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States in some ordered structures and axioms of choice

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Axiom of Choice

AC : Given an infinite family $(A_i)_{i \in I}$ of non-empty sets, the product $\prod_{i \in I} A_i$ is non-empty.

We work in **ZF**, set theory **without the Axiom of Choice**.

We consider the **Hahn-Banach axiom (HB)**, a weak form of the Axiom of Choice. Remark : in **ZF**, **HB** is not provable and **HB** does not imply **AC** (see Howard and Rubin's book, [3]).

Theorem 1 : in **ZF**, **HB** implies the following statement **S_g**

S_g : For every abelian ordered group G with a positive order unit e and every subgroup H of G such that $e \in H$, every e -state on H can be extended into a e -state on G .

Remark : the converse statement **S_g** \Rightarrow **HB** also holds in **ZF**.

Hahn-Banach Axiom **HB** : a weak form of the **AC**

HB : Given a real vector space E , a sublinear mapping $p : E \rightarrow \mathbb{R}$ (i.e. a subadditive mapping such that for all $t \in \mathbb{R}^+$ and for all $x \in E, p(tx) = tp(x)$), a vector subspace S of E and a linear mapping $f : S \rightarrow \mathbb{R}$ such that $f \leq p|_S$, there exists a linear mapping $g : E \rightarrow \mathbb{R}$ extending f such that $g \leq p$.

Corollary 1 : in **ZF**, **HB** implies the classical following statement

Given a real normed vector space $(E, \| \cdot \|)$ and $a \in E \setminus \{0\}$, there exists a linear form $\varphi : E \rightarrow \mathbb{R}$ continuous of norm 1 such that $\varphi(a) = \|a\|$.

Proof : apply **HB** to the sublinear mapping $p := \| \cdot \|$, the vector

subspace $S := \text{Vect}(a)$ and the linear form

$$f : S \rightarrow \mathbb{R}$$
$$\lambda a \mapsto \lambda \|a\| .$$

1 On partially ordered groups :

Let G be an abelian ordered group. A non-zero element e of G is an **order unit** of G if : $\forall x \in G \exists k \in \mathbb{Z} -ke \leq x \leq ke$.

2 On partially ordered vector spaces :

Let E be an ordered vector space over \mathbb{R} .

- An element $e \in E \setminus \{0\}$ is an *order unit* of E if it is an order unit of the ordered group $(E, +)$. For an order unit e of E : $e \in E^+$ or $-e \in E^+$.
- Given a positive order unit $e \in E^+$ we associate a *semi-norm* $\| \cdot \|_e$ defined by :

$$\forall x \in E \quad \|x\|_e := \inf\{t \in \mathbb{R}^+, -te \leq x \leq te\}$$

- The semi-norms associated to two positive order units are equivalent and then, they define the same topology on E .

1 On ordered groups :

Let G be an abelian ordered group. A group morphism $f : G \rightarrow \mathbb{R}$ is *positive* if it is *increasing* i.e. $\forall x, y \in G, (x \leq y \Rightarrow f(x) \leq f(y))$.

2 On ordered vector spaces :

Lemma 1 (Characterisation)

Let $f : E \rightarrow \mathbb{R}$ be a linear form on an ordered vector space E with an order unit $e \in E^+$. Then :

f is positive (i.e. increasing) if and only if f is continuous of norm $f(e)$.

Proof :

- Assume that f is positive. Let $x \in E$: there exists $s \in \mathbb{R}_+^*$ such that $-se \leq x \leq se$, so $|f(x)| \leq s|f(e)|$ and then f is continuous and of norm $f(e)$.
- Now assume that f is continuous of norm $f(e)$. Let $x \in E^+$, show that $f(x) \geq 0$:
 - If $x \leq e$ then $0 \leq e - x \leq e$ so $f(e - x) \leq \|f\| \cdot \|e - x\|_e \leq f(e)$ and finally $f(x) \geq 0$.
 - If $x \not\leq e$, there exists $s \in \mathbb{R}_+^*$ such that $-se \leq x \leq se$ then, apply the previous case to $\frac{1}{s}x$.

Concurrent relations (Luxemburg, [4])

Let X and Y be two sets. Given a binary relation \mathcal{R} on $X \times Y$, for every $x \in X$, we define $\mathcal{R}(x) := \{y \in Y \mid x\mathcal{R}y\}$.

- The relation \mathcal{R} is *concurrent* if for every finite subset $F := \{x_1, \dots, x_n\}$ of X , the intersection $\mathcal{R}(x_1) \cap \dots \cap \mathcal{R}(x_n)$ is non-empty.
- If \mathcal{R} is a concurrent relation on $X \times Y$, we can define the *filter* \mathcal{F} on Y generated by the sets $\mathcal{R}(x)$, $x \in X$:

$$\mathcal{F} := \{A \subseteq Y \mid \exists x_1, \dots, x_n \in X \ \mathcal{R}(x_1) \cap \dots \cap \mathcal{R}(x_n) \subseteq A\}$$

Definition

Let E be a real vector space. Consider a set T and \mathcal{F} a filter over T . Denote by Z the following vector subspace of the vector space E^T :

$$Z := \{(x_t)_{t \in T} \in E^T \mid \{t \in T \mid x_t = 0\} \in \mathcal{F}\}$$

The **reduced power** E^T/\mathcal{F} of E by the filter \mathcal{F} is the quotient vector space E^T/Z . We denote by \bar{z} the class of an element $z \in E^T$ and we

consider the canonical embedding :

$$\begin{array}{ccc} \text{can} : E & \rightarrow & E^T/\mathcal{F} \\ x & \mapsto & \overline{(x)_{t \in T}} \end{array}$$

Remarks : if E is an ordered vector space :

- The vector space E^T endowed with the product order is an ordered vector space and the vector subspace Z is **order-convex** i.e. for every $v, w \in E^T$, $[v, w] := \{x \in E^T \mid v \leq x \leq w\} \subseteq Z$.

Thus, the reduced power E^T/\mathcal{F} is also an ordered vector space.

- Moreover, if E has an order unit e , then the set :

$$\mathcal{L}_0(E^T/\mathcal{F}) := \{z \in E^T/\mathcal{F} \mid \exists \alpha, \beta \in \mathbb{R} \quad \alpha \text{can}(e) \leq z \leq \beta \text{can}(e)\}$$

is an ordered vector space with order unit $\text{can}(e)$.

A group morphism $f : G \rightarrow \mathbb{R}$ on an abelian ordered group G with positive order unit e is a e -state if f is positive and $f(e) = 1$.

We want to prove the following result :

Theorem 1 : in **ZF**, **HB** implies the following statement S_g

S_g : For every abelian ordered group G with a positive order unit e and every subgroup H of G such that $e \in H$, every e -state on H can be extended into a e -state on G .

The proof is in two steps : first, extending to “one dimension” and then extending to G .

Step 1 : extending to one dimension, in ZF

Let G be an abelian ordered group, H be a subgroup of G and $f : H \rightarrow \mathbb{R}$ a positive group morphism on H .

Extending to one dimension : If H is **cofinal** (i.e. for every $x \in G$, there exists $y \in H$ such that $x \leq y$) and if $x \in G$, we consider :

- $p_H(x) = \sup \left\{ \frac{f(y)}{m} \mid m \in \mathbb{N}^*, y \in H, y \leq mx \right\} \in \mathbb{R}$
- $r_H(x) = \inf \left\{ \frac{f(z)}{n} \mid n \in \mathbb{N}^*, z \in H, nx \leq z \right\} \in \mathbb{R}$

Remark : $p_H(x) \leq r_H(x)$.

Lemma 2 (Goodearl [2], extending to one dimension)

Let G be an abelian ordered group, H be a cofinal subgroup of G and $f : H \rightarrow \mathbb{R}$ a positive group morphism on H . Let $x \in G$:

- 1 For every positive group morphism $g : H + \mathbb{Z}x \rightarrow \mathbb{R}$ extending f , we have : $p_H(x) \leq g(x) \leq r_H(x)$.
- 2 For every $t \in [p_H(x), r_H(x)]$, it is possible to extend f to a positive group morphism $g : H + \mathbb{Z}x \rightarrow \mathbb{R}$ such that $g(x) = t$.

Corollary 2 : extending to a finite number of dimensions

Let G be an abelian ordered group, H be a cofinal subgroup of G and $f : H \rightarrow \mathbb{R}$ a positive group morphism on H . Let $x_1, \dots, x_n \in G$. There exists a positive group morphism $g : H + \mathbb{Z}x_1 + \dots + \mathbb{Z}x_n$ extending f such that $p_H \leq g \leq r_H$.

Proof : apply the preceding Lemma and remark that if H_1 is a subgroup of G such that $H \subseteq H_1$, $p_H \leq p_{H_1} \leq r_{H_1} \leq r_H$.

Step 2 : Proof of Theorem 1 *i.e.* $\mathbf{HB} \Rightarrow \mathbf{S}_g$

Extending to \mathbf{G} : Let G be an abelian ordered group with positive order unit e , H a subgroup of G such that $e \in H$ (then H is cofinal) and $f : H \rightarrow \mathbb{R}$ a e -state on H .

1. Concurrent relation :

- Denote by $\mathcal{P}_{fin}(G)$ the set of finite subsets of G and $T := \{g \in \mathbb{R}^G \mid p_H \leq g \leq r_H\}$.
- Let \mathcal{R}_f be the binary relation defined by $\forall (F, g) \in \mathcal{P}_{fin}(G) \times T :$

$$\mathcal{R}_f(F, g) : \begin{cases} \forall a, b \in F (a + b \in F \Rightarrow g(a + b) = g(a) + g(b)) \\ \forall a \in F (-a \in F \Rightarrow g(-a) = -g(a)) \\ \forall a, b \in F (a \leq b \Rightarrow g(a) \leq g(b)) \\ \forall a \in F (a \in H \Rightarrow g(a) = f(a)) \\ \forall a \in F p_H(a) \leq g(a) \leq r_H(a) \end{cases}$$

- Using Corollary 2, we prove that if $F \in \mathcal{P}_{fin}(G)$ there exists $g \in T$ extending f ; then $\mathcal{R}_f(F) \neq \emptyset$.
- \mathcal{R}_f is concurrent because if $F_1, \dots, F_n \in \mathcal{P}_{fin}(G)$ then $\emptyset \neq \mathcal{R}_f(F_1 \cup \dots \cup F_n) \subseteq \mathcal{R}_f(F_1) \cap \dots \cap \mathcal{R}_f(F_n)$.
- Thus, we consider the filter \mathcal{F} on T generated by the sets $\mathcal{R}_f(F)$, $F \in \mathcal{P}_{fin}(G)$.

2. Reduced power of \mathbb{R} :

- Consider the reduced power \mathbb{R}^T/\mathcal{F} (if $z \in \mathbb{R}^T$, we note \bar{z} the class of z in \mathbb{R}^T/\mathcal{F}) and $\text{can} : \mathbb{R} \rightarrow \mathbb{R}^T/\mathcal{F}$ the canonical embedding.

- Consider $\varphi : G \rightarrow \mathbb{R}^T/\mathcal{F}$
 $x \mapsto \frac{(g(x))_{g \in T}}{g \in T}$: φ is a positive group morphism.

- For all $x \in G$, $\varphi(x) \in \mathcal{L}_0(\mathbb{R}^T/\mathcal{F})$ because :

$$\forall g \in T \quad r_H \leq g \leq p_H$$

Then :

$$\forall x \in G \quad r_H(x) \text{can}(1) \leq \varphi(x) \leq p_H(x) \text{can}(1)$$

First positive group morphism

$$\varphi : G \rightarrow \mathcal{L}_0(\mathbb{R}^T/\mathcal{F})$$

3. Use of \mathbf{HB} :

- **Normed vector space $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N$:**

- $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})$ is an ordered vector space with an order unit $e_1 := \text{can}(1)$.
- Thus it is endowed with a semi-norm $\|\cdot\|_{e_1}$.
- Let N be the vector subspace $N := \{x \in \mathcal{L}_0(\mathbb{R}^T/\mathcal{F}) \mid \|x\|_{e_1} = 0\}$.
- The quotient vector space $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N$ is endowed with the associated quotient norm.

- Apply \mathbf{HB} (Corollary 1) to $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N$: there exists a linear form $\psi : \mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N \rightarrow \mathbb{R}$ continuous of norm 1 such that $\psi(e_1 + N) = \|e_1 + N\| = 1$.

Then,
$$\Gamma : \mathcal{L}_0(\mathbb{R}^T/\mathcal{F}) \rightarrow \mathbb{R}$$

$$z \mapsto \psi(z + N)$$
 is continuous of norm 1

and $\Gamma(e_1) = 1$: with Lemma 1, Γ is a e_1 -state.

- $\Gamma \circ \text{can} = \text{Id}_{\mathbb{R}}$.

Second positive group morphism

$$\Gamma : \mathcal{L}_0(\mathbb{R}^T/\mathcal{F}) \rightarrow \mathbb{R}$$

4. Existence of state on G :

Extension of f

$$\tilde{f} := \Gamma \circ \varphi : G \rightarrow \mathbb{R}$$

- \tilde{f} is a e-state.
- \tilde{f} extends f because if $x \in H$ then :
 - $\tilde{f}(x) = \Gamma \circ \varphi(x) = \Gamma(\overline{(g(x))_{g \in T}})$.
 - But $\overline{(g(x))_{g \in T}} = \text{can}(f(x))$ because $\mathcal{R}_f(\{x\}) \subseteq \{g \in T \mid g(x) = f(x)\} \in \mathcal{F}$.
 - Then $\tilde{f}(x) = \Gamma(\text{can}(f(x))) = f(x)$ because $\Gamma \circ \text{can} = \text{Id}_{\mathbb{R}}$

We worked on several structures : abelian ordered group with positive order unit, real vector spaces with positive order unit, or unital C^* -algebras.

Given an abelian ordered group G (resp. a real ordered vector space E) with a positive order unit e , a *pure state* on G (resp. on E) is an extreme point of the convex set of e -states on G (resp. on E).

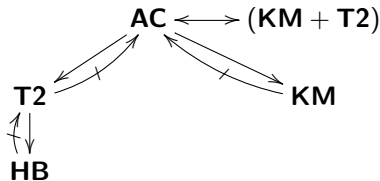
Question

Which consequence of axiom of choice do we need to prove the existence of states or pure states on ordered groups or ordered vector spaces with order unit ?

Consider the two other following weak forms of the Axiom of Choice :

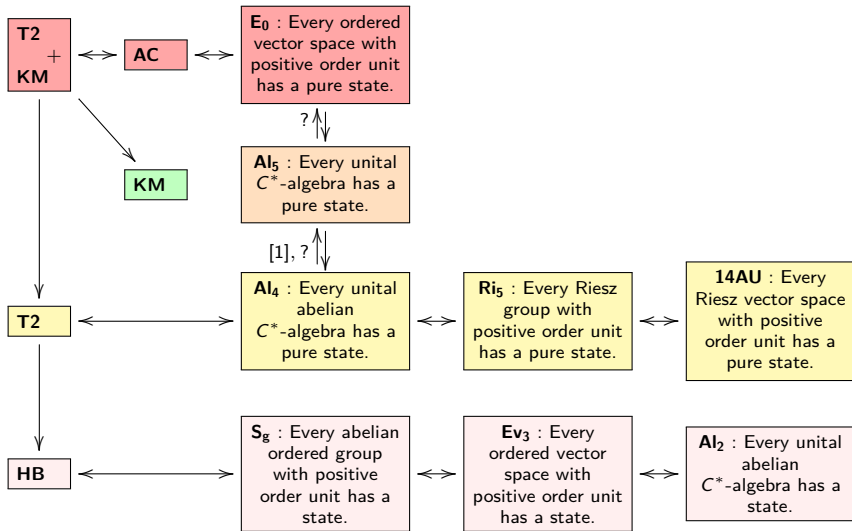
- **KM** (*Krein-Milman axiom*) : Let K be a non-empty compact convex subset of a topological locally convex Hausdorff real vector space X . Then K has an extreme point.
- **T2** (*Tychonov's axiom*) : For every family $(X_i)_{i \in I}$ of compact Hausdorff spaces, the product $\prod_{i \in I} X_i$ is compact.

We have the following diagram :



We obtained the following results :

Diagram : states and axioms of choice



Thank you for listening.



G. Buskes and ACM van Rooij.

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