \leq C\left(\left\|\vec{F}\right\|_{\mathbf{X}'} + \|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d}\right)\|\delta\vec{w}\|^2_{\mathbf{X}}.$$

Therefore, there exists $\eta > 0$ such that if $\left\|\vec{F}\right\|_{\mathbf{X}} + \|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d} \leq \eta$, then

$$\left|\langle D_{\mathcal{G}}(\vec{w},p)[\delta\vec{w},\delta p],\delta\vec{w}\rangle_{\mathbf{X}'\times\mathbf{X}}\right| \leq \mathrm{Re}^{-1}\varepsilon_0\,\|\delta\vec{w}\|^2_{\mathbf{X}}.$$

Using now (8), we obtain that the bilinear form

$$A(\delta\vec{w},\vec{v}) := a(\varepsilon;\delta\vec{w},\vec{v}) - \langle D_{\mathcal{G}}(\vec{w},p)[\delta\vec{w},\delta p],\vec{v}\rangle_{\mathbf{X}'\times\mathbf{X}}\,,$$

is coercive and continuous on $\mathbf{X}\times\mathbf{X}$. Lemma 2 gives that $b(\varepsilon;\cdot,\cdot)$ satisfies an inf-sup condition and [12, II.1, Proposition 1.3] then show that (35) is well-posed and the solutions satisfy a bound similar to those of Theorem 3. This proves that $D_{\mathcal{F}}(\vec{w},p) : \mathbf{X}\times L^2(\Omega) \to \mathbf{X}\times L^2(\Omega)$ is an isomorphism with bounded inverse.

We now show the properties needed to apply the results from [10].

Theorem 13. *Assume that the solution $(\vec{w},p)$ to (5) is in $H^{s+1}(\Omega)^d\times H^s(\Omega)$. Assume also that h is small enough so that $\mathcal{S}_h$ is well-defined. Then we have the following properties*

(i) The next error estimate is valid

$$\left\|(\mathcal{S}_h-\mathcal{S})\,\vec{F}\right\|_{\boldsymbol{X}\times L^2(\Omega)} \le Ch^s\left\|\mathcal{S}\vec{F}\right\|_{H^{s+1}(\Omega)^d\times H^s(\Omega)}.$$

(ii) There exists a constant $C(\vec{w},p)>0$ that does not depend on h so that

$$\|\mathcal{F}_h(\vec{w},p)\|_{\boldsymbol{X}\times L^2(\Omega)} \le C(\vec{w},p)h^s.$$

(iii) There exists $\eta>0$ such that if $\left\|\vec{F}\right\|_{\boldsymbol{X}}+\|\vec{u}_{\rm in}\|_{H^{1/2}_{00}(\Gamma_{\rm in})^d}\le\eta$, then $D_{\mathcal{F}_h}(\vec{w},p)$ is an isomorphism of $\boldsymbol{X}\times L^2(\Omega)$ and the norm of its inverse is bounded independently of h.

(iv) There exists a neighborhood $\mathcal{U}$ of $(\vec{w},p)\in\boldsymbol{X}\times L^2(\Omega)$ and a constant $L>0$ such that

$$\forall(\vec{v},q)\in\mathcal{U},\ \|D_{\mathcal{F}_h}(\vec{w},p)-D_{\mathcal{F}_h}(\vec{v},p)\|_{\mathcal{L}(\boldsymbol{X}\times L^2(\Omega))}\le L\,\|(\vec{w}-\vec{v},p-q)\|_{\boldsymbol{X}\times L^2(\Omega)}\,.$$

Proof. The proof of (i) follows from (32), the regularity of $(\vec{w},p)$ and [16, p. 61, Corollary 1.110]. To get (ii), we note that $\mathcal{F}(\vec{w},p)=0$. Using then (31), we obtain that

$$\mathcal{F}_h(\vec{w},p)=\mathcal{F}_h(\vec{w},p)-\mathcal{F}(\vec{w},p)=(\mathcal{S}_h-\mathcal{S})\,\mathcal{G}(\vec{w},p).$$

Lemma 4 and (i) then prove (ii). Regarding (iii), the invertibility of $D_{\mathcal{F}_h}(\vec{w},p)$ can be obtained as in the proof of Lemma 12. The fact that the inverse of $D_{\mathcal{F}_h}(\vec{w},p)$ has a norm that does not depend on h follows from the fact that the coercivity constant of $a(\varepsilon;\cdot,\cdot)$ and the inf-sup constant of $b(\varepsilon;\cdot,\cdot)$ does not depend on the mesh-size if h is small enough.

We now prove (iv). The differential $D_{\mathcal{F}_h}(\vec{w},p)$ of $\mathcal{F}_h$ at $(\vec{w},p)$ is given by

$$D_{\mathcal{F}_h}(\vec{w},p)[\delta\vec{w},\delta p]=(\delta\vec{w},\delta p)-\mathcal{S}_hD_{\mathcal{G}}(\vec{w},p)[\delta\vec{w},\delta p],$$

where $D_{\mathcal{G}}(\vec{w},p)$ is defined in (34). From (32), we only have to study the local Lipschitz property of the application $(\vec{v},q) \to D_{\mathcal{G}}(\vec{v},q)$. Using (34), one can see that the three first terms appearing in $D_{\mathcal{G}}(\vec{v},q)$ are locally Lipschitz. It only remains to prove a that the next application $\Psi : (\vec{v},q) \in \mathbf{X} \times L^2(\Omega) \mapsto \Psi(\vec{v},q) \in \mathbf{X}'$ defined for all $\vec{u} \in \mathbf{X}$ by

$$\langle \Psi(\vec{v},q), \vec{u} \rangle_{\mathbf{X}' \times \mathbf{X}} = - \int_\Omega \beta(\varepsilon) \left\{ \frac{\vec{v}+\vec{V}}{\left|\vec{v}+\vec{V}\right|} \cdot \delta\vec{w} \right\} ((\vec{v}+\vec{V}) \cdot \vec{u})\, dx,$$

is Lipschitz in a neighborhood of $(\vec{w},p)$ satisfying $\mathcal{F}(\vec{w},p) = 0$.
We start by the case where $(\vec{w},p) = (-\vec{V},p)$. It is worth noting that

$$\|\Psi(\vec{v},q)\|_{\mathbf{X}'} \leq \|\beta\|_{L^\infty(\Omega)} \left\|\vec{v}+\vec{V}\right\|_{\mathbf{X}} \|\delta\vec{w}\|_{\mathbf{X}},$$

and thus $\Psi(-\vec{V},p) = 0$. This shows that

$$\left\|\Psi(\vec{v},q) - \Psi(-\vec{V},p)\right\|_{\mathbf{X}'} \leq \|\beta\|_{L^\infty(\Omega)} \left\|\vec{v} - (-\vec{V})\right\|_{\mathbf{X}} \|\delta\vec{w}\|_{\mathbf{X}},$$

and thus the application $(\vec{v},q) \to \Psi(\vec{v},q)$ is Lipchitz in a neighborhood of $(\vec{w},p) = (-\vec{V},p)$.
The application $(\vec{v},q) \to \Psi(\vec{v},q)$ is smooth on $\left(\mathbf{X} \setminus \{-\vec{V}\}\right) \times L^2(\Omega)$. The differential of Ψ for all $\vec{w} \neq -\vec{V}$ is:

$$\begin{aligned}
\langle D_\Psi(\vec{w},p)[\delta\vec{u}], \vec{v} \rangle_{\mathbf{X}' \times \mathbf{X}} \quad = \quad & - \int_\Omega \beta(\varepsilon) \left\{ \frac{\delta\vec{u}}{\left|\vec{w}+\vec{V}\right|} \cdot \delta\vec{w} \right\} ((\vec{w}+\vec{V}) \cdot \vec{v})\, dx \\
& - \int_\Omega \beta(\varepsilon) \left\{ \frac{\vec{w}+\vec{V}}{\left|\vec{w}+\vec{V}\right|} \cdot \delta\vec{w} \right\} (\delta\vec{u} \cdot \vec{v})\, dx \\
& + \int_\Omega \beta(\varepsilon) \left\{ (\vec{w}+\vec{V} \cdot \delta\vec{w}) \right\} ((\vec{w}+\vec{V}) \cdot \vec{v}) \left(\frac{(\vec{w}+\vec{V}) \cdot \delta\vec{u}}{\left|\vec{w}+\vec{V}\right|^3} \right) dx.
\end{aligned}$$

This yields

$$\|D_\Psi(\vec{w},p)[\delta\vec{u}]\|_{\mathbf{X}'} \leq 3\, \|\beta\|_{L^\infty(\Omega)} \|\delta\vec{w}\|_{\mathbf{X}} \|\delta\vec{u}\|_{\mathbf{X}},$$

and a Taylor expansion finally shows that $(\vec{v},q) \to \Psi(\vec{v},q)$ is also locally Lipschitz in a neighborhood of $(\vec{w},p)$ satisfying $\mathcal{F}(\vec{w},p) = 0$ if $\vec{w} \neq -\vec{V}$.

Thanks to Lemma 12 and Theorem 13, we can use [18, p. 302, Theorem 3.1] (see also [10], [21, p. 14, Section 4.2], [22]) to get the following error estimate.

Theorem 14. *Assume that the assumptions from Lemmas 6 and 12 and of Theorem 13 are valid. Then there exists a constant $C(\vec{w}, p) > 0$ such that*

$$\|\vec{w}_h - \vec{w}\|_{\boldsymbol{X}} + \|p_h - p\|_{L^2(\Omega)} \leq C(\vec{w}, p) h^s.$$

In addition, if $(\vec{u}, p)$ denotes the solution to (4) then its finite element approximation $(\vec{u}_h, p_h)$ satisfies the error estimate

$$\|\vec{u}_h - \vec{u}\|_{\boldsymbol{X}} + \|p_h - p\|_{L^2(\Omega)} \leq Ch^s \left(\|\vec{w}\|_{H^{s+1}(\Omega)} + \|p\|_{H^s(\Omega)} \right) + \left\| \vec{V} - \mathcal{I}_{\boldsymbol{X}_h} \vec{V} \right\|_{\boldsymbol{X}}.$$

Theorem 14 gives optimal error estimate. Note nevertheless that the $O(h^s)$ can be deteriorated if the divergence-free lifting $\vec{V}$ is not regular enough.

The non-linear problem using a discrete porosity

We prove here some error estimates for the finite element approximation of $(\vec{w}, p)$ which satisfies (5). To ease the presentation, we introduce some notations. The solution to (5) is denoted by $\Phi(\varepsilon) = (\vec{w}(\varepsilon), p(\varepsilon))$, $\Phi_h(\varepsilon_h) = (\vec{w}_h(\varepsilon_h), p_h(\varepsilon_h))$ satisfy the discrete problem (13) so that $\Phi_h(\varepsilon) = (\vec{w}_h(\varepsilon), p_h(\varepsilon))$ is the solution to the non-linear discrete problem (29). We now write

$$\Phi(\varepsilon) - \Phi_h(\varepsilon_h) = (\Phi(\varepsilon) - \Phi_h(\varepsilon)) + (\Phi_h(\varepsilon) - \Phi_h(\varepsilon_h)) = E_1 + E_2.$$

It is worth noting that Theorem 14 can be used to bound E_1 and yields

$$\|E_1\|_{\mathbf{X} \times L^2(\Omega)} = \|\vec{w}_h - \vec{w}\|_{\mathbf{X}} + \|p_h - p\|_{L^2(\Omega)} \leq C(\vec{w}, p) h^s. \tag{36}$$

Regarding the second error term E_2, we recall that $\Phi_h(\varepsilon)$ is a solution to $\mathcal{H}(\varepsilon, \Phi_h) = 0$ where

$$\mathcal{H}(\varepsilon, \Phi_h) = \mathcal{L}(\varepsilon; \vec{w}_h, p_h) - \mathcal{G}(\varepsilon; \vec{w}_h, p_h), \text{ with } \mathcal{G}(\varepsilon; \vec{w}, p) = (G(\varepsilon; \vec{w}_h), 0).$$

Above, $\mathcal{L}(\varepsilon; \cdot, \cdot) : (\vec{w}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h \mapsto \mathbf{X}' \times L^2(\Omega)$ is the operator associated to (30) defined by

$$\langle \mathcal{L}(\varepsilon; \vec{w}, p), (\vec{v}, q) \rangle = (a(\varepsilon; \vec{w}, \vec{v}) + b(\varepsilon; \vec{v}, p), b(\varepsilon; \vec{w}, q)).$$

We are now in position to study the regularity of the mapping $\varepsilon \mapsto \Phi_h(\varepsilon)$.

Theorem 15. *Assume that the assumption under which Problem (29) has a unique solution are valid (see Theorem 8). Then there exists h_0 such that if $h < h_0$, we have the next estimate*

$$\|E_2\|_{\boldsymbol{X} \times L^2(\Omega)} \leq C \|\varepsilon - \varepsilon_h\|_{L^\infty(\Omega) \cap W^{1,4}(\Omega)},$$

where $C > 0$ does not depend on h.

PROOF. Let $\mathbf{Y} = L^\infty(\Omega) \cap W^{1,4}(\Omega)$ and

$$U = \{\varepsilon \in \mathbf{Y} \mid \varepsilon_0 \le \varepsilon(x) \le 1 \ \forall x \in \Omega\}.$$

For any $\varepsilon \in U$ and if $\vec{F}$ and $\vec{u}_{\text{in}}$ have small enough norms, we have the existence and uniqueness of $\Phi_h \in \mathbf{X}_h \times \mathbf{M}_h$ satisfying $\mathcal{H}(\varepsilon, \Phi_h) = 0$ together with the estimate

$$\|\Phi_h\|_{\mathbf{X}_h \times \mathbf{M}_h} \le C \left(\left\|\vec{F}\right\|_{\mathbf{X}} + \|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})} \right),$$

where the constant $C > 0$ does not depend on h since the coercivity and inf-sup constants does not depend on the mesh-size for h small enough. Using similar techniques as those from the demonstration of Lemma 12, one can show that $\delta\Phi \mapsto D_{\mathcal{H}}(\varepsilon, \Phi_h)[0, \delta\Phi]$ is an isomorphism (this actually amount to solve a discrete version of (35)). We emphasize that this application is also an isomorphism even if Φ_h is not a solution to $\mathcal{H}(\varepsilon, \Phi_h) = 0$, as soon as $\|\Phi_h\|_{\mathbf{X} \times L^2(\Omega)}$, $\left\|\vec{F}\right\|_{\mathbf{X}}$ and $\|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d}$ are small enough. In addition, since the coercivity and inf-sup constants does not depend on the mesh-size the norm of the inverse of the application $\delta\Phi \mapsto D_{\mathcal{H}}(\varepsilon, \Phi_h)[0, \delta\Phi]$ is bounded with a constant independent of h.

We now apply the implicit function theorem around some fixed $\varepsilon \in U$. Since the application $\mathcal{H} : \mathbf{Y} \times (\mathbf{X}_h \times \mathbf{M}_h) \to \mathbf{X}' \times L^2(\Omega)$ is continuous Fréchet differentiable we can use the implicit function theorem. This yields two neighborhoods $\mathcal{O} \subset U \subset \mathbf{Y}$ of ε and $\mathcal{V}_h \subset \mathbf{X}_h \times \mathbf{M}_h$ of $\Phi_h(\varepsilon)$ such that the application $\varepsilon \in \mathcal{O} \mapsto \Phi_h(\varepsilon) \in \mathcal{V}_h$ is Fréchet differentiable and that

$$\forall (\varepsilon, \Phi) \in \mathcal{O} \times \mathcal{V}_h, \ \mathcal{H}(\varepsilon, \Phi_h(\varepsilon)) = 0. \tag{37}$$

We also have some $\delta > 0$ such that the ball centered at $\Phi_h(\varepsilon)$ of radius δ is included into $\mathcal{V}_h$. This yields

$$\forall \varepsilon \in \mathcal{O}, \ \|\Phi_h(\varepsilon)\| \le \delta + C \left(\left\|\vec{F}\right\|_{\mathbf{X}} + \|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d} \right). \tag{38}$$

Differentiating (37) with respect to ε gives

$$D_{\mathcal{H}}(\varepsilon, \Phi_h(\varepsilon)) \left[0, D_{\Phi_h}(\varepsilon)[\delta\varepsilon]\right] = -D_{\mathcal{H}}(\varepsilon, \Phi_h(\varepsilon))[\delta\varepsilon, 0].$$

A direct calculation gives that

$$\begin{aligned}
\langle D_{\mathcal{H}}(\varepsilon, \Phi)[\delta\varepsilon, 0], (\vec{v}, q) \rangle &= (2\mathrm{Re}^{-1} \int_\Omega \delta\varepsilon S(\vec{u}) \mathbin{:} S(\vec{v}) + \alpha'(\varepsilon)(\delta\varepsilon)\vec{u} \cdot \vec{v} \, dx \\
&- \int_\Omega p \mathrm{div}(\delta\varepsilon \vec{v}) \, dx \\
&+ \int_\Omega \delta\varepsilon (\vec{u} \cdot \nabla)\vec{v} \cdot \vec{w} + \beta'(\varepsilon)(\delta\varepsilon)|\vec{u}|\vec{v} \cdot \vec{w} \, dx \\
&, \quad - \int_\Omega q \mathrm{div}(\delta\varepsilon \vec{u}) \, dx),
\end{aligned}$$

where $\Phi = (\vec{u}, p)$. From the Hölder inequality, we obtain

$$\|D_{\mathcal{H}}(\varepsilon, \Phi)[\cdot, 0]\|_{\mathcal{L}(\mathbf{Y}, \mathbf{X}' \times L^2(\Omega))} \leq C \max \left\{ \|\alpha'(\varepsilon)\|_{L^\infty(\Omega)}, \|\beta'(\varepsilon)\|_{L^\infty(\Omega)} \right\} \|\Phi\|_{\mathbf{X} \times L^2(\Omega)},$$

where $C > 0$ only depend on Ω. From (38), we can take $\delta, \vec{F}, \vec{u}_{\text{in}}$ small enough so that $\delta\Phi \mapsto D_{\mathcal{H}}(\varepsilon, \Psi_h)[0, \delta\Phi]$ is an isomorphism. It is worth noting that its inverse is bounded independently of h. This yields

$$\begin{aligned} \sup_{\varepsilon \in \mathcal{O}} \|D_{\Phi_h}(\varepsilon)\|_{\mathcal{L}(\mathbf{Y}, \mathbf{X}_h \times \mathbf{M}_h)} &\leq C \sup_{\varepsilon \in \mathcal{O}} \|D_{\mathcal{H}}(\varepsilon, \Phi)[\cdot, 0]\|_{\mathcal{L}(\mathbf{Y}, \mathbf{X}' \times L^2(\Omega))} \\ &\leq C \sup_{\varepsilon \in \mathcal{O}} \left(\max \left\{ \|\alpha'(\varepsilon)\|_{L^\infty(\Omega)}, \|\beta'(\varepsilon)\|_{L^\infty(\Omega)} \right\} \right). \end{aligned}$$

Owning to (12) we can chose h_0 such that $\|\varepsilon - \varepsilon_h\|_{W^{1,\infty}(\Omega) \cap W^{l+1,r}(\Omega)}$ is small enough so that $\varepsilon_h \in \mathcal{O}$ for any $h < h_0$. Since both functions $s \in [\varepsilon_0, 1] \mapsto \alpha'(s) \in \mathbb{R}^+$ and $s \in [\varepsilon_0, 1] \mapsto \beta'(s) \in \mathbb{R}^+$ are bounded, one gets

$$\begin{aligned} \|\Phi_h(\varepsilon) - \Phi_h(\varepsilon_h)\|_{\mathbf{X} \times L^2(\Omega)} &\leq \sup_{\gamma \in \mathcal{O}} \|D_{\Phi_h}(\gamma)\|_{\mathcal{L}(\mathbf{Y}, \mathbf{X}_h \times \mathbf{M}_h)} \|\varepsilon - \varepsilon_h\|_{L^\infty(\Omega) \cap W^{1,4}(\Omega)} \\ &\leq C \|\varepsilon - \varepsilon_h\|_{L^\infty(\Omega) \cap W^{1,4}(\Omega)}, \end{aligned}$$

and we have finally proved the desired result.

Assuming that $(\vec{w}, p) \in H^{s+1}(\Omega)^d \times H^s(\Omega)$ and using Theorem (15) and (36), we have proved that there exist $\eta > 0$ and h_0 such that if $h < h_0$ and

$$\left\|\vec{F}\right\|_{\mathbf{X}} + \|\vec{u}_{\text{in}}\|_{H_{00}^{1/2}(\Gamma_{\text{in}})^d} \leq \eta,$$

then

$$\begin{aligned} \|\vec{u}_h - \vec{u}\|_{\mathbf{X}} + \|p_h - p\|_{L^2(\Omega)} &\leq C(\vec{w}, p) h^s + \left\|\vec{V} - \mathcal{I}_{\mathbf{X}_h} \vec{V}\right\| \\ &+ C \|\varepsilon - \varepsilon_h\|_{L^\infty(\Omega) \cap W^{1,4}(\Omega)}, \end{aligned} \qquad (39)$$

where $(\vec{u}, p)$ satisfies (4) and $(\vec{u}_h, p_h) = (\vec{w}_h + \mathcal{I}_{\mathbf{X}_h} \vec{V}, p_h)$ satisfies (13). We then get optimal error estimates for the finite element approximation using a discretization of the porosity of the solution to the Darcy-Forcheimer-Brinkman model with mixed boundary conditions.

4. Numerical analysis of the DBF model

In this section, we present the numerical analysis of the DBF model. First, we present the method used to solve the non-linear discrete problem. The latter relies on a fixed-point method also known as Picard iteration and we are going to prove its convergence. We consider next a smoothly varying porosity, such as those appearing in packed beds (see e.g. [45, 1, 40]), to illustrate the convergence properties of the finite element method.

4.1. Picard-like iteration for solving the non-linear discrete problem

We recall that the non-linear discrete problem can be written as

$$\begin{aligned}
&\textit{Find } (\vec{u}_h, p_h) \in \mathbf{X}_{1,h} \times \mathbf{M}_h \textit{ such that forall } (\vec{v}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h\\
&\vec{u}_h|_{\Gamma_{\text{in}}} = \vec{u}_{\text{in}} \text{ and}\\
&\begin{cases} a(\varepsilon; \vec{u}_h, \vec{v}_h) + \delta c(\varepsilon; \vec{u}_h, \vec{u}_h, \vec{v}_h) + b(\varepsilon; \vec{v}_h, p_h) &= \left\langle \vec{F}, \vec{v}_h \right\rangle_{\mathbf{X}' \times \mathbf{X}}, \\ b(\varepsilon; \vec{u}_h, q_h) &= 0, \end{cases}
\end{aligned} \tag{40}$$

where one could use either ε or the finite element interpolant of the porosity, namely taking $\varepsilon = \varepsilon_h$ in (40) and $\delta \in \{0, 1\}$ allows to go from the linear ($\delta = 0$) to the non-linear problem ($\delta = 1$). The Picard-like iteration used to solve (40) with $\delta = 1$ is obtained by computing, for some n, $(\vec{u}_{h,n}, p_{h,n}) \in \mathbf{X}_{1,h} \times \mathbf{M}_h$ such that $\vec{u}_{h,n}|_{\Gamma_{\text{in}}} = \vec{u}_{\text{in}}$ and

$$\begin{cases} & \forall (\vec{v}_h, p_h) \in \mathbf{X}_h \times \mathbf{M}_h \\ a(\varepsilon; \vec{u}_{h,n}, \vec{v}_h) + c(\varepsilon; \vec{u}_{h,n-1}, \vec{u}_{h,n}, \vec{v}_h) + b(\varepsilon; \vec{v}_h, p_{h,n}) &= \left\langle \vec{F}, \vec{v}_h \right\rangle_{\mathbf{X}' \times \mathbf{X}}, \\ b(\varepsilon; \vec{u}_{h,n}, q_h) &= 0, \end{cases} \tag{41}$$

We now assume there is no volumic right hand side to lighten the overall expressions, and study below the convergence of the iterative method (41).

Theorem 16. *We consider $\varepsilon = \varepsilon_h$ in (41). Assume that $\|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d}$ is small enough so that*

$$\left(\frac{\|a\|}{\alpha_0} + 1 \right) M(\varepsilon) \, \|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d} \leq \frac{\text{Re}^{-1} \varepsilon_0}{C_{\text{NL}}},$$

where $M(\varepsilon)$ is given in Lemma 18. Assume also that

$$\|\alpha(\varepsilon)\|_{L^\infty(\Omega)} \leq C\text{Re}^{-1}, \ \|\beta(\varepsilon)\|_{L^\infty(\Omega)} \leq C,$$

where C are generic positive constants that does not depend on Re^{-1}. Then there exists a generic constant $C_{\text{CV}} > 0$ that may depend on ε such that if

$$\|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d} \leq C_{\text{CV}} \text{Re}^{-1},$$

then the sequence $(\vec{u}_{h,n}, p_{h,n})$ generated by (41) converges toward the solution to (40) in the strong topology of $\boldsymbol{X} \times L^2(\Omega)$.

Proof. Note first that $(\vec{u}_{h,1}, p_{h,1}) = (\vec{w}_{h,1} + V, p_{h,1})$ where $(\vec{w}_{h,1}, p_{h,1}) \in \mathbf{X}_h \times \mathbf{M}_h$ satisfies (14) with the next right hand side

$$\left\langle \vec{F}, \vec{v}_h \right\rangle_{\mathbf{X}' \times \mathbf{X}} = -a(\varepsilon; V, v_h).$$

Using [12, II.1, Proposition 1.3], we get that $(\vec{w}_{h,1}, p_{h,1})$ exists uniquely and satisfies

$$\|\vec{w}_{h,1}\|_{\text{X}} \leq \frac{\|a\|}{\alpha_0} \|V\|_{\mathbf{X}}, \ \|p_{h,1}\|_{L^2(\Omega)} \leq \frac{1}{\beta_2(h, \varepsilon)} \left(1 + \frac{\|a\|}{\alpha_0} \right) \|a\| \, \|V\|_{\mathbf{X}},$$

from which we infer

$$\|\vec{u}_{h,1}\|_{\mathbf{X}} \leq \left(\frac{\|a\|}{\alpha_0}+1\right)\|V\|_{\mathbf{X}} \leq \left(\frac{\|a\|}{\alpha_0}+1\right) M(\varepsilon)\,\|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d}\,.$$

If we now assume that $\|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d}$ is small enough so that

$$\|\vec{u}_{h,n-1}\|_{\mathbf{X}} \leq \frac{\mathrm{Re}^{-1}\varepsilon_0}{C_{\mathrm{NL}}}, \tag{42}$$

then the bilinear form $a(\varepsilon;\cdot,\cdot)+c(\varepsilon;\vec{u}_{h,n-1},\cdot,\cdot)$ is coercive with coercivity constant $\mathrm{Re}^{-1}\varepsilon_0$. As a result, $\vec{w}_{h,n}+V, p_{h,n} \in \mathbf{X}_h \times \mathbf{M}_h$ defined as $(\vec{u}_{h,n},p_{h,n}) = (\vec{w}_{h,n}+V,p_{h,n})$ is well-defined and satisfies

$$\begin{aligned}
\|\vec{w}_{h,n}\|_{\mathrm{X}} &\leq \frac{\|a\|+C_{\mathrm{NL}}\,\|\vec{u}_{h,n-1}\|_{\mathbf{X}}}{\mathrm{Re}^{-1}\varepsilon_0}\,\|V\|_{\mathbf{X}}\,,\\
\|p_{h,n}\|_{L^2(\Omega)} &\leq \frac{1}{\beta_2(h,\varepsilon)}\left(1+\frac{\|a\|+C_{\mathrm{NL}}\,\|\vec{u}_{h,n-1}\|_{\mathbf{X}}}{\mathrm{Re}^{-1}\varepsilon_0}\right)\left(\|a\|+C_{\mathrm{NL}}\,\|\vec{u}_{h,n-1}\|_{\mathbf{X}}\right)\|V\|_{\mathbf{X}}\,,
\end{aligned}$$

from which we infer that

$$\|\vec{u}_{h,n}\|_{\mathbf{X}} \leq \left(\frac{\|a\|+C_{\mathrm{NL}}\,\|\vec{u}_{h,n-1}\|_{\mathbf{X}}}{\mathrm{Re}^{-1}\varepsilon_0}+1\right)\|V\|_{\mathbf{X}}\,. \tag{43}$$

Now setting $(\vec{\Phi}_{h,n},\pi_{h,n}) = (\vec{u}_{h,n}-\vec{u}_{h,n-1}, p_{h,n}-p_{h,n-1})$, we get that $(\vec{\Phi}_{h,n},\pi_{h,n}) \in \mathbf{X}_h \times \mathbf{M}_h$ satisfies the following discrete linear saddle-point problem

$$\left\{\begin{aligned}
a(\varepsilon;\vec{\Phi}_{h,n},\vec{v}_h)+c(\varepsilon;\vec{u}_{h,n-1},\vec{\Phi}_{h,n},\vec{v}_h)+b(\varepsilon;\vec{v}_h,\pi_{h,n}) &= \left\langle \vec{G}_n,\vec{v}_h\right\rangle_{\mathbf{X}'\times\mathbf{X}}\,,\\
b(\varepsilon;\vec{\Phi}_{h,n},q_h) &= 0,
\end{aligned}\right. \tag{44}$$

with $\vec{G}_n \in \mathbf{X}'$ defined for all $\vec{v} \in \mathbf{X}$ as follow

$$\left\langle \vec{G}_n,\vec{v}\right\rangle_{\mathbf{X}'\times\mathbf{X}} = c(\varepsilon;\vec{u}_{h,n-2},\vec{u}_{h,n-1},\vec{v}) - c(\varepsilon;\vec{u}_{h,n-1},\vec{u}_{h,n-1},\vec{v}).$$

From (42), we know that (44) is well-posed and that $(\vec{\Phi}_{h,n},\pi_{h,n})$ satisfies

$$\begin{aligned}
\|\Phi_{h,n}\|_{\mathbf{X}} &\leq \frac{1}{\mathrm{Re}^{-1}\varepsilon_0}\left\|\vec{G}_n\right\|_{\mathbf{X}'}\,,\\
\|\pi_{h,n}\|_{L^2(\Omega)} &\leq \frac{1}{\beta_2(h,\varepsilon)}\left(1+\frac{\|a\|+C_{\mathrm{NL}}\,\|\vec{u}_{h,n-1}\|_{\mathbf{X}}}{\mathrm{Re}^{-1}\varepsilon_0}\right)\left\|\vec{G}_n\right\|_{\mathbf{X}'}\,.
\end{aligned}$$

A computation gives

$$\sup_{\|v\|_{\mathbf{X}}=1}\left|\left\langle \vec{G}_n,\vec{v}\right\rangle_{\mathbf{X}'\times\mathbf{X}}\right| \leq C(\Omega)\max\left\{1,\|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\}\|\Phi_{h,n-1}\|_{\mathbf{X}}\,\|\vec{u}_{h,n-1}\|_{\mathbf{X}}\,,$$

where $C(\Omega)$ is a generic constant. We can thus finally infer that

$$\begin{aligned}
&\|\Phi_{h,n}\|_{\mathbf{X}} \le C(\Omega)\frac{1}{\mathrm{Re}^{-1}\varepsilon_0}\max\left\{1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\}\|\vec{u}_{h,n-1}\|_{\mathbf{X}}\|\Phi_{h,n-1}\|_{\mathbf{X}},\\
&\|\pi_{h,n}\|_{L^2(\Omega)} \le C(\Omega)\frac{1}{\beta_2(h,\varepsilon)}\left(1+\frac{\|a\|+C_{\mathrm{NL}}\|\vec{u}_{h,n-1}\|_{\mathbf{X}}}{\mathrm{Re}^{-1}\varepsilon_0}\right)\\
&\times\max\left\{1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\}\|\Phi_{h,n-1}\|_{\mathbf{X}}\|\vec{u}_{h,n-1}\|_{\mathbf{X}}.
\end{aligned} \tag{45}$$

The $O(.)$ below are used to highlight the dependance of some parameters with respect to Re^{-1}. Note that $\|a\| = C_K^2\|\alpha(\varepsilon)\|_{L^\infty(\Omega)} + 2\mathrm{Re}^{-1} = O(\mathrm{Re}^{-1})$ and $\alpha_0 = 2\mathrm{Re}^{-1}\varepsilon_0$ (see (8) and that $C_{\mathrm{NL}} = C(\Omega)\max(1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}) = O(1)$. As a result, one has that $\|a\|/\alpha_0 = O(1)$ and if $\|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})} \le C_{\mathrm{CV}}\mathrm{Re}^{-1}$ with

$$C_{\mathrm{CV}} \le \frac{\alpha_0\varepsilon_0}{C_{\mathrm{NL}}M(\varepsilon)(\alpha_0+\|a\|)},$$

then (42) is satisfied by $\vec{u}_{h,1}$. Now, if we assume that

$$C_{\mathrm{CV}} \le \frac{1}{2}\frac{\alpha_0\varepsilon_0}{C_{\mathrm{NL}}M(\varepsilon)(\alpha_0+\|a\|)}, \tag{46}$$

we get from (43) that $\|u_{h,2}\|_{\mathbf{X}} \le \mathrm{Re}^{-1}\varepsilon_0/C_{\mathrm{NL}}$ hence it satisfies (42) and by induction the whole sequence is determined and satisfies (42) for any $n \in \mathbb{N}^*$. Now combining (43) with (45) give

$$\begin{aligned}
\|\Phi_{h,n}\|_{\mathbf{X}} \;\le\;& C(\Omega)\frac{1}{\mathrm{Re}^{-1}\varepsilon_0}\max\left\{1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\}\left(\frac{\|a\|+C_{\mathrm{NL}}\|\vec{u}_{h,n-2}\|_{\mathbf{X}}}{\mathrm{Re}^{-1}\varepsilon_0}+1\right)\\
&\times M(\varepsilon)\|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d}\|\Phi_{h,n-1}\|_{\mathbf{X}}\\
\le\;& 2C_{\mathrm{CV}}C(\Omega)\frac{M(\varepsilon)\mathrm{Re}^{-1}}{\mathrm{Re}^{-1}\varepsilon_0}\max\left\{1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\}\left(\frac{\|a\|}{\alpha_0}+1\right)\|\Phi_{h,n-1}\|_{\mathbf{X}}.
\end{aligned}$$

As a result, if in addition to (46), we have

$$2C_{\mathrm{CV}}C(\Omega)\frac{M(\varepsilon)}{\varepsilon_0}\max\left\{1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\}\left(\frac{\|a\|}{\alpha_0}+1\right) < 1, \tag{47}$$

then the sequence $(\vec{u}_{h,n})_{n\in\mathbb{N}^*} \subset \mathbf{X}_1$ is a Cauchy sequence and therefore converges toward some $\vec{u}_h \in \mathbf{X}_1$. We actually have $\vec{u}_h \in \mathbf{X}_{1,h}$ since for all $n \in \mathbb{N}^*$ $(\vec{u}_{h,n})_{n\in\mathbb{N}^*} \subset \mathbf{X}_{1,h}$.
To get the convergence for the pressures $(p_{h,n})_{n\in\mathbb{N}^*}$, we note that (45) and (42) give the next bound

$$\|\pi_{h,n}\|_{L^2(\Omega)} \le C(\Omega)\frac{\mathrm{Re}^{-1}\varepsilon_0}{C_{\mathrm{NL}}\beta_2(h,\varepsilon)}\left(2+\frac{\|a\|}{\mathrm{Re}^{-1}\varepsilon_0}\right)\max\left\{1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\}\|\Phi_{h,n-1}\|_{\mathbf{X}}. \tag{48}$$

Since $(\Phi_{h,n})_{n\in\mathbb{N}^*}$ is a sequence that converges to 0 in $\mathbf{X}$, we infer that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, one has $\|\pi_{h,n}\|_{L^2(\Omega)} \leq 1/2$ and thus the sequence $(p_{h,n})_{n\in\mathbb{N}^*} \subset \mathbf{M}_h$ is a Cauchy sequence. This gives the existence of some $p_h \in \mathbf{M}_h$ such that $(p_{h,n})_{n\in\mathbb{N}^*}$ converges toward p_h. It now only remains to pass to the limit as $n \to +\infty$ to get that $(\vec{u}_h, p_h) \in \mathbf{X}_{1,h} \times \mathbf{M}_h$ converges toward the solution to (40).

Remark 17. *If ε is used then $\beta(h,\varepsilon)$ appears in the bounds where the pressure is involved (see (45) and (48)).*

We emphasize that the constant C_{CV} satisfy (46,47) which reduces to

$$C_{\mathrm{CV}} < \frac{\alpha_0\varepsilon_0}{2(\|a\| + \alpha_0)} \min\left\{\frac{1}{C_{\mathrm{NL}}M(\varepsilon)}, \frac{1}{C(\Omega)M(\varepsilon)\max\left\{1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\}}\right\},$$

where $C(\Omega)$ only depends on the geometry of the domain. It is also worth noting that the proof of Theorem 16 can also be used to prove the existence of solution to the continuous and discrete problems (4), (40) for $\vec{F} = 0$ and $\alpha(\varepsilon) = O(\mathrm{Re}^{-1})$ as soon as h is small enough to ensure that $\beta_2(h,\varepsilon) \geq \beta_*(\varepsilon)$. Nevertheless, we found that it was easier to rely on (31) which fit in the framework of [18, p. 302, Theorem 3.1] (see also [10], [21, p. 14, Section 4.2], [22]) to get explicit error estimates for the finite element method.

One can also compute the speed of convergence of the fixed-point iteration (41). Let us assume that C_{CV} is given as above and that $C_{\mathrm{CV}} \leq q < 1$. Since $\|\Phi_{h,n}\|_{\boldsymbol{X}} \leq q\,\|\Phi_{h,n-1}\|_{\boldsymbol{X}}$, we have

$$\|\vec{u}_{h,n} - \vec{u}\|_{\boldsymbol{X}} \leq \frac{q^{n-1}}{1-q}\,\|\vec{u}_{h,1} - \vec{u}_{h,2}\|_{\boldsymbol{X}},$$

where $\vec{u}_h$ satisfy (40). In addition, we can prove by induction that $\|\Phi_{h,n-1}\|_{\boldsymbol{X}} \leq q^{n-2}\,\|\Phi_{h,2}\|_{\boldsymbol{X}}$ and (48) thus gives

$$\|\pi_{h,n}\|_{L^2(\Omega)} \leq C(\Omega)\frac{\mathrm{Re}^{-1}\varepsilon_0}{C_{\mathrm{NL}}\beta_2(h,\varepsilon)}\left(2 + \frac{\|a\|}{\mathrm{Re}^{-1}\varepsilon_0}\right)\max\left\{1, \|\beta(\varepsilon)\|_{L^\infty(\Omega)}\right\} q^{n-2}\,\|\Phi_{h,2}\|_{\boldsymbol{X}}.$$

Note that the fixed-point iteration defined in (41) is well-defined if $\vec{u}_{\mathrm{in}}$ have a small enough norm. We also emphasize that, under these assumptions, this method is globally convergent. It is worth noting that the method do not converge otherwise and that the upper bound above which the method diverges also depends on the Reynolds number. As a result, divergence may occurs if Re or $\|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})^d}$ are too large. Nevertheless note that, for any Reynolds number, one can find some $\vec{u}_{\mathrm{in}}$ for which (41) actually converges and conversely, for any $\vec{u}_{\mathrm{in}}$, we have a Re_0 such that for any $\mathrm{Re} < \mathrm{Re}_0$ the method converges. We illustrate this behavior in our numerical experiments.

4.2. Numerical experiments

For this test case, we choose the following smooth porosity

$$\varepsilon(x,y) = 0.45\left(1 + \frac{1-0.45}{0.45}\exp(-(1-y))\right),$$

and recall that this can be obtained when considering packed beds such as those studied in [45, 1]. The Darcy and Forchheimer terms are defined in [29, p. 3, Eq (8,9)] (see also [45]) and read

$$\alpha(\varepsilon) = \frac{150}{\mathrm{Re}} \left(\frac{1-\varepsilon}{\varepsilon} \right)^2, \ \beta(\varepsilon) = 1.75 \left(\frac{1-\varepsilon}{\varepsilon} \right). \tag{49}$$

It is easy to see that they satisfy all the assumptions (3). We set $\Omega = (0,2) \times (0,1)$, $\Gamma_{\mathrm{in}} = \{0\} \times [0,1]$, $\Gamma_{\mathrm{out}} = \{2\} \times [0,1]$ and $\Gamma_w = [0,2] \times (\{0\} \cup \{1\})$. For the inlet velocity, we take the parabolic profile

$$\vec{u}_{\mathrm{in}}(y) = c_{\mathrm{in}} y(1-y)\vec{e}_x,$$

where c_{in} is a constant which is going to be tuned in order for $\|\vec{u}_{\mathrm{in}}\|_{H^{1/2}_{00}(\Gamma_{\mathrm{in}})}$ to be small enough to ensure that the discrete problem has a unique solution and also that (41) converges.

All the following numerical computations are done with FreeFem [23]. We use the Crout solver to get solve the linear problems (41) which needs every sub-matrices to be invertible. We thus add the term ηp in the incompressibility condition with $\eta = 1e-07$. We also used a $\mathbb{P}_1$ finite element approximation of ε, that is ε_h defined as the finite element interpolant of the porosity. We also emphasize that all the convergence theorems proved in the previous section apply to the considered test case. Finally, the mesh is obtained thanks to the Freefem command *buildmesh* $(a(N) + b(N) + c(N) + d(N))$ with being N the number of vertices on each part of the boundary denoted by a, b, c, d. As a result, the mesh-size h is

$$h = \frac{\sqrt{2}}{N},$$

and we can consider only N.

To set the constant c_{in}, we are going to compute the error between the last two iterates of (41) after $it_{\max}$ iterations have been performed. This amount to compute the following quantity

$$\mathrm{err}(\mathrm{Re}, it_{\max}) = \max\left\{ \|p_{h,it_{\max}} - p_{h,it_{\max}-1}\|_{L^2(\Omega)}, \|\vec{u}_{h,it_{\max}} - \vec{u}_{h,it_{\max}-1}\|_{L^2(\Omega)^2} \right\}.$$

We now want to compute numerically $c_{\mathrm{in,min}}$ such that

$$\forall c > c_{\mathrm{in,min}}, \ \mathrm{err}(\mathrm{Re}, it_{\max}) > tol,$$

for a given itmax and tolerance. Since the fixed point iteration (41) converges if $\vec{u}_{\mathrm{in}}$ has small enough norm, finding such $c_{\mathrm{in,min}}$ is useful to setup the parameters of our numerical experiments, namely the Reynolds number and the inlet velocity.

In Figure 1 are shown values of $c_{\mathrm{in,min}}$ for $it_{\max} = 10$, $N = 40, 120$ and $tol = 1e-5, 1e-10$ for several values of the Reynolds number. We used a discrete porosity defined as the $\mathbb{P}_1$ finite element interpolation of ε. For all

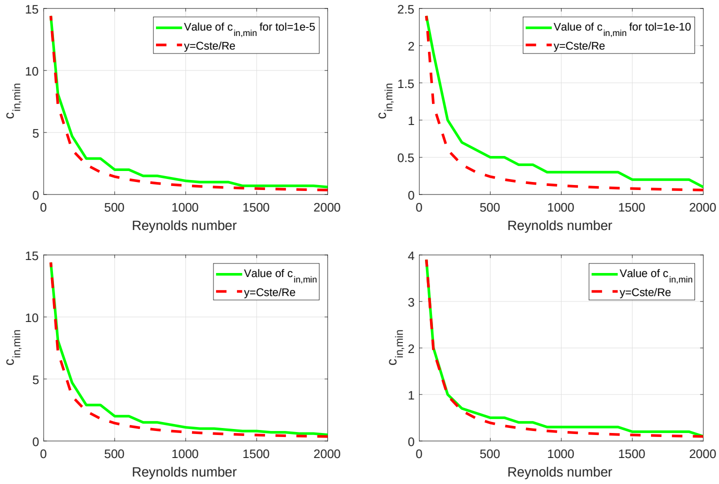

Figure 1: Value of $c_{\text{in,min}}$ with $it_{\max} = 10$. Top: $N = 40$, Bottom: $N = 120$, Left row: $tol = 1e - 5$, Right row: $tol = 1e - 10$.

cases considered, the value of $c_{\text{in,min}}$ behaves like $C \times Re^{-1}$. We emphasize that this is in agreement with Theorem 16 which shows that one needs to have $\|\vec{u}_{\text{in}}\|_{H^{1/2}_{00}(\Gamma_{\text{in}})} \leq C_{\text{CV}}/Re$ for the fixed point iteration to converge. It is also worth noting that, if one keeps the same number of iterations while diminishing the tolerance, then one gets a value of $c_{\text{in,min}}$ that is greatly reduced. This is actually expected from the theoretical speed of convergence of the iteration (41) computed in Remark 17 since a smaller value for c_{in} means a smaller value for C_{CV} which is directly linked to the speed of convergence. Note also that the value of $c_{\text{in,min}}$ slightly depends on the mesh-size. This is again expected from Remark 17 since the speed of convergence of the pressure depend on the discrete inf-sup constant $\beta_2(h, \varepsilon)$ which depends on h.

We now give some illustrations of the convergence order of the finite element approximation toward the continuous solution. Note that we do no have an explicit solution and we are thus going to compare the numerical solutions with one computed on a finer mesh. We first discuss the regularity of the weak solution to the Darcy-Brinkman-Forchheimer problem. Since ε is smooth and bounded over $\overline{\Omega}$, any solution to (1) also satisfies a Stokes problem

$$\left\{ \begin{array}{rcll} -2\mathrm{Re}^{-1}\mathrm{div}(S(\vec{u})) + \nabla p & = & \vec{F}, & \text{in } \Omega \\ \mathrm{div}\,(\vec{u}) & = & -\varepsilon^{-1}\nabla\varepsilon \cdot \vec{u}, & \text{in } \Omega, \end{array} \right. \tag{50}$$

with

$$\vec{F} = -\varepsilon^{-1}\alpha(\varepsilon)\vec{u} - \varepsilon^{-1}\beta(\varepsilon)\vec{u}|\vec{u}| - (\vec{u} \cdot \nabla)\,\vec{u} - 2\mathrm{Re}^{-1}\varepsilon^{-1}S(\vec{u})\nabla\varepsilon.$$

Regarding the boundary condition on Γ_{out}, one can prove as in the demonstration of Lemma 18 that $\mu \in H^{1/2}_{00}(\Gamma_{\text{out}}) \mapsto \varepsilon\mu \in H^{1/2}_{00}(\Gamma_{\text{out}})$ is a continuous linear

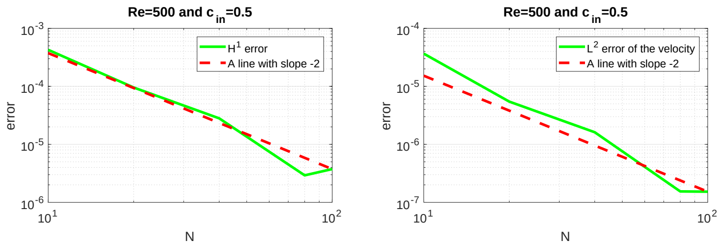

Figure 2: $(Re, c_{\rm in}) = (500, 0.5)$: (Left) Err_{tot}, (Right) L^2 error for the velocity.

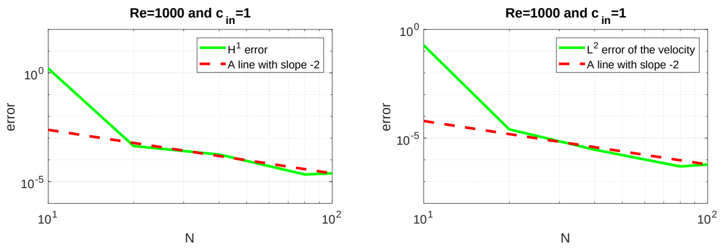

Figure 3: $(Re, c_{\rm in}) = (1000, 1)$: (Left) Err_{tot}, (Right) L^2 error for the velocity.

mapping with inverse mapping given as $\mu \in H^{1/2}_{00}(\Gamma_{\rm out}) \mapsto \varepsilon^{-1}\mu \in H^{1/2}_{00}(\Gamma_{\rm out})$. The boundary condition on $\Gamma_{\rm out}$ thus reduces to $\left(2{\rm Re}^{-1}S(\vec{u}) - p\right)\vec{n} = 0$ in $H^{1/2}_{00}(\Gamma_{\rm out})'$ which corresponds to a traction boundary condition on $\Gamma_{\rm out}$. As a result, any weak solution to (1) satisfies a Stokes problem with mixed boundary conditions, a right hand side $\vec{F}$ and inhomogeneous divergence ${\rm div}(\vec{u}) \in L^2(\Omega)$. Since $\vec{u}_{\rm in} = c_{\rm in} y(1-y)$ is smooth with $\vec{u}_{\rm in}|_{\partial\Gamma_{\rm in}} = 0$, it is actually at least in $H^{3-1/2}_{00}(\Gamma_{\rm in})$. The results from [27] (see also [28]) then ensure that any weak solution $(\vec{u}, p)$ to (1) is at least in $H^3(\Omega) \times H^2(\Omega)$. The convergence Theorem 14 (see also (39)) then gives

$$\begin{aligned} Err_{tot} := \|\vec{u}_h - \vec{u}\|_{\mathbf{X}} + \|p_h - p\|_{L^2(\Omega)} \quad &\leq \quad Ch^2\left(\|\vec{w}\|_{H^3(\Omega)} + \|p\|_{H^2(\Omega)}\right). \\ &+ \left\|\vec{V} - \mathcal{I}_{\mathbf{X}_h}\vec{V}\right\|_{\mathbf{X}} + C\,\|\varepsilon - \varepsilon_h\|_{L^\infty(\Omega)\cap W^{1,4}(\Omega)}. \end{aligned}$$

This error estimate is optimal when no finite element approximation of the porosity is used. Since we do not have explicit solution, we note $(\vec{u}_{\rm ex}, p_{\rm ex})$ the solution obtained with $N = 200$ and we compute the error between the discrete solution for $N \leq 100$ and $(\vec{u}_{\rm ex}, p_{\rm ex})$.

The errors are shown in Figures 2 for $(Re, c_{\rm in}) = (500, 0.5)$ and in Figure 3 for $(Re, c_{\rm in}) = (1000, 1)$. The optimal order of convergence, namely $Err_{tot} = O(h^2)$, is obtained. Since we used an approximate porosity $\varepsilon_h \in \mathbf{M}_h$ the convergence order is actually expected to be smaller. Nevertheless, the smoothness of ε ensures that $\|\varepsilon - \varepsilon_h\|_{L^\infty(\Omega)} = O(h^2)$ and that $\|\varepsilon - \varepsilon_h\|_{W^{1,4}(\Omega)} = O(h)$. We could therefore conclude that the theoretical error estimates involving the gradient of

$(\varepsilon_h - \varepsilon)$ is not optimal and that only the L^∞ norm of $(\varepsilon - \varepsilon_h)$ should appear in Err_{tot}. It is worth noting that this could be achieved by considering the bilinear form $\widetilde{b}(\varepsilon; \vec{u}, q) = \int_\Omega \varepsilon \nabla p \cdot \vec{u}\, dx$ instead of $b(\varepsilon; \cdot, \cdot)$ which is well-defined as $p \in H^1(\Omega)$. Note also that the Reynolds number does not have a significant effect on the total error. Regarding the L^2 error of the velocity, one could have expected one extra order of convergence as in the case of Stokes flow (see e.g. [16, p. 185, Proposition 4.18]) or elliptic problems. Note nevertheless that the convergence order in the L^2-norm is the same as the one of the total error. Once again, we have $\|\varepsilon - \varepsilon_h\|_{L^\infty(\Omega)} = O(h^2)$ which may cause the L^2 error to be second order accurate even if $\mathbb{P}_2$ element to approximate the velocity.

Appendix A. Divergence-free lifting

We provide here the existence of a divergence-free lifting of the inhomogeneous Dirichlet boundary condition.

Lemma 18. *There exists a vector field $\vec{V}$ solution of*

$$\begin{cases} \operatorname{div}\left(\varepsilon \vec{V}\right) &= 0 \quad \text{in } \Omega, \\ \vec{V} &= \vec{u}_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \\ \vec{V} &= 0 \quad \text{on } \Gamma_{\text{w}}, \end{cases}$$

which satisfies

$$\left\| \vec{V} \right\|_{\boldsymbol{X}} \le M(\varepsilon) \left\| \vec{u_{\text{in}}} \right\|_{H_{00}^{1/2}(\Gamma_{\text{in}})^d}.$$

where $M(\varepsilon) = C\left\{\varepsilon_0^{-1} + \varepsilon_0^{-1} \|\nabla \varepsilon\|_{L^3(\Omega)}\right\}$.

Proof. Let $\vec{a}$ be defined as follows

$$\vec{a} = \begin{cases} \varepsilon^{-1}\vec{u}_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ 0 & \text{on } \Gamma_{\text{w}}, \\ \vec{a}_{\text{out}} & \text{on } \Gamma_{\text{out}}, \end{cases}$$

where

$$\vec{a}_{\text{out}} := \left(-\int_{\Gamma_{\text{in}}} \varepsilon^{-1} \vec{u}_{\text{in}} \cdot \vec{n}\, d\sigma \right) \vec{u}_{\text{out}}$$

with $\vec{a}_{\text{out}} \in H_{00}^{1/2}(\Gamma_{\text{out}})$ being a given function such that

$$\int_{\Gamma_{\text{out}}} \vec{a}_{\text{out}} \cdot \vec{n}\, d\sigma = 1.$$

Note that, since $\vec{u}_{\text{in}} \in H_{00}^{1/2}(\Gamma_{\text{in}})$, one has $E_0(\vec{u}_{\text{in}}) \in H^{1/2}(\partial\Omega)$ and thus there exists $\vec{\Phi}_{\text{in}} \in H^1(\Omega)$ such that $\vec{\Phi}_{\text{in}}|_{\partial\Omega} = E_0(\vec{u}_{\text{in}})$. Since $\varepsilon^{-1}\vec{\Phi}_{\text{in}} \in H^1(\Omega)^d$, $\varepsilon^{-1}\vec{\Phi}_{\text{in}} \in H^{1/2}(\partial\Omega)$ and then $\varepsilon^{-1}\vec{\Phi}_{\text{in}}|_{\Gamma_{\text{in}}} = \varepsilon^{-1}\vec{u}_{\text{in}} \in H^{1/2}(\Gamma_{\text{in}})$. Note that

$$\int_{\Gamma_{\text{in}}} \frac{\varepsilon(s)^{-2} |\vec{u}_{\text{in}}|^2}{\operatorname{dist}(s, \partial\Gamma_{\text{in}})} < +\infty,$$

since $\vec{u}_{\text{in}} \in H^{1/2}_{00}(\Gamma_{\text{in}})$ and $\varepsilon \in L^\infty(\overline{\Omega})$ and we obtain $\varepsilon^{-1}\vec{u}_{\text{in}} \in H^{1/2}_{00}(\Gamma_{\text{in}})$.
Since $\vec{a}_{\text{out}} \in H^{1/2}_{00}(\Gamma_{\text{out}})$, we have some $\vec{\Phi}_{\text{out}} \in H^1(\Omega)$ such that $\vec{\Phi}_{\text{out}}|_{\partial\Omega} = E_0(\vec{u}_{\text{out}})$. Now setting $\vec{\Phi} = \vec{\Phi}_{\text{out}} + \vec{\Phi}_{\text{in}} \in H^1(\Omega)$, we have that $\vec{\Phi}|_{\partial\Omega} = \vec{a}$ and thus $\vec{a} \in H^{1/2}(\partial\Omega)$. Since

$$\int_{\partial\Omega} \vec{a}\cdot\vec{n}\, d\sigma = 0,$$

we can use [17, p. 176, Exercise III 3.5] (Ω needs to be bounded and locally Lipschitz) to get the existence of $\vec{U} \in H^1(\Omega)$ that satisfies

$$\left\{\begin{array}{rcl} \operatorname{div}(\vec{U}) & = & 0 \text{ in } \Omega, \\ \vec{U} & = & \vec{a} \text{ on } \partial\Omega, \end{array}\right. \tag{A.1}$$

together with the bound

$$\left\|\vec{U}\right\|_{H^1(\Omega)^d} \leq C\left\|\vec{a}\right\|_{H^{1/2}(\partial\Omega)^d} \leq C\left\|\vec{u_{\text{in}}}\right\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d}$$

From [7, p. 3, Lemma 2.1], the application $\vec{u} \in H^1(\Omega)^d \mapsto \varepsilon\vec{u} \in H^1(\Omega)^d$ is an isomorphism. Therefore, there exists $\vec{V} \in \mathbf{X}$ such that $\vec{U} = \varepsilon\vec{V}$ and

$$\left\{\begin{array}{rcll} \operatorname{div}\left(\varepsilon\vec{V}\right) & = & 0 & \text{in } \Omega, \\ \vec{V} & = & \vec{u}_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ \vec{V} & = & 0 & \text{on } \Gamma_{\text{w}}, \\ \vec{V} & = & \varepsilon\vec{a}_{\text{out}} & \text{on } \Gamma_{\text{out}}. \end{array}\right. \tag{A.2}$$

A computation also gives the bound (see also [40, p. 3, Theorem 2]):

$$\begin{array}{rcl} \left\|\vec{V}\right\|_{H^1(\Omega)^d} & = & \left\|\varepsilon^{-1}\vec{U}\right\|_{H^1(\Omega)^d} \leq C\left\{\varepsilon_0^{-1} + \varepsilon_0^{-2}\left\|\nabla\varepsilon\right\|_{L^3(\Omega)}\right\}\left\|\vec{U}\right\|_{H^1(\Omega)^d} \\ & \leq & C\left\{\varepsilon_0^{-1} + \varepsilon_0^{-1}\left\|\nabla\varepsilon\right\|_{L^3(\Omega)}\right\}\left\|\vec{u_{\text{in}}}\right\|_{H^{1/2}_{00}(\Gamma_{\text{in}})^d}. \end{array}$$

The proof is then finished by using the Korn inequality.

Appendix B. Regularity of the boundary stress tensor

We now give here a result about the regularity of the boundary stress tensor that justify the equivalence between (1,2) and its weak formulation (4). The latter can also be used when considering inhomogeneous traction boundary conditions such as

$$\varepsilon\left(2\mathrm{Re}^{-1}S(\vec{u}) - p\right)\vec{n} = \vec{\varphi},$$

on some part of the boundary. It is worth noting that all the results proved in the paper apply if such boundary conditions are considered since this only changes the right-hand-side. The regularity of the boundary stress tensor $\varepsilon\left(2\mathrm{Re}^{-1}S(\vec{u}) - p\right)\vec{n}$ is given in the next result.

Lemma 19. *Assume that $\varepsilon \in L^\infty(\Omega) \cap W^{1,4}(\Omega)$ and that $\alpha(\varepsilon), \beta(\varepsilon) \in L^\infty(\Omega)$. Let $\vec{f} \in L^2(\Omega)^d$ and $(\vec{u}, p) \in H^1(\Omega)^d \times L^2(\Omega)$ satisfying (1). Then, for any $\Gamma_c \subset \partial\Omega$, we have that*

$$\mathbb{G}\vec{n} := \varepsilon \left(2\mathrm{Re}^{-1} S(\vec{u}) - p\right) \vec{n} \in \left(H_{00}^{1/2}\left(\Gamma_{\mathrm{c}}\right)^d\right)'.$$

Proof. From Remark 1 and the incompressibility condition, the non-linear term can be written as

$$\mathrm{div}\left(\varepsilon \vec{u} \otimes \vec{u}\right) = \varepsilon(\vec{u} \cdot \nabla)\vec{u}.$$

Using $\varepsilon \nabla p = \nabla(\varepsilon p) - p\nabla\varepsilon$, one has

$$-\mathrm{div}\left(2\mathrm{Re}^{-1}\varepsilon S(\vec{u}) - \varepsilon p \mathbb{I}\right) = p\nabla\varepsilon + \varepsilon(\vec{u}\cdot\nabla)\vec{u} - \alpha(\varepsilon)\vec{u} - \beta(\varepsilon)\vec{u}|\vec{u}| + \varepsilon\vec{f}.$$

For $p = 4/3$, the Hölder inequality gives

$$\begin{aligned}
\left\|\mathrm{div}\left(2\mathrm{Re}^{-1}\varepsilon S(\vec{u}) - \varepsilon p\mathbb{I}\right)\right\|_{L^p(\Omega)} \quad \leq \quad & \|\varepsilon\|_{L^\infty(\Omega)} \left(\left\|\vec{f}\right\|_{L^p(\Omega)} + \|\vec{u}\|_{L^4(\Omega)} \|\nabla\vec{u}\|_{L^2(\Omega)}\right) \\
& + \|\alpha(\varepsilon)\|_{L^\infty(\Omega)} \|\vec{u}\|_{L^p(\Omega)} + \|\beta(\varepsilon)\|_{L^\infty(\Omega)} \|\vec{u}\|^2_{L^{2p}(\Omega)} \\
& + \|p\|_{L^2(\Omega)} \|\nabla\varepsilon\|_{L^4(\Omega)},
\end{aligned}$$

From the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$, one gets

$$\mathrm{div}\left(2\mathrm{Re}^{-1}\varepsilon S(\vec{u}) - \varepsilon p\mathbb{I}\right) \in L^{4/3}\left(\Omega\right)^d.$$

For any $\vec{v} \in W^{1,q}(\Omega, \mathbb{R}^d)$, we have the following Green's identity

$$\int_\Omega \vec{v}\cdot \mathrm{div}\,\mathbb{G}dx + \int_\Omega \mathbb{G}\mathbf{:}\nabla\vec{v}dx = \langle \mathbb{G}\cdot\vec{n}, \vec{v}\rangle, \tag{B.1}$$

where $\langle ., . \rangle$ is the duality product between $\left(W^{1-1/q,q}(\partial\Omega)\right)'$ and $W^{1-1/q,q}(\partial\Omega)$. Now let $\mu \in H_{00}^{1/2}(\Gamma_{\mathrm{c}})^d$ then $E_0\mu \in H^{1/2}(\partial\Omega)^d$ and, since the trace operator $\tau : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ has a continuous bounded right inverse $\tau^{-1} : H^{1/2}(\partial\Omega) \to H^1(\Omega)$, there exists a $\vec{v} = \tau^{-1}\left(E_0\mu\right) \in H^1(\Omega)^d$ such that $\vec{v}|_{\partial\Omega} = E_0\mu$ and

$$\|\vec{v}\|_{H^1(\Omega)^d} \leq C(\Omega)\,\|E_0\mu\|_{H^{1/2}(\partial\Omega)^d} \leq C\,\|\mu\|_{H_{00}^{1/2}(\Gamma_{\mathrm{c}})^d},$$

for a generic constant $C(\Omega) > 0$. Observe now that

$$\langle \mathbb{G}\vec{n}, \mu\rangle_{\left(H_{00}^{1/2}(\Gamma_{\mathrm{c}})^d\right)' \times H_{00}^{1/2}(\Gamma_{\mathrm{c}})^d} = \langle \mathbb{G}\vec{n}, E_0\mu\rangle = \langle \mathbb{G}\vec{n}, \vec{v}\rangle.$$

The Green's formula (B.1) together with Hölder inequality and the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$ then gives that

$$\begin{aligned}
\left|\langle \mathbb{G}\vec{n}, \mu\rangle_{\Gamma_{\mathrm{c}}}\right| \quad & \leq \quad \|\vec{v}\|_{H^1(\Omega)} \|\mathbb{G}\|_{L^2(\Omega)^{d\times d}} + \|\mathrm{div}(\mathbb{G})\|_{L^{4/3}(\Omega)^d} \|\vec{v}\|_{L^4(\Omega)^d} \\
& \leq \quad C(\Omega)\,\|\vec{v}\|_{H^1(\Omega)} \left(\|\mathbb{G}\|_{L^2(\Omega)^{d\times d}} + \|\mathrm{div}(\mathbb{G})\|_{L^{4/3}(\Omega)^d}\right) \\
& \leq \quad C(\Omega)\,\|\mu\|_{H_{00}^{1/2}(\Gamma_{\mathrm{c}})^d} \left(\|\mathbb{G}\|_{L^2(\Omega)^{d\times d}} + \|\mathrm{div}(\mathbb{G})\|_{L^{4/3}(\Omega)^d}\right),
\end{aligned}$$

which finally shows that $\mathbb{G} \cdot \vec{n} \in \left(H_{00}^{1/2}(\Gamma_c)^d\right)'$ by taking the supremum over all $\mu \in H_{00}^{1/2}(\Gamma_c)^d$.

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