Heterogeneous Expectations, Dynamics, and Stability of Markets

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Abstract

This paper examines the role of heterogeneous beliefs in a cobweb model. For that purpose, we study the price dynamics resulting from the interaction of agents whose price expectations differ. We proceed in two stages. First, two groups of agents are distinguished. They are either fundamentalists, or chartists. The latter specify the expected price from an adaptive process, the former consider the expected price as the steady state price, they then display "rational behaviour". Second, we enrich the model by allowing that agents may choose between rational expectations and a simple adaptive process.

Our work shows how market stability alters as the proportion of fundamentalists relative to chartists varies. We demonstrate two propositions. The market behaviour of fundamentalists compared to chartists promotes market stability. The existence of market stability depends on the specification of the expectations and the intensity of switching between the two behaviours.

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1. Introduction

The aim of this paper is to explore the links between heterogeneous beliefs and market stability. In economic systems, the introduction and implications of behavioural diversity are of great importance.

In a behavioural heterogeneity context, if globalisation encourages firms all to maximise anticipated profit, using a simple naïve cobweb approach, instability is likely to be generated if the underlying supply curve is relatively elastic. If in the absence of globalisation "satisficing" behaviours are more frequent, market stability could be present. The proportion of types of behaviour is an important factor in the analysis of market stability. Satisficers basically change as a result of globalisation from relatively unresponsive supplies to responsive ones (Lasselle *et al.*, 2001a). In an expectational framework, the problem is different but related. Indeed, the assumption of heterogeneous beliefs is usually not sufficient *per se* to explain the large and persistent movements of some economic market variables.

When analysing financial asset prices, Levy and Levy (1996) claimed that "…unacceptable market inefficiencies are observed when homogeneous expectations are assumed." They compare stock price dynamics in models with homogeneous expectations and heterogeneous expectations and show that the results obtained in the model with heterogeneous expectations are much more realistic. So, appropriate analysis of the asset price dynamics undoubtedly requires heterogeneous beliefs. How can these be considered?

Several authors such as Frankel and Froot (1990) assume that two types of traders exist in the market. There are traders who form their expectations of the future price by their observation of the past and current prices, i.e. the so-called “chartists”. There are traders who think of the future price according to a model that would be exactly correct if there were no chartists, i.e. the so-called “fundamentalists”. The dynamical study of the price depends on the relative weight of the population of one group relative to the other group. The study of the dynamics of such markets can be done in two ways.

First, the co-existence of the two types of agents in the economy (some include a third class of agents the so-called “portfolio managers” who mix the behaviour of the two groups) can be considered. Their proportion does not vary endogenously but is exogenously determined.

Second, a more sophisticated -endogenous- process of switching between the two groups can be allowed. In other words, the forecasts of the market participants are drawn from competing views and their switching from one behaviour to another is possible. This process is far more satisfying but, as we shall see, quite difficult technically speaking to study. It involves highly nonlinear equations and the nature of the process can often be only assessed by use of simulation models.

The study of heterogeneous beliefs can be done in two ways. Either it tries to reflect the reality by analysing empirical data and seeks to elaborate new forecasting rules that stick to the real environment of the economy, or the economy is analysed in a more theoretical manner. In this paper, we adopt the second perspective.

Conclusions (to be drawn) are as follows. First, when all agents are fundamentalists, the market is stable. Second, when all agents are chartists, the market is unstable. Third, in between these two extreme cases, there are a full range of mixture of these two behaviours which can lead either to market stability or instability. It then can be shown that the more chartists are in the market, the more unstable the market is.

One of the most powerful tools to study the influence of heterogeneous expectations and switching on the dynamics of prices is the cobweb model (Goeree and Hommes, 2000; Hommes, 2000; Lasselle *et al.*, 2001b). Even if this model is quite simple, it has become a classical example in economics dynamics since adaptive (Nerlove, 1958) and rational expectations were first introduced in it. When the cobweb model is considered in this literature, it may lead to lengthy and not always straightforward calculations, because of highly nonlinear equations (connecting with hyperbolic functions). This is the reason that the theoretical analysis is based on numerical simulations.

In this paper, the analysis of the cobweb model is developed in two stages.

In the first part of our study, we assume that two groups of producers – fundamentalists and chartists – co-exist on the market. We study the stability of the dynamical price path when the relative weight of both groups in the population is changing. Depending on the specification of the adaptive process, we can then show that fundamentalists market behaviour as compared to that of the chartists tends to promote market stability. Chartist behaviour leads to instability, allowing the existence of cycles. Although this model gives interesting and valuable insights, it might lead to criticism since there is no "endogenous" switching between the two groups of agents. In a second stage, endogenous switching is allowed for by extending Brock and Hommes's (1997) specification.

Brock and Hommes (1997) first offered a sensible specification of the switching function. This function takes into account the incentives of each group of agents might have to change their behaviour. Their specification is done in terms of last level of net realized profits. They considered the cobweb model with rational versus naïve expectations. As Chiarella and He (2001) point out the key aspect of that model is that it exhibits expectations feedback. Agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance. They enrich the model by allowing among other things the agents to have different risk attitudes. Another possible extension of Brock and Hommes (1997) is offered by Branch (2002). He considers that the agents can choose between three predictors: rational, naïve and adaptive beliefs. He explains that the characteristics of the predictor set affect the asymptotic stability conditions. As adaptive expectations incorporate past information, their influence tends to dampen price oscillations.

In this paper, agents are assumed to have the choice between two predictors: a costly rational predictor and a costless adaptive process. We proceed in two stages. First, two groups of agents are distinguished. They are either fundamentalists, or chartists. The latter specify the expected price from an adaptive process, the former consider the expected price as the steady state price, they then display “rational behaviour". Second, we enrich the model by allowing that agents may choose between rational expectations and a simple adaptive process.

By connecting the first stage of our study to the second stage, we are then able to deduce some important results. We indeed show that when agents can choose between rational expectations and adaptive process:

- Market stability exists when the intensity of choice between the two behaviours is not too large whatever the specification of the simple adaptive process. The chartist behaviour is the less destabilising when they take into account the present price without ignoring the past prices.
- Bifurcations occur when the weight of the chartists to that of the fundamentalists reaches a certain value. This depends on the expectation specification and the intensity of choice.

In other words, we establish the stability/instability conditions of the stationary point of the cobweb model with rational versus adaptive expectations.

The paper is organised as follows. In section 2, we present the simple cobweb model with heterogeneous expectations and an exogenous switching. In section 3, we enrich the model by considering an endogenous switching process inspired from that of Brock and Hommes (1997). Section 4 concludes.

2. The Linear Cobweb Model with Heterogeneous Expectations and Exogenous Switching

2.1 The Model

We consider the cobweb model and we study the dynamical path of prices in the market of a non storable good that takes one time unit to produce. For simplicity we assume that the supply and demand functions are linear. Let $D(p_t) = a - b\, p_t$ be the demand and $S\left(p_t^e\right) = d\ p_t^e$ the supply of the good, where p_t is the actual price and p_t^e the producers expected price, made at the beginning of period *t*. All the parameters are strictly positive. Supply is derived from firms maximizing profits with a cost function $c(q) = q^2 / (2d)$, so

$$S\left(p_{t+1}^e\right) = \text{argmax}\left(p_{t+1}^e\, q - c(q)\right) = (c')^{-1}\left(p_{t+1}^e\right) = d\ p_{t+1}^e \qquad (1)$$

In case of homogeneous expectations all producers use the same expectations or predictor function $p_{t+1}^e = H\left(\vec{P}_t\right)$, where $\vec{P}_t = (p_t, p_{t-1}, \ldots, p_{t-L})$ is a vector of past prices. Equilibrium price dynamics is then described by

$$D(p_{t+1}) = S\left(H\left(\vec{P}_t\right)\right) \text{ or}$$

$$p_{t+1} = D^{-1} S\left(H\left(\overrightarrow{P_t} \right) \right) = \frac{a}{b} - \frac{d}{b} p^e_{t+1} \tag{2}$$

with D^{-1} the inverse demand function. Since we assume that demand is decreasing and supply S is increasing, (2) is well defined.

In this paper we consider heterogeneous expectations. Two types of agents co-exist: the chartists (c) and the fundamentalists (f). In this section, the weighting of both groups is considered exogenous and we denote by $n_f \in [0, 1]$ the relative weight associated with fundamentalists. The market equilibrium in the cobweb with two groups of agents is then determined by

$$D(p_{t+1}) = n_f \, S\left(H_f\left(\overrightarrow{P_t} \right) \right) + (1 - n_f) S\left(H_c\left(\overrightarrow{P_t} \right) \right) \tag{3}$$

Fundamentalists have perfect foresight, therefore $H_f\left(\overrightarrow{P_t} \right) = p_{t+1}$.

Chartists or trend traders have expectations evaluated by a simple adaptive process as in Hommes (1991). This adaptive process is a weighted average of the two most recent prices, so $H_c\left(\overrightarrow{P_t} \right) = \tau \, p_t + (1 - \tau) p_{t-1}$ with $0 < \tau < 1$. The parameter τ is called the expectations weight factor. When $\tau = 1$, expectations are naïve.

Using linear demand and supply and the predictor functions, market equilibrium in (3) becomes:

$$p_{t+1} = \left(a - n_f\, d\, p_{t+1} - (1 - n_f) d \left(\tau\, p_t + (1-\tau)\, p_{t-1}\right)\right)/b \qquad (4)$$

Note that fundamentalists are assumed to have perfect knowledge about market equilibrium equations, prices, and also about the chartists' behaviour. Without loss of generality we change coordinates and choose the steady state price $p^* = a/(b+d)$, the intersection of demand and supply, as the new origin, so that p_t represents (positive or negative) deviations from the steady state. For our model this simply means fixing $a = 0$ in (4). Let $A(n_f) = d\,(n_f - 1)/(b + d\, n_f)$.

$A(n_f)$ is non positive and $\partial A(n_f)/\partial n_f > 0$. Solving equation (4) for p_{t+1} then yields

$$p_{t+1} = A(n_f)(\tau\, p_t + (1-\tau)\, p_{t-1}) \qquad (5)$$

Equation (5) is a second-order linear difference equation which can be rewritten as a system (5’) of two first-order difference equations:

$$(5') \quad \begin{cases} h_{t+1} = p_t \\ p_{t+1} = A(n_f)(\tau\, p_t + (1-\tau) h_t) \end{cases}$$

This system defines a global dynamics including the stationary point equilibrium $E = (0,0)$.

2.2. Stability and Cycles

The stability or the instability of the stationary point *E* issued from the two-dimensional system (5') can be directly investigated by looking at the corresponding characteristic polynomial $Q(\lambda)$ denoted by:

$$Q(\lambda) \equiv \lambda^2 - T\,\lambda + D = 0 \tag{6}$$

Its two characteristic roots are denoted (λ_1, λ_2) and the following definitions apply: $T = \lambda_1 + \lambda_2$ and $D = \lambda_1 . \lambda_2$, in which we denote the determinant by *D* and the trace by *T* of the Jacobian matrix of (5') taken at the stationary point *E*. The characteristic polynomial associated with (5') is:

$$Q(\lambda) = \lambda^2 - \tau\,A(n_f)\lambda - (1-\tau)\,A(n_f) = 0 \tag{6'}$$

From equation (6') and the previous definitions based on the characteristic roots, we get $T(n_f, \tau) = \tau\,A(n_f) = -d(1-n_f)\tau/(b + d\,n_f)$ and $D(n_f, \tau) = -(1-\tau)\,A(n_f) = d(1-n_f)(1-\tau)/(b + d\,n_f)$.

Proposition 1 - Properties of the trace and the determinant

$T(n_f, \tau) \leq 0$ and $D(n_f, \tau) \geq 0$

$\partial T(n_f, \tau)/\partial n_f > 0$ and $\partial T(n_f, \tau)/\partial \tau < 0$

$\partial D(n_f, \tau)/\partial n_f < 0$ and $\partial D(n_f, \tau)/\partial \tau < 0$

Proof: the proof is left to the reader.

It is easy to deduce the following relation between the trace and the determinant:

$$D = (1 - 1/\tau)T \qquad (7)$$

As shown by de la Fuente (2000), a simple geometrical way to look at stability is to locate in the plane (T,D) the position of equation (7), (*cf.* Figure 1). In the open region above (below) the parabola of equation $D = 1/4T^2$, both roots are complex (real). Any couple (T,D) defining by (7) which lies in the interior of the triangle *ABC* makes the stationary point stable.

i) Let us assume first that $\tau = 1/2$ and $2b < d < 8b$, i.e. we study the position of (7) in the plane (T,D) with $T(n_f, 1/2)$ and $D(n_f, 1/2)$. Since d and n_f are both non negative, (7) is defined by a segment $[OG]$ in the (T,D) plane (*cf.* Figure 1).

Figure 1

On the one hand, the stationary point *E* is stable in the dynamics defined by (5') for any couple (T,D) belonging to $[OF[$ which lies in the interior of the triangle *ABC*. On the other hand, *E* is unstable in the dynamics defined by (5') for any couple (T,D) belonging to $[FG]$. Since $[OG]$ crosses the segment $[AB]$, one generates in this way a change of stability in which the two characteristic roots (λ_1,λ_2) are complex conjugate and cross the unit circle in the complex plane. When $[OG]$ crosses $[AB]$ of equation $D=1$, a three-period cycle occurs for a value of the weight $n_f^* = (d-2b)/(3d)$. Note that this value is positive (and therefore exists) if and only if $d > 2b$.

On the other hand, since we have $T(n_f,1/2) = A(n_f,1/2)/2 = -d(1-n_f)/(2(b+d\,n_f))$ and $D(n_f,1/2) = -A(n_f,1/2)/2 = d(1-n_f)/(2(b+d\,n_f))$ and recall Proposition 1, there are two cases depending on the relative weight of both groups:

- for $0 < n_f \leq n_f^*$, the equilibrium is unstable,
- for $n_f^* < n_f < 1$, the equilibrium is stable.

In other words, fundamentalists market behaviour as compared to that of chartists tends to promote market stability. When the weight of the chartists is high and greater than $(1-n_f^*)$, non convergent oscillations emerge.

ii) Second, we can assume any value for τ. The slope of the segment $[OG]$ changes, when τ varies. Indeed, when τ is close to 1, the segment $[OG]$ is close to the horizontal axis; when τ is close to zero, $[OG]$ is close to the vertical axis. In other words, the line defines by $[OG]$ may not only cross the curve defined by $D = 1/4T^2$ but also the segment $[AC]$. One generates in this way another change of stability in which one of the two real characteristic roots (λ_1, λ_2) crosses -1.

iii) Third, let us assume that $d/b > 1$. If all agents are chartists, the stationary point is unstable.

To summarise, one can say:

For a given $\overline{\tau}$, when n_f varies, the analysis is done along $[OF]$. The higher n_f is, the closer to the origin the combination $\left(T(n_f, \overline{\tau}), D(n_f, \overline{\tau})\right)$ defining (7) is.

For a given $\overline{n}_f$, when τ varies, the analysis is done from the slope of $[OF]$. The higher τ is, the higher the slope is.

The stability of the stationary point depends not only on fundamentalists market behaviour as compared to that of chartists but also on the expectations function of chartists, namely on the weight they give in their expectations to the current price relatively to the past price.

It can be computed and illustrated (*cf*. Figure 2) that

when $\tau > 2/3$, the stationary point is unstable if $n_f < n_f^{**}$, where $n_f^{**} = 1 - (b+d)/(2\tau d)$.

when $\tau < 2/3$, the stationary point is unstable if $n_f < n_f^{*}$, where $n_f^{*} = (b + d(\tau - 1))/(d(\tau - 2))$.

Figure 2

Proposition 2 - Stability properties of the stationary point *E*

Assume that the slopes of the supply and the demand satisfy $d/b > 1$. The stationary point is stable when:

(i) $2/3 < \tau < 1$

(i)1. and $1/(1-2\tau) < A(n_f) < 4(\tau - 1)/\tau^2$. The two characteristic roots λ_1 and λ_2 are real and take values between –1 and 0.

(i)2. and $A(n_f) > 4(\tau - 1)/\tau^2$, the two characteristic roots λ_1 and λ_2 are complex with modulus less than 1.

(ii) $0 < \tau < 2/3$ and $A(n_f) > 1/(\tau - 1)$, the two characteristic roots λ_1 and λ_2 are complex with modulus less than 1.

In both cases, the dynamical path of prices takes the form of damped oscillations.

Proof: See Appendix.

Corollary[1]

(i) When $A(n_f^{**}) = 1/(1 - 2\tau)$ (with $p_1 = -p_0$) and $\tau > 2/3$, a two-period cycle occurs.

(ii) When $A(n_f^{*}) = 1/(\tau - 1)$, a cycle appears. For instance, when $\tau = 1/2$, $n_f^{*} = (d - 2b)/(3d)$ (with $d > 2b$) and $A(n_f^{*}) = -1$, a three-period cycle appears.

Proof: See Appendix.

Figure 3

[1] The period of oscillations is not limited to 2 or 3. See Proof in the Appendix.

Figure 3 illustrates Proposition 2. The shaded areas represent the combination of τ and $A(n_f)$ where the stability properties are fulfilled. The darkest (lightest) area represents the combination of τ and $A(n_f)$ where the eigenvalues are complex (real). Although the model is simple, three facts can be drawn from Figure 3 and they will be relevant in the next section.

First, if $n_f = 0$, $A(n_f) = -d/b < -1$, the stationary point is unstable.

Second, since $\partial A(n_f)/\partial n_f > 0$, fundamentalists market behaviour as compared to that of chartists tends to promote market stability. Indeed for any given τ, the higher n_f is, the higher $A(n_f)$ is and the closer $A(n_f)$ is to zero. As shown in Figure 3, the zones of stability are close to the horizontal axis.

Third, chartists behaviour destabilises less the market when τ is around $2/3$. This is an interesting result due to our specific adaptive process. This result tells us that chartist behaviour is more destabilising when chartists weight very heavily one of the two prices on which they based their expectations, i.e. when τ is close either to zero, or to one. When their expectations are based on a sufficiently mix of the current and the past price, then the stability conditions are wider; these reach their maximum when the weight of the current price is equal to $2/3$.

3. The Cobweb Model with Heterogeneous Expectations and Endogenous Switching

We are going to enrich the model by specifying a switching function which takes into account the incentives that each group of agents might have in changing behaviour. For that purpose and for practicality, we introduce the specification introduced by Brock and Hommes (1997) in terms of net realized profits in the last period as the performance measure for predictor selection.

3.1. Endogenous Switching Process

Realized net profits $\pi_j\left(p_{t+1}, H_j\left(\vec{P_t}\right)\right)$, $j = f, c$, from using predictor $H_j\left(\vec{P_t}\right)$ when the actual equilibrium price becomes p_{t+1} equals

$$\pi_j\left(p_{t+1}, H_j\left(\vec{P_t}\right)\right) = p_{t+1} S\left(H_j\left(\vec{P_t}\right)\right) - \frac{\left(S\left(H_j\left(\vec{P_t}\right)\right)\right)^2}{2d} - C_j \qquad (8)$$

where $C_j \geq 0$ are information costs for obtaining predictor H_j. For the chartists, information has no costs. For the fundamentalists, $C_f = C$ is positive.

Using linear demand and supply, rational expectations versus adaptive process, and realized net profits in the last period, the general performance measure is

$$\pi_f(p_{t+1}, p_{t+1}) = \frac{d}{2} p_{t+1}^2 - C \qquad \text{(9a)}$$

$$\pi_c(p_{t+1}, p_t, p_{t-1}) = \frac{d}{2}(\tau p_t + (1-\tau)p_{t-1})(2 p_{t+1} - (\tau p_t + (1-\tau)p_{t-1})) \qquad \text{(9b)}$$

After observing the equilibrium price p_{t+1}, the updated fractions of agents using rational expectations or an adaptive process in the next period are:

$$n_{f,t+1} = \text{Exp}\left[\beta\left(\frac{d}{2} p_{t+1}^2 - C\right)\right] \Big/ Z_{t+1} \qquad \text{(10a)}$$

$$n_{c,t+1} = \text{Exp}\left[\beta \frac{d}{2}(\tau p_t + (1-\tau)p_{t-1})(2 p_{t+1} - (\tau p_t + (1-\tau)p_{t-1}))\right] \Big/ Z_{t+1} \qquad \text{(10b)}$$

where Z_{t+1} is the sum of the numerators, β is the intensity of choice, measuring how fast agents switch predictors. Since $n_{f,t+1} + n_{c,t+1} = 1$, (10a) can be rewritten[2] as:

$$n_{f,t+1} = 1 \Big/ \left(1 + \text{Exp}\left(\beta\left\{\frac{d}{2}\left((\tau p_t + (1-\tau)p_{t-1})^2 \left(A(n_{f,t}) - 1\right)^2\right) - C\right\}\right)\right) \qquad \text{(11)}$$

where $A(n_{f,t}) = d\,(n_{f,t} - 1) / (b + d\, n_{f,t})$

Using linear demand and supply and the predictor functions, market equilibrium in (3) becomes:

$$p_{t+1} = A\left(n_{f,t}\right)\left(\tau\, p_t + (1-\tau)p_{t-1}\right) \tag{12}$$

Note that this equation is equivalent to equation (5), but n_f is now endogeneised and depends on the performance of the agents, i.e. it varies over time and its different values are given by equation (11).

We are then able to deduce the adaptive rational equilibrium dynamics of the cobweb model with rational expectations versus adaptive process. It is described by the following three-dimensional system of two non linear difference equations given by (11) and (12) and can be summarised as follows:

$$p_{t+1} = \phi\left(p_t, p_{t-1}, n_{f,t}\right)$$

$$n_{f,t+1} = \varphi\left(p_t, p_{t-1}, n_{f,t}\right)$$

where $\phi\left(p_t, p_{t-1}, n_{f,t}\right) = A\left(n_{f,t}\right)\left(\tau\, p_t + (1-\tau)p_{t-1}\right)$,

$$\varphi\left(p_t, p_{t-1}, n_{f,t}\right) = \frac{1}{1+\mathrm{Exp}\left[-\beta\left\{\frac{d}{2}\left(\left(\tau\, p_t + (1-\tau)p_{t-1}\right)^2 \left(A\left(n_{f,t}\right)-1\right)^2\right) - C\right\}\right]}$$

It can be rewritten as a system of three first-order difference equations:

$$h_{t+1} = p_t \tag{13a}$$

$$p_{t+1} = \phi\left(h_t, p_t, n_{f,t}\right) \tag{13b}$$

$$n_{f,t+1} = \varphi\left(h_t, p_t, n_{f,t}\right) \tag{13c}$$

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[2] See Appendix for the computations.

The stability or the instability of the stationary point issued from the system (13) can be directly investigated by looking at the Jacobian matrix of (13) taken at the stationary point. These properties will be studied in the following sub-section.

Our model represents the cobweb model of Brock and Hommes (1997) with rational versus adaptive expectations when $\tau = 1$. However, our adaptive process is simpler than that of Branch (2002). Branch considers a possible costly predictor defined from many past prices. We consider a costless predictor depending on p_t and p_{t-1}. As we shall see in the next sub-section, it allows us to make a systematic and tractable dynamical study.

3.2 Stability and Cycles

A simple computation shows that the system (13) has a unique stationary point $E' = \left(0, 0, \bar{n}_f(\beta) = 1/\left(1 + \mathrm{Exp}[\beta C]\right)\right)$. To ease the presentation, let us assume that $C = 0$ or $C = 1$. When $C = 0$, the agents have free access to the sophisticated predictor.

The stability (or the instability) properties of the stationary point are now studied when n_f is endogeneised. In what follows, we proceed in the same manner as in the previous section. We make our study depending on the variation of τ. We could also make the analysis when the intensity of choice β varies. Indeed $A(n_f)$ has becomed $A(\overline{n}_f(\beta))$.

Proposition 3

$$\partial A(\overline{n}_f(\beta))/\partial \beta < 0$$

<u>Proof:</u> The proof is left to the reader.

The stability results are then as follows:

Proposition 4 – Stability properties of the stationary point *E'*

Assume that $d/b > 1$.

(i) When the information costs are $C = 0$, the stationary point $E' = (0,0,1/2)$ is globally stable.

(ii) When the information costs are $C = 1$, the stationary point $E' = (0,0,\overline{n}_f(\beta))$, where $\overline{n}_f(\beta) = 1/(1 + Exp\beta)$, is locally asymptotically stable when

Either $2/3 < \tau < 1$

(ii)1. and $1/(1-2\tau) < A(\overline{n}_f(\beta)) < 4(\tau - 1)/\tau^2$ (14)

There are three eigenvalues: λ_1 and λ_2 are real and take values between –1 and 0, λ_3 is zero.

(ii)2. and $A\left(\overline{n}_f(\beta)\right) > 4(\tau-1)/\tau^2$. There are three

eigenvalues: λ_1 and λ_2 are complex with modulus less than 1, λ_3 is zero.

Or $0 < \tau < 2/3$ and $A\left(\overline{n}_f(\beta)\right) > 1/(\tau-1)$ (15)

There are three eigenvalues: λ_1 and λ_2 are complex with modulus less than 1, λ_3 is zero.

Proof: See Appendix.

The stationary point is locally asymptotically stable when all eigenvalues of the Jacobian matrix at the stationary point have moduli strictly less than one (Azariadis (1993), p. 59). Propositions 4 and 2 are closely related. Indeed the analysis of the stability properties of E' can be resumed from Proposition 2. The dynamical system (13) has three eigenvalues: 0, λ_1 and λ_2, where λ_1 and λ_2 are identical to those of Proposition 2. Proposition 4 is illustrated by Figure 4. From Proposition 3, we know that when the intensity of choice β is lower, $\overline{n}_f$ and $A\left(\overline{n}_f\right)$ are higher. It is then more likely that the stationary point is stable.

Proposition 5 – Possibility of Bifurcations

(i) If equality holds in the left-hand side of (14), and β (or τ) is used as bifurcation parameter, then the dynamic system undergoes a Flip bifurcation.

(ii) If equality holds in (15), and β (or τ) is used as bifurcation parameter, then the dynamic system undergoes a Neimark-Sacker bifurcation, with $\tau \neq 0.5$.

Proof: See Appendix.

Figure 4

Figure 4 illustrates Propositions 4 and 5. On the one hand, the shaded areas show the stability zones of the stationary equilibrium. As in Figure 3, the darkest area represents the combination of τ and $A\left(\bar{n}_f(\beta)\right)$ where the eigenvalues are complex. The curves which separate shaded areas from non shaded areas are given by the curves describing the Flip bifurcation and the Neimark-Sacker bifurcation.

Figure 5

Figure 5 allows us to plot the bifurcation curves in the (β, τ)-plane for specific values of the parameters, $b = 1$ and $d = 5$. The two curves intersect for $\tau = 2/3$. The intensity of choice varies negatively with n_f. The stability areas are "smaller" for low values of the intensity of choice (or for large values of n_f) and "extreme" values of τ.

As in Brock and Hommes (1997), when the costs for rational expectations are not zero, the equilibrium can be unstable. Our analysis of the unstable stationary point enriches the results of Brock and Hommes into two ways, technical and economical.

The technical problem we face is rather different. Indeed, the introduction of the adaptive process has added another dimension in the model. Our model is three-dimensional, recall that of Brock and Hommes is only two-dimensional. As a consequence we do not deal with the same kind of bifurcation and new phenomena occur. Their paper emphasises the possibility of a period-doubling bifurcation. In our case, not only a Flip bifurcation occurs, but also does a Neimark-Sacker bifurcation. Depending on the specification of the adaptive process, degenerate Neimark-Sacker bifurcations may arise and strong resonances may occur.

In economic terms, the introduction of the adaptive process allows new phenomena to happen because either expectations change, or the intensity of choice varies.

Stability is ensured for some "relative" small values of β. As β increases, it is less likely that the system exhibits stable paths even when τ varies (*cf*. Figure 5). As we mention in Section 2, when the expectations are based on a sufficiently mix of the current price and the past price, the stability conditions are wider with respect to the intensity of choice, i.e. chartists behaviour is less destabilising for the market. Chartists have indeed more information to make predictions, fewer fundamentalists are required in the economy to stabilise it. In that respect, our result does differ from theorem 8 of Branch (2002). This result is also in contrast of the usual thought that the chartists' behaviour is more stabilising in the economy when the recent past prices are weighted more heavily. The knowledge of the "critical value" of τ ($2/3$) is of great importance. When the cost of adaptive process is nil, the range of parameters under which the system is stable increases (decreases) from the case of rational versus naïve expectations as τ varies from 1 to $2/3$ ($2/3$ to 0). The range will be at its maximum around $\tau = 2/3$ and at its minimum at the extreme values of τ.

Therefore excessive weight implies that the behaviour of chartists is destabilising. In other words, a balanced, or somewhat dampened approach in which sufficient weighting is given to both sets of information about observed prices is necessary to create stability.

Second, instability occurs depending on values taken by the expectations weight factor or by the intensity of choice, leading to the possibility of bifurcation. As chartists put less weight on today's price or the intensity of choice is getting larger, one of the eigenvalues can cross –1 or the modulus of the complex eigenvalues can cross the unit circle. (*Cf.* Figure 5). In our model, the bifurcations occur either when the expectations weight factor decreases, for a given intensity of choice, or when the intensity of choice increases for a given expectations weight factor. Let us first consider a decrease of τ. In Figure 4, when the expectations are naïve as in Brock and Hommes ($\tau = 1$), Flip bifurcation can only happen for a chosen value of $A(\beta)$. As τ decreases (for a given intensity of choice and the other parameters remaining constant), the system can undergo the two types of bifurcation previously mentioned. The same phenomena occur when the intensity of choice is getting larger, for a given τ and the other parameters remaining constant).

Let us finally add that the possibility of the period-doubling bifurcation is obtained for lower $A(\beta)$, i.e. n_f is lower but the intensity of choice is higher as τ decreases.

4. Concluding Comments

Even if heterogeneous beliefs do not affect the existence of the stationary equilibrium in the cobweb model, the local stability conditions of the latter are largely dependent on the former. Our work has emphasised that the nature of the chartists' behaviour is as relevant as their relative weight in the market. In the behavioural heterogeneity rather than expectational heterogeneity context, the nature of the behavioural response of the heterogeneous groups might be as equally important as their proportions. We can therefore draw two main features of behavioural diversity when two types of expectations co-exist.

On the one hand, when switching from one behaviour to another is possible, its nature needs to be well specified. When its switching is purely exogenous, this article clearly shows that chartist behaviour promotes market instability. When it is endogenous, the resulting dynamics is nonlinear and the instability previously observed persists but may take various forms. Different types of bifurcations may occur in response to variations of specific parameters, such as the intensity of the agents' switching from one behaviour to another.

On the other hand, the definition of chartists' expectations function is as important as the relative weight of chartists in generating instability. Extending Brock and Hommes (1997) simple case of naïve expectations, we have shown that the form taken by the chartists' adaptive process is crucial. Instability is more likely to occur when chartists rely very heavily on one of the past prices used in their expectation function.

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Appendix

Proof of Proposition 2

The model is given by:

$$(5') \quad \begin{cases} h_{t+1} = p_t \\ p_{t+1} = -\dfrac{d\left(1-n_f\right)}{b+d\,n_f}\left(\tau\,p_t + (1-\tau)h_t\right) \end{cases}$$

Let $A\left(n_f\right) = -\dfrac{d\left(1-n_f\right)}{b+d\,n_f}$. At the stationary point $E = \left(h^*, p^*\right) = (0,0)$, the Jacobian Matrix $J(E)$ is:

$$J(E) = \begin{pmatrix} 0 & 1 \\ A\left(n_f\right)(1-\tau) & A\left(n_f\right)\tau \end{pmatrix}$$

The characteristic polynomial associated with this matrix is:

$$\lambda^2 - A\left(n_f\right)\tau\lambda - A\left(n_f\right)(1-\tau) = 0$$

<u>Real roots</u>

If $A\left(n_f\right)\tau^2 + 4(1-\tau) < 0 \quad \Leftrightarrow \quad A\left(n_f\right) < -4\dfrac{(1-\tau)}{\tau^2}$, the characteristic roots are real and equal to $\lambda_1 = \dfrac{A\left(n_f\right)\tau - \sqrt{A\left(n_f\right)\left(A\left(n_f\right)\tau^2 + 4(1-\tau)\right)}}{2}$

and $\lambda_2 = \dfrac{A\left(n_f\right)\tau + \sqrt{A\left(n_f\right)\left(A\left(n_f\right)\tau^2 + 4(1-\tau)\right)}}{2}$.

Study of λ_1

$$\lambda_1 < -1 \Leftrightarrow \frac{A(n_f)\tau - \sqrt{A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right)}}{2} < -1$$

$$\Leftrightarrow A(n_f)\tau + 2 < \sqrt{A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right)}$$

If $A(n_f)\tau + 2 < 0 \Leftrightarrow A(n_f) < -\frac{2}{\tau}$, the above inequality is always true and then $\lambda_1 < -1$ whatever τ.

Let us now assume that $A(n_f) \geq -\frac{2}{\tau}$ and let us find the conditions for which $-1 < \lambda_1 < 0$. We have:

$$-A(n_f)\tau - 2 < -\sqrt{A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right)}$$

$$\Leftrightarrow \sqrt{A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right)} < 2 + A(n_f)\tau$$

$$\Leftrightarrow A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right) < \left(A(n_f)\tau + 2\right)^2$$

$$\Leftrightarrow 4A(n_f)(1-2\tau) < 4$$

$$\Leftrightarrow A(n_f) > -\frac{1}{2\tau - 1} \text{ if } \tau > \frac{1}{2}$$

Note that $-\frac{1}{2\tau-1} > -\frac{2}{\tau}$ when $\tau > \frac{2}{3}$.

$$-\frac{2}{\tau} + \frac{1}{2\tau - 1} < 0 \Leftrightarrow \frac{-3\tau + 2}{\tau(2\tau - 1)} < 0 \text{ if } \tau > \frac{2}{3}.$$

So we have shown that when $-\frac{2}{\tau} < -\frac{1}{2\tau - 1} < A(n_f) < -4\frac{(1-\tau)}{\tau^2}$ and $\tau > \frac{2}{3}$, then $-1 < \lambda_1 < 0$.

Study of λ_2

$$-1 < \lambda_2 < 0 \iff -1 < \frac{A(n_f)\tau + \sqrt{A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right)}}{2} < 0$$

$$\iff -2 < A(n_f)\tau + \sqrt{A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right)} < 0$$

$-2 - A(n_f)\tau < \sqrt{A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right)}$ is always true.

$\sqrt{A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right)} < -A(n_f)\tau$ is always true, so $-1 < \lambda_2 < 0$.

<u>Complex roots</u>

If $A(n_f)\tau^2 + 4(1-\tau) > 0$, the characteristic roots are complex and equal to $\lambda_{1,2} = \frac{A(n_f)\tau \pm i\sqrt{-A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right)}}{2}$

Study of the modulus

$$|\lambda_{1,2}| = \sqrt{\left(\frac{A(n_f)\tau}{2}\right)^2 + \frac{1}{4}\left(-A(n_f)\left(A(n_f)\tau^2 + 4(1-\tau)\right)\right)} = \sqrt{-(1-\tau)A(n_f)}$$

$$|\lambda_{1,2}| > 1 \iff A(n_f) < -\frac{1}{1-\tau}$$

Note that $-\frac{1}{1-\tau} > -4\frac{(1-\tau)}{\tau^2}$ when $0 < \tau < \frac{2}{3}$.

Q.E.D.

Sketch of the proof of Corollary

Cycle of period 2 $\{p_1, p_2\}$ (here $\{p_1, -p_1\}$)

$$p_2 = A(n_f)(\tau p_1 + (1-\tau) p_2) \iff p_2 = \frac{A(n_f)\tau p_1}{1 - A(n_f)(1-\tau)}$$

and $p_2 = \left[\frac{A(n_f)\tau}{1 - A(n_f)(1-\tau)}\right]^2 p_2$

$$\iff \left|\frac{A(n_f)\tau}{1 - A(n_f)(1-\tau)}\right| = 1 \text{ with } A(n_f) \neq -\frac{1}{1-\tau}$$

Note that $\frac{A(n_f)\tau}{1 - A(n_f)(1-\tau)} = 1$ is impossible.

A few computations yield $A(n_f) = -\frac{1}{2\tau - 1}$.

In this case, $\lambda_1 = -1$ and $-1 \leq \lambda_2 = \frac{\tau - 1}{2\tau - 1} \leq 0$ (recall $\tau > 2/3$).

Cycle of period 3 (Allen (1956), p. 196)

Since the trace of the matrix $J(E)$ is always negative and its determinant is always positive, θ is between $\pi/2$ and π (with $\theta \neq \pi/2$

since $\tau \neq 0$ and $\theta = \pi$ is impossible since $\tau \neq 1$). The period of oscillations lies between 2 and 4. Let us take two examples.

When $\tau = 1/2$, $n_f = \dfrac{d-2b}{3d}$, then $A(n_f) = -1$ and we have

$p_{t+1} = -p_t - p_{t-1}$.

We can rewrite the roots $\lambda_{1,2} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$ in polar co-ordinates as

$cos\,\theta \pm i\, sin\,\theta$ where $\theta = 2\pi/3$. The period of the cycle is equal to

$2\pi/(2\pi/3) = 3$.

Another simple period of oscillation is to be found when $\theta = 3\pi/4$, τ is then equal to $\sqrt{2}/\left(1+\sqrt{2}\right) \cong 0.5857$. The period of the cycle is equal to $8/3$, i.e. the dynamical path is cyclical every 8 periods of unit time.

Generally speaking, cycles appear when $A(n_f) = -\dfrac{1}{1-\tau}$ (*cf.* proof of Proposition 1) and $\tau < 2/3$. In that case, the complex roots are equal to:

$$\lambda_{1,2} = -\frac{\tau}{2(1-\tau)} \pm i\frac{\sqrt{(2-\tau)(2-3\tau)}}{2(1-\tau)}.$$

Q.E.D.

<u>Some computations for Section 3</u>

Computation of $n_{f,t+1}$:

$$n_{f,t+1} = \mathrm{Exp}\left[\beta\left(\frac{d}{2}p_{t+1}^2 - C\right)\right] \Big/ Z_{t+1}$$

$$n_{c,t+1} = \mathrm{Exp}\left[\beta\frac{d}{2}(\tau p_t + (1-\tau)p_{t-1})(2p_{t+1} - (\tau p_t + (1-\tau)p_{t-1}))\right] \Big/ Z_{t+1}$$

where

$$Z_{t+1} = \mathrm{Exp}\left[\beta\left(\frac{d}{2}p_{t+1}^2 - C\right)\right] + \mathrm{Exp}\left[\beta\frac{d}{2}(\tau p_t + (1-\tau)p_{t-1})(2p_{t+1} - (\tau p_t + (1-\tau)p_{t-1}))\right]$$

$$n_{f,t+1} = \frac{1}{1 + \dfrac{\mathrm{Exp}\left[\beta\frac{d}{2}(\tau p_t + (1-\tau)p_{t-1})(2p_{t+1} - (\tau p_t + (1-\tau)p_{t-1}))\right]}{\mathrm{Exp}\left[\beta\left(\frac{d}{2}p_{t+1}^2 - C\right)\right]}}$$

$\Leftrightarrow$

$$n_{f,t+1} = \frac{1}{1 + \mathrm{Exp}\left(-\beta\left\{\frac{d}{2}p_{t+1}(p_{t+1} - 2(\tau p_t + (1-\tau)p_{t-1})) + (\tau p_t + (1-\tau)p_{t-1})^2 - C\right\}\right)}$$

$\Leftrightarrow$

$$n_{f,t+1} = \frac{1}{1 + \mathrm{Exp}\left(-\beta\left\{\begin{array}{l}\frac{d}{2}A(n_{f,t})(\tau p_t + (1-\tau)p_{t-1}) \\ (A(n_{f,t})(\tau p_t + (1-\tau)p_{t-1}) - 2(\tau p_t + (1-\tau)p_{t-1})) \\ + (\tau p_t + (1-\tau)p_{t-1})^2 - C\end{array}\right\}\right)}$$

$\Leftrightarrow$

$$n_{f,t+1} = \frac{1}{1 + \mathrm{Exp}\left(-\beta\left\{\frac{d}{2}(\tau p_t + (1-\tau)p_{t-1})^2(A(n_{f,t}) - 1)^2 - C\right\}\right)}$$

Proof of Proposition 4(ii)

We just need to study the stability properties of the stationary point $E' = \left(0, 0, \bar{n}_f(\beta) = 1/(1+\text{Exp}\beta)\right)$. We look for the conditions for which all the absolute values of the real eigenvalues or all the modulus of the complex eigenvalues of the Jacobian matrix at E' are less than 1.

The Jacobian Matrix at E':

$$J(E') = \begin{pmatrix} 0 & 1 & 0 \\ A(\bar{n}_f(\beta))(1-\tau) & A(\bar{n}_f(\beta))(\tau) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In what follows, we will denote $\bar{n}_f(\beta)$ by $\bar{n}_f$, keeping in mind that the relative weight of fundamentalists depends on the intensity of choice β.

If $A(\bar{n}_f)\tau^2 + 4(1-\tau) > 0$, then there are three eigenvalues: 0 and

$$\lambda_{1,2} = \frac{A(\bar{n}_f)\tau \pm i\sqrt{-A(\bar{n}_f)\left(A(\bar{n}_f)\tau^2 + 4(1-\tau)\right)}}{2}$$

If $A(\bar{n}_f)\tau^2 + 4(1-\tau) < 0$, then there are three eigenvalues 0 and

$$\lambda_{1,2} = \frac{A(\bar{n}_f)\tau \pm \sqrt{A(\bar{n}_f)\left(A(\bar{n}_f)\tau^2 + 4(1-\tau)\right)}}{2}.$$

When all the modulus of the complex eigenvalues and all the absolute value of the real eigenvalues are less than 1 (*cf.* proof of Proposition 2), the stationary point is asymptotically stable.

<u>Proof of Proposition 5</u> (we follow Kuznetsov (2000))

Our system (13) is three-dimensional and needs to be rewritten so that the stationary point is at the origin.

$$h_{t+1} = p_t$$

$$p_{t+1} = \phi\left(h_t, p_t, n_{f,t}\right)$$

$$n_{f,t+1} = \varphi\left(h_t, p_t, n_{f,t}\right)$$

Let us denote $m_t = n_{f,t} - \overline{n}_f$. Then the system becomes system (16):

$$h_{t+1} = p_t \tag{16a}$$

$$p_{t+1} = \phi\left(h_t, p_t, m_t + \overline{n}_f\right) \tag{16b}$$

$$m_{t+1} = \varphi\left(h_t, p_t, m_t + \overline{n}_f\right) - \overline{n}_f = \psi\left(h_t, p_t, m_t\right) \tag{16c}$$

The stationary point is then $(0,0,0)$.

Let us denote (16) as a discrete –time dynamical system:

$$x \rightarrow f(x)$$

We can write this system as:

$$\tilde{x} = J\,x + F(x),\ x \in R^3, \tag{17}$$

where J is the Jacobian matrix of (16) at the stationary point and $F(x) = O\left(\|x\|^2\right)$ is a smooth function. Let us represent its Taylor expansion in the form

$$F(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O\left(\|x\|^4\right),$$

where $B(x,y)$ and $C(x,y,z)$ are multilinear functions.

Let us first consider the Flip case.

The Jacobian matrix J of (16) at the stationary point is:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\tau - 1}{2\tau - 1} & -\frac{\tau}{2\tau - 1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There are three eigenvalues: 0, -1 and $(\tau - 1)/(2\tau - 1)$. The corresponding critical eigenspace T^c is one dimensional and spanned by an eigenvector $q \in R^3$ such that $Jq = -q$, where $q^T = \left(1/\sqrt{2}, -1/\sqrt{2}, 0\right)$. Let $s \in R^3$ be the adjoint eigenvector, that is, $J^T s = -s$, where J^T is the transposed matrix of J. Normalise s with respect to q such that $\langle s, q \rangle = 1$, where $s^T = \frac{\sqrt{2}}{2 - 3\tau}(1 - \tau, 2\tau - 1, 0)$. Let T^{su} denote a 2-dimensional linear eigenspace of J corresponding to the eigenvalues other than –1. It can be shown that $y \in T^{su}$ if and only if $\langle s, y \rangle = 0$.

The first term of $F(.)$ is the Hessian matrix B of (16) at the stationary point. B can be partitioned into three elements:

$$B^T = \begin{pmatrix} B_h^T & B_p^T & B_m^T \end{pmatrix}$$

where B_h is the first partitioned Hessian matrix associated with (16a), B_p is the second partitioned Hessian matrix associated with (16b), B_m is the third partitioned Hessian matrix associated with (16c).

These partitioned matrices are computed as follows:

$$B_z = 0\,,\ B_p = \begin{pmatrix} 0 & 0 & B_p(1,3) \\ 0 & 0 & B_p(2,3) \\ B_p(1,3) & B_p(2,3) & 0 \end{pmatrix} \text{ and}$$

$$B_m = \begin{pmatrix} B_m(1,1) & B_m(1,2) & 0 \\ B_m(1,2) & B_m(2,2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $B_p = m^T B_p(1,1)h + m^T B_p(2,3)p + h^T B_p(1,3)m + p^T B_p(2,3)m$

and $B_m = h^T B_m(1,1)h + p^T B_m(1,2)h + h^T B_m(1,2)p + p^T B_m(2,2)p$

We left to the reader to show that none of $C(.)$ is a relevant element for us.

Now we "decompose" any vector $x \in R^3$ as

$$x = uq + y,$$

where $uq \in T^c$, $y \in T^{su}$, and

$$\begin{cases} u = \langle s, x \rangle = \dfrac{\sqrt{2}}{2-3\tau}\left((1-\tau)h_t + (2\tau - 1)p_t\right), \\ y = x - \langle p, x \rangle q = \begin{pmatrix} \left(\dfrac{1-2\tau}{2-3\tau}\right)(h_t + p_t) \\ \left(\dfrac{1-\tau}{2-3\tau}\right)(h_t + p_t) \\ m_t \end{pmatrix}. \end{cases}$$

In the coordinates (u, y), (17) can be written as

$$\begin{cases} u = -u + \langle s, F(uq + y) \rangle, \\ y = Jy + F(uq + y) - \langle s, F(uq + y) \rangle q. \end{cases}$$

Using Taylor expansions, we can write the above system as:

$$\begin{cases} \tilde{u} = -u + \dfrac{1}{2}\sigma u^2 + u\langle \gamma, y \rangle + \dfrac{1}{6}\delta u^3 + \ldots, \\ \tilde{y} = Jy + \dfrac{1}{2}\alpha u^2 + \ldots. \end{cases} \tag{18}$$

where $u \in R$, $y \in R^3$, $\delta, \sigma \in R$, $\alpha, \gamma \in R^3$ and $\langle \gamma, y \rangle = \sum_{i=1}^{3} \gamma_i \, y_i$ is the standard scalar product in R^3. σ, δ and α are given by:

$$\sigma = \langle s, B(q,q) \rangle = s^T B(q,q) = 0$$

$$\delta = \langle s, C(q,q,q) \rangle = 0$$

$$\alpha = B(q,q) - \langle s, B(q,q) \rangle q = B(q,q) = \begin{pmatrix} 0 & 0 & D(1-2\tau)^2/2 \end{pmatrix}^T \quad (19)$$

where $D = \frac{\beta \, d \, Exp\beta}{\overline{n}_f} \left[-\frac{b+d}{b+d\overline{n}_f} \right]^2$.

The scalar product $\langle \gamma, y \rangle$ can be expressed as $\langle \gamma, y \rangle = \langle s, B(q,y) \rangle$.

The center manifold of (18) has the representation

$$y = V(u) = \frac{1}{2} w_2 u^2 + O(u^3),$$

where $w_2 \in T^{su} \subset R^3$, so that $\langle s, w_2 \rangle = 0$. The vector w_2 satisfies an equation in R^3:

$$(J - I) w_2 + \alpha = 0 .$$

In our case, $w_2^T = \left(0,\, 0,\, \frac{D}{2}(1-2\tau)^2 \right)$ and I is the identity matrix. The center manifold takes the form

$$\widetilde{u} = -u + \frac{1}{6}\left(\delta - 3\left\langle s, B\left(q, (J-I)^{-1}\alpha\right) \right\rangle \right) u^3 + O(u^4),$$

This restricted map can be simplified further. Using (19) and the identity

$$(J-I)^{-1}q=-0.5q\,,$$

we can write the restricted map as

$$\tilde{u}=-u+\alpha(0)u^2+\gamma(0)u^3+O(u^4), \qquad (20)$$

with $\alpha(0)=0.5\langle s,B(q,q)\rangle=0$

$$\gamma(0)=\frac{1}{6}\langle s,C(q,q,q)\rangle-\frac{1}{4}(\langle s,B(q,q)\rangle)^2-\frac{1}{2}\left\langle s,B\left(q,(J-I)^{-1}B(q,q)\right)\right\rangle.$$

$$\gamma(0)=-\frac{1}{2}\left\langle s,B\left(q,(J-I)^{-1}B(q,q)\right)\right\rangle=-\frac{1}{4}\frac{(1-2\tau)^4}{(2-3\tau)}D\,A'(\overline{n}_f)$$

where $A'(\overline{n}_f)=\dfrac{4\tau^2\,d^2}{(2\tau d-1)(b+d)(2\tau-1)}$.

It can be shown that (20) can be transformed to the normal form

$$\tilde{\varepsilon}=-\varepsilon+\chi(0)\,\varepsilon^3+O(\varepsilon^4),$$

where $\chi(0)=\alpha^2(0)+\gamma(0)$

Thus, the critical normal form coefficient $\chi(0)$, that determines the nondegeneracy of the Flip bifurcation and allows us to predict the direction of bifurcation of the two-period cycle, is given the following invariant formula:

$$\chi(0)=\frac{1}{6}\langle s,C(q,q,q)\rangle-\frac{1}{2}\left\langle s,B\left(q,(J-I)^{-1}B(q,q)\right)\right\rangle=-\frac{1}{4}\frac{(1-2\tau)^4}{(2-3\tau)}D\,A'(\overline{n}_f)$$

Let us now consider the Neimark-Sacker case.

The Jacobian matrix J of (16) at the stationary point is:

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -\frac{\tau}{1-\tau} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There are three eigenvalues: 0 and $\lambda_{1,2} = -\frac{\tau}{2(1-\tau)} \pm i\frac{\sqrt{(2-\tau)(2-3\tau)}}{2(1-\tau)} = Re(\lambda) \pm i\, Im(\lambda)$. J has a simple pair of eigenvalues on the unit circle $\lambda_{1,2} = e^{\pm i\theta_0}$ with $\pi/2 < \theta_0 < \pi$ and $\theta_0 \neq 2\pi/3$. Let $q \in C^3$ be a complex eigenvector corresponding to λ_1:

$$J\, q = e^{i\theta_0}\, q\ ,\ J\, \overline{q} = e^{-i\theta_0}\, \overline{q}\ ,$$

$q^T = (1, Re(\lambda) + i\, Im(\lambda), 0)$ and $\overline{q}^T = (1, Re(\lambda) - i\, Im(\lambda), 0)$. Introduce also the adjoint eigenvector $s \in C^3$ having the properties

$$J^T s = e^{-i\theta_0}\, s \text{ and } J^T \overline{s} = e^{i\theta_0}\, \overline{s}\ ,$$

and satisfying the normalisation

$$\langle s, q \rangle = 1\ ,$$

where $\langle s, q \rangle = \sum_{i=1}^{3} \overline{s}_i q_i$ is the standard product in C^3, $\overline{s}^T = \frac{1}{2\, Im(\lambda)(Im(\lambda) - i\, Re(\lambda))}(1, -Re(\lambda) - i\, Im(\lambda), 0)$. The critical real eigenspace T^c corresponding to $\lambda_{1,2}$ is two-dimensional and is spanned by $\{Re\,q, Im\,q\}$. The real eigenspace T^{su} corresponding to the other eigenvalue is one dimensional. It can be shown that $y \in T^{su}$ if and only if $\langle s, y \rangle = 0$.

Now we "decompose" any vector $x \in R^3$ as

$$x = zq + \overline{z}\overline{q} + y,$$

where $z \in C^1$, and $zq + \overline{z}\overline{q} + y \in T^c$, $y \in T^{su}$. The complex variable z is a coordinate on T^c. We have

$$\begin{cases} z = \langle s, x \rangle, \\ y = x - \langle s, x \rangle q - \langle \overline{s}, x \rangle \overline{q}. \end{cases}$$

In these coordinates, (17) can be written as

$$\begin{cases} \tilde{z} = e^{i\theta_0} z + \langle s, F(zq + \overline{z}\overline{q} + y) \rangle, \\ y = Jy + F(zq + \overline{z}\overline{q} + y) - \langle s, F(zq + \overline{z}\overline{q} + y) \rangle q - \langle \overline{s}, F(zq + \overline{z}\overline{q} + y) \rangle \overline{q}. \end{cases}$$

This system is 5-dimensional, but we have to remember the two real constraints imposed on y. The system can be written in a similar form to (18), namely (21)

$$\begin{cases}\tilde{z} = e^{i\theta_0} z + \frac{1}{2} G_{20} z^2 + G_{11} z\bar{z} + \frac{1}{2} G_{02} \bar{z}^2 + \frac{1}{2} G_{21} z^2 \bar{z} + \langle G_{10}, y\rangle z + \langle G_{01}, y\rangle \bar{z} + \ldots, \\ \tilde{y} = Jy + \frac{1}{2} H_{20} z^2 + H_{11} z\bar{z} + \frac{1}{2} H_{02} \bar{z}^2 \ldots . \end{cases}$$

where G_{20}, G_{11}, G_{02}, G_{21} $\in C^1$, G_{01}, G_{10}, H_{ij} $\in C^3$ and the scalar product in C^3 is used. The complex numbers and vectors involved in (21) can be computed as follows:

$$\langle G_{10}, y\rangle = \langle s, B(q, y)\rangle, \ \langle G_{01}, y\rangle = \langle s, B(\bar{q}, y)\rangle$$

The center manifold of (21) has the representation

$$y = V(z, \bar{z}) = \frac{1}{2} w_{20} z^2 + w_{11} z\bar{z} + \frac{1}{2} w_{20} \bar{z}^2 + O\left(|z|^3\right),$$

where $\langle s, w_{ij}\rangle = 0$. The vectors $w_{ij} \in C^3$ can be found from the linear equations

$$\begin{cases}\left(e^{2i\theta_0} I - J\right) w_{20} = H_{20} \\ (I - J) w_{11} = H_{11} \\ \left(e^{-2i\theta_0} I - J\right) w_{02} = H_{02}\end{cases} \qquad (22).$$

These equations have unique solutions. The matrix $I - J$ is invertible because 1 is not an eigenvalue of *J,* $e^{i\theta_0} \neq 1$. If $e^{3i\theta_0} \neq 1$, then $\left(e^{\pm 2i\theta_0} I - J\right)$ are also invertible in C^3. Thus, generically, the restricted map can be written as (21)

$$\widetilde{z} = i\omega_0 z + + \frac{1}{2} G_{20} z^2 + G_{11} z\overline{z} + \frac{1}{2} G_{02} \overline{z}^2 + \frac{1}{2} \left(G_{21} + 2\left\langle s, B\left(q, (I-J)^{-1} H_{11}\right)\right\rangle \right)$$
$$+ \left\langle s, B\left(\overline{q}, \left(e^{2i\theta_0} I - J\right)^{-1} H_{20} \right)\right\rangle z^2 \overline{z} + \ldots$$

Using (21) and the identity

$$(I-J)^{-1} q = \frac{1}{\left(1-e^{i\theta_0}\right)} q \,, \quad \left(e^{2i\theta_0} I - J\right)^{-1} q = \frac{e^{-i\theta_0}}{\left(e^{i\theta_0} - 1\right)} q \,,$$

$$(I-J)^{-1} \overline{q} = \frac{1}{\left(1-e^{-i\theta_0}\right)} \overline{q} \,, \quad \left(e^{2i\theta_0} I - J\right)^{-1} \overline{q} = \frac{e^{i\theta_0}}{\left(e^{3i\theta_0} - 1\right)} \overline{q}$$

transforms (22) into the map

$$\widetilde{z} = e^{2i\theta_0} z + \frac{1}{2} g_{20} z^2 + g_{11} z\overline{z} + \frac{1}{2} g_{02} \overline{z}^2 + \frac{1}{2} g_{21} z^2 \overline{z} + \ldots$$

where $g_{20} = \langle s, B(q,q) \rangle = 0$, $g_{11} = \langle s, B(q,\overline{q}) \rangle = 0$, and

$g_{02} = \langle s, B(\overline{q},\overline{q}) \rangle = 0$

and

$$g_{21} = \langle s, C(q,q,\overline{q}) \rangle + 2\left\langle s, B\left(q, (I-J)^{-1} B(q,\overline{q})\right)\right\rangle + \left\langle s, B\left(\overline{q}, \left(e^{2i\theta_0} I - J\right)^{-1} B(q,q) \right)\right\rangle$$
$$+ \frac{e^{-i\theta_0}\left(1-2e^{i\theta_0}\right)}{1-e^{i\theta_0}} \langle s, B(q,q) \rangle \langle s, B(q,\overline{q}) \rangle - \frac{2}{1-e^{-i\theta_0}} \left| \langle s, B(q,\overline{q}) \rangle \right|^2 - \frac{e^{i\theta_0}}{e^{3i\theta_0} - 1} \left| \langle s, B(\overline{q},\overline{q}) \rangle \right|^2$$

$$g_{21} = 2\left\langle s, B\left(q, (I-J)^{-1} B(q,\overline{q})\right)\right\rangle + \left\langle s, B\left(\overline{q}, \left(e^{2i\theta_0} I - J\right)^{-1} B(q,q) \right)\right\rangle$$
$$= \frac{-\left(Re(\lambda) + i\, Im(\lambda)\right)}{2\, Im(\lambda)\left(Im(\lambda) - i\, Re(\lambda)\right)} \left(A'\left(\overline{n}_f\right) D \left(1 - \tau + \tau\, Re(\lambda) + i\tau\, Im(\lambda)\right)^3 \right)$$

In the absence of strong resonances, i.e.

$$e^{ik\theta_0} \neq 1, \text{ for } k = 1,2,3,4$$

we can write the restricted map as

$$\tilde{z} = e^{i\theta_0} z\left(1 + \kappa(0)|z|^2\right) + O\left(u^4\right), \qquad (23)$$

with $\alpha(0) = Re\,\kappa(0) = 0$, that determines the direction of the bifurcation of a closed invariant curve, can be computed by the formula:

$$\alpha(0) = Re\left(\frac{e^{-i\theta_0} g_{21}}{2}\right) - Re\left(\frac{\left(1 - 2e^{i\theta_0}\right)e^{-2i\theta_0}}{2\left(1 - e^{i\theta_0}\right)} g_{20} g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2$$

Therefore, $\alpha(0) = Re\left(\frac{e^{-i\theta_0} g_{21}}{2}\right)$.

This compact formula allows us to verify the nondegeneracy of the nonlinear terms at a nonresonant Neimark-Sacker bifurcation of our 3-dimensional map.

Let us now see what happens when $\theta_0 = 2\pi/3$. Recall the findings of the corollary. When $\tau = 0.5$ (and $A(\bar{n}_f) = -1$), the stationary equilibrium undergoes a strong resonance 1:3 as $\theta = 2\pi/3$, see Kuznetsov (2000) p. 397.

Finally, when $\tau = 2/3$, the two curves of the Neimark-Sacker bifurcation and of the Flip bifurcation intersect. The stationary point has a double –1 eigenvalue, a codim-2 bifurcation occurs (See Frouzakis *et al*., 1991, p. 85).

Q.E.D.

Figure 1 when $\tau = 0.5$

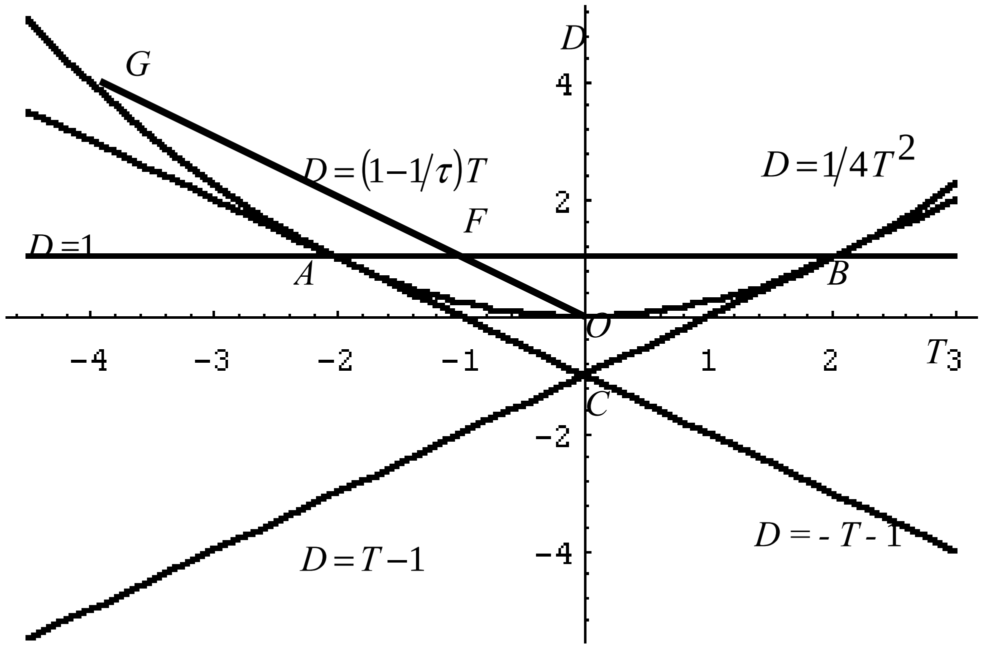


Figure 2

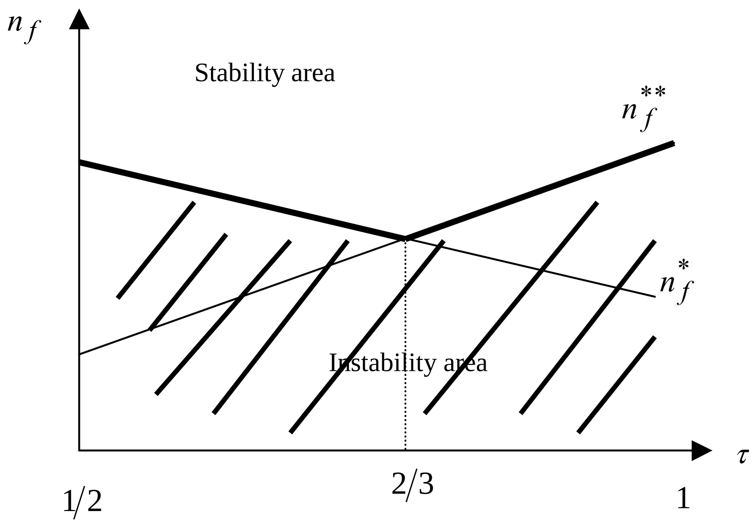
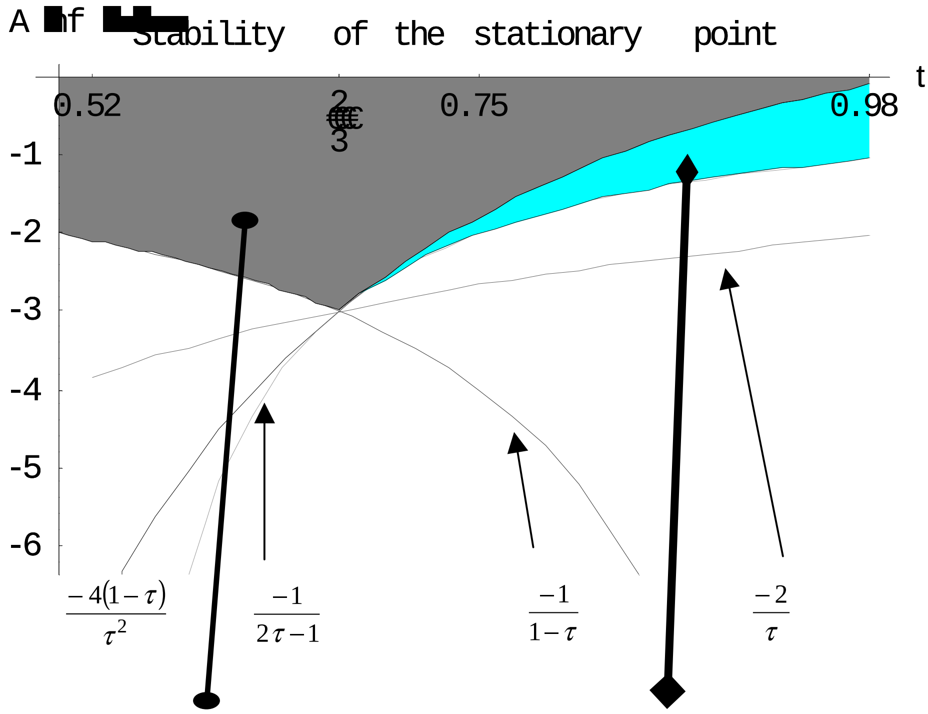


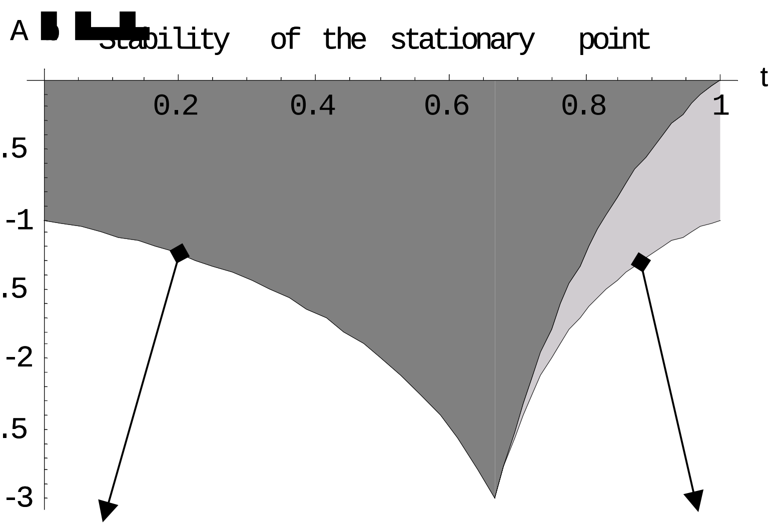
Figure 3



Stability zone for complex roots

Stability zone for real roots

Figure 4



Possibility of Neimark-Sacker bif.

Possibility of Flip bifurcation

Figures 5 is drawn for d = 5 and b = 1 and show the stability properties of the stationary point.

Figure 5