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# States in some ordered structures and axioms of choice

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## Axiom of Choice

**AC** : Given an infinite family  $(A_i)_{i \in I}$  of non-empty sets, the product  $\prod_{i \in I} A_i$  is non-empty.

We work in **ZF**, set theory **without the Axiom of Choice**.

We consider the **Hahn-Banach axiom (HB)**, a weak form of the Axiom of Choice. Remark : in **ZF**, **HB** is not provable and **HB** does not imply **AC** (see Howard and Rubin's book, [3]).

**Theorem 1** : in **ZF**, **HB** implies the following statement **S<sub>g</sub>**

**S<sub>g</sub>** : For every abelian ordered group  $G$  with a positive order unit  $e$  and every subgroup  $H$  of  $G$  such that  $e \in H$ , every  $e$ -state on  $H$  can be extended into a  $e$ -state on  $G$ .

Remark : the converse statement **S<sub>g</sub>**  $\Rightarrow$  **HB** also holds in **ZF**.

Hahn-Banach Axiom **HB** : a weak form of the **AC**

**HB** : Given a real vector space  $E$ , a sublinear mapping  $p : E \rightarrow \mathbb{R}$  (i.e. a subadditive mapping such that for all  $t \in \mathbb{R}^+$  and for all  $x \in E$ ,  $p(tx) = tp(x)$ ), a vector subspace  $S$  of  $E$  and a linear mapping  $f : S \rightarrow \mathbb{R}$  such that  $f \leq p|_S$ , there exists a linear mapping  $g : E \rightarrow \mathbb{R}$  extending  $f$  such that  $g \leq p$ .

**Corollary 1** : in **ZF**, **HB** implies the classical following statement

Given a real normed vector space  $(E, \| \cdot \|)$  and  $a \in E \setminus \{0\}$ , there exists a linear form  $\varphi : E \rightarrow \mathbb{R}$  continuous of norm 1 such that  $\varphi(a) = \|a\|$ .

**Proof** : apply **HB** to the sublinear mapping  $p := \| \cdot \|$ , the vector

subspace  $S := \text{Vect}(a)$  and the linear form

$$f : S \rightarrow \mathbb{R}$$
$$\lambda a \mapsto \lambda \|a\| .$$

## 1 On partially ordered groups :

Let  $G$  be an abelian ordered group. A non-zero element  $e$  of  $G$  is an **order unit** of  $G$  if :  $\forall x \in G \exists k \in \mathbb{Z} -ke \leq x \leq ke$ .

## 2 On partially ordered vector spaces :

Let  $E$  be an ordered vector space over  $\mathbb{R}$ .

- An element  $e \in E \setminus \{0\}$  is an *order unit* of  $E$  if it is an order unit of the ordered group  $(E, +)$ . For an order unit  $e$  of  $E$  :  $e \in E^+$  or  $-e \in E^+$ .
- Given a positive order unit  $e \in E^+$  we associate a *semi-norm*  $\| \cdot \|_e$  defined by :

$$\forall x \in E \quad \|x\|_e := \inf\{t \in \mathbb{R}^+, -te \leq x \leq te\}$$

- The semi-norms associated to two positive order units are equivalent and then, they define the same topology on  $E$ .

## 1 On ordered groups :

Let  $G$  be an abelian ordered group. A group morphism  $f : G \rightarrow \mathbb{R}$  is *positive* if it is *increasing* i.e.  $\forall x, y \in G, (x \leq y \Rightarrow f(x) \leq f(y))$ .

## 2 On ordered vector spaces :

### Lemma 1 (Characterisation)

Let  $f : E \rightarrow \mathbb{R}$  be a linear form on an ordered vector space  $E$  with an order unit  $e \in E^+$ . Then :

$f$  is positive (i.e. increasing) if and only if  $f$  is continuous of norm  $f(e)$ .

### Proof :

- Assume that  $f$  is positive. Let  $x \in E$  : there exists  $s \in \mathbb{R}_+^*$  such that  $-se \leq x \leq se$ , so  $|f(x)| \leq s|f(e)|$  and then  $f$  is continuous and of norm  $f(e)$ .
- Now assume that  $f$  is continuous of norm  $f(e)$ . Let  $x \in E^+$ , show that  $f(x) \geq 0$  :
  - If  $x \leq e$  then  $0 \leq e - x \leq e$  so  $f(e - x) \leq \|f\| \cdot \|e - x\|_e \leq f(e)$  and finally  $f(x) \geq 0$ .
  - If  $x \not\leq e$ , there exists  $s \in \mathbb{R}_+^*$  such that  $-se \leq x \leq se$  then, apply the previous case to  $\frac{1}{s}x$ .

# Concurrent relations (Luxemburg, [4])

Let  $X$  and  $Y$  be two sets. Given a binary relation  $\mathcal{R}$  on  $X \times Y$ , for every  $x \in X$ , we define  $\mathcal{R}(x) := \{y \in Y \mid x\mathcal{R}y\}$ .

- The relation  $\mathcal{R}$  is *concurrent* if for every finite subset  $F := \{x_1, \dots, x_n\}$  of  $X$ , the intersection  $\mathcal{R}(x_1) \cap \dots \cap \mathcal{R}(x_n)$  is non-empty.
- If  $\mathcal{R}$  is a concurrent relation on  $X \times Y$ , we can define the *filter*  $\mathcal{F}$  on  $Y$  generated by the sets  $\mathcal{R}(x)$ ,  $x \in X$  :

$$\mathcal{F} := \{A \subseteq Y \mid \exists x_1, \dots, x_n \in X \ \mathcal{R}(x_1) \cap \dots \cap \mathcal{R}(x_n) \subseteq A\}$$

## Definition

Let  $E$  be a real vector space. Consider a set  $T$  and  $\mathcal{F}$  a filter over  $T$ . Denote by  $Z$  the following vector subspace of the vector space  $E^T$  :

$$Z := \{(x_t)_{t \in T} \in E^T \mid \{t \in T \mid x_t = 0\} \in \mathcal{F}\}$$

The **reduced power**  $E^T/\mathcal{F}$  of  $E$  by the filter  $\mathcal{F}$  is the quotient vector space  $E^T/Z$ . We denote by  $\bar{z}$  the class of an element  $z \in E^T$  and we

consider the canonical embedding :

$$\begin{array}{ccc} \text{can} : & E & \rightarrow & E^T/\mathcal{F} \\ & x & \mapsto & \overline{(x)_{t \in T}} \end{array}$$

**Remarks** : if  $E$  is an ordered vector space :

- The vector space  $E^T$  endowed with the product order is an ordered vector space and the vector subspace  $Z$  is **order-convex** i.e. for every  $v, w \in E^T$ ,  $[v, w] := \{x \in E^T, v \leq x \leq w\} \subseteq E^T$ .

Thus, the reduced power  $E^T/\mathcal{F}$  is also an ordered vector space.

- Moreover, if  $E$  has an order unit  $e$ , then the set :

$$\mathcal{L}_0(E^T/\mathcal{F}) := \{z \in E^T/\mathcal{F} \mid \exists \alpha, \beta \in \mathbb{R} \quad \alpha \text{can}(e) \leq z \leq \beta \text{can}(e)\}$$

is an ordered vector space with order unit  $\text{can}(e)$ .



A group morphism  $f : G \rightarrow \mathbb{R}$  on an abelian ordered group  $G$  with positive order unit  $e$  is a  $e$ -state if  $f$  is positive and  $f(e) = 1$ .

We want to prove the following result :

**Theorem 1** : in **ZF**, **HB** implies the following statement  $S_g$

$S_g$  : For every abelian ordered group  $G$  with a positive order unit  $e$  and every subgroup  $H$  of  $G$  such that  $e \in H$ , every  $e$ -state on  $H$  can be extended into a  $e$ -state on  $G$ .

The proof is in two steps : first, extending to “one dimension” and then extending to  $G$ .

# Step 1 : extending to one dimension, in ZF

Let  $G$  be an abelian ordered group,  $H$  be a subgroup of  $G$  and  $f : H \rightarrow \mathbb{R}$  a positive group morphism on  $H$ .

**Extending to one dimension :** If  $H$  is **cofinal** (i.e. for every  $x \in G$ , there exists  $y \in H$  such that  $x \leq y$ ) and if  $x \in G$ , we consider :

- $p_H(x) = \sup \left\{ \frac{f(y)}{m} \mid m \in \mathbb{N}^*, y \in H, y \leq mx \right\} \in \mathbb{R}$
- $r_H(x) = \inf \left\{ \frac{f(z)}{n} \mid n \in \mathbb{N}^*, z \in H, nx \leq z \right\} \in \mathbb{R}$

**Remark :**  $p_H(x) \leq r_H(x)$ .

**Lemma 2** (Goodearl [2], extending to one dimension)

*Let  $G$  be an abelian ordered group,  $H$  be a cofinal subgroup of  $G$  and  $f : H \rightarrow \mathbb{R}$  a positive group morphism on  $H$ . Let  $x \in G$  :*

- 1 *For every positive group morphism  $g : H + \mathbb{Z}x \rightarrow \mathbb{R}$  extending  $f$ , we have :  $p_H(x) \leq g(x) \leq r_H(x)$ .*
- 2 *For every  $t \in [p_H(x), r_H(x)]$ , it is possible to extend  $f$  to a positive group morphism  $g : H + \mathbb{Z}x \rightarrow \mathbb{R}$  such that  $g(x) = t$ .*

## Corollary 2 : extending to a finite number of dimensions

Let  $G$  be an abelian ordered group,  $H$  be a cofinal subgroup of  $G$  and  $f : H \rightarrow \mathbb{R}$  a positive group morphism on  $H$ . Let  $x_1, \dots, x_n \in G$ . There exists a positive group morphism  $g : H + \mathbb{Z}x_1 + \dots + \mathbb{Z}x_n$  extending  $f$  such that  $p_H \leq g \leq r_H$ .

**Proof :** apply the preceding Lemma and remark that if  $H_1$  is a subgroup of  $G$  such that  $H \subseteq H_1$ ,  $p_H \leq p_{H_1} \leq r_{H_1} \leq r_H$ .

## Step 2 : Proof of Theorem 1 *i.e.* $\mathbf{HB} \Rightarrow \mathbf{S}_g$

**Extending to  $\mathbf{G}$**  : Let  $G$  be an abelian ordered group with positive order unit  $e$ ,  $H$  a subgroup of  $G$  such that  $e \in H$  (then  $H$  is cofinal) and  $f : H \rightarrow \mathbb{R}$  a  $e$ -state on  $H$ .

### 1. Concurrent relation :

- Denote by  $\mathcal{P}_{fin}(G)$  the set of finite subsets of  $G$  and  $T := \{g \in \mathbb{R}^G \mid p_H \leq g \leq r_H\}$ .
- Let  $\mathcal{R}_f$  be the binary relation defined by  $\forall (F, g) \in \mathcal{P}_{fin}(G) \times T$  :

$$\mathcal{R}_f(F, g) : \begin{cases} \forall a, b \in F (a + b \in F \Rightarrow g(a + b) = g(a) + g(b)) \\ \forall a \in F (-a \in F \Rightarrow g(-a) = -g(a)) \\ \forall a, b \in F (a \leq b \Rightarrow g(a) \leq g(b)) \\ \forall a \in F (a \in H \Rightarrow g(a) = f(a)) \\ \forall a \in F p_H(a) \leq g(a) \leq r_H(a) \end{cases}$$

- Using Corollary 2, we prove that if  $F \in \mathcal{P}_{fin}(G)$  there exists  $g \in T$  extending  $f$  ; then  $\mathcal{R}_f(F) \neq \emptyset$ .
- $\mathcal{R}_f$  is concurrent because if  $F_1, \dots, F_n \in \mathcal{P}_{fin}(G)$  then  $\emptyset \neq \mathcal{R}_f(F_1 \cup \dots \cup F_n) \subseteq \mathcal{R}_f(F_1) \cap \dots \cap \mathcal{R}_f(F_n)$ .
- Thus, we consider the filter  $\mathcal{F}$  on  $T$  generated by the sets  $\mathcal{R}_f(F)$ ,  $F \in \mathcal{P}_{fin}(G)$ .

# Step 2 : Proof of Theorem 1 *i.e.* $\mathbf{HB} \Rightarrow \mathbf{S}_g$ (cont'd)

## 2. Reduced power of $\mathbb{R}$ :

- Consider the reduced power  $\mathbb{R}^T/\mathcal{F}$  (if  $z \in \mathbb{R}^T$ , we note  $\bar{z}$  the class of  $z$  in  $\mathbb{R}^T/\mathcal{F}$ ) and  $\text{can} : \mathbb{R} \rightarrow \mathbb{R}^T/\mathcal{F}$  the canonical embedding.

- Consider  $\varphi : G \rightarrow \frac{\mathbb{R}^T/\mathcal{F}}{(g(x))_{g \in T}}$  :  $\varphi$  is a positive group morphism.

- For all  $x \in G$ ,  $\varphi(x) \in \mathcal{L}_0(\mathbb{R}^T/\mathcal{F})$  because :

$$\forall g \in T \quad r_H \leq g \leq p_H$$

Then :

$$\forall x \in G \quad r_H(x) \text{can}(1) \leq \varphi(x) \leq p_H(x) \text{can}(1)$$

First positive group morphism

$$\varphi : G \rightarrow \mathcal{L}_0(\mathbb{R}^T/\mathcal{F})$$

## 3. Use of $\mathbf{HB}$ :

- **Normed vector space  $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N$  :**

- $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})$  is an ordered vector space with an order unit  $e_1 := \text{can}(1)$ .
- Thus it is endowed with a semi-norm  $\| \cdot \|_{e_1}$ .
- Let  $N$  be the vector subspace  $N := \{x \in \mathcal{L}_0(\mathbb{R}^T/\mathcal{F}) \mid \|x\|_{e_1} = 0\}$ .
- The quotient vector space  $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N$  is endowed with the associated quotient norm.

- Apply  $\mathbf{HB}$  (Corollary 1) to  $\mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N$  : there exists a linear form  $\psi : \mathcal{L}_0(\mathbb{R}^T/\mathcal{F})/N \rightarrow \mathbb{R}$  continuous of norm 1 such that  $\psi(e_1 + N) = \|e_1 + N\| = 1$ .

Then,  $\Gamma : \mathcal{L}_0(\mathbb{R}^T/\mathcal{F}) \rightarrow \mathbb{R}$  is continuous of norm 1  
 $z \mapsto \psi(z + N)$

and  $\Gamma(e_1) = 1$  : with Lemma 1,  $\Gamma$  is a  $e_1$ -state.

- $\Gamma \circ \text{can} = \text{Id}_{\mathbb{R}}$ .

## Second positive group morphism

$$\Gamma : \mathcal{L}_0(\mathbb{R}^T/\mathcal{F}) \rightarrow \mathbb{R}$$

## 4. Existence of state on $G$ :

### Extension of $f$

$$\tilde{f} := \Gamma \circ \varphi : G \rightarrow \mathbb{R}$$

- $\tilde{f}$  is a e-state.
- $\tilde{f}$  extends  $f$  because if  $x \in H$  then :
  - $\tilde{f}(x) = \Gamma \circ \varphi(x) = \Gamma(\overline{(g(x))_{g \in T}})$ .
  - But  $\overline{(g(x))_{g \in T}} = \text{can}(f(x))$  because  $\mathcal{R}_f(\{x\}) \subseteq \{g \in T \mid g(x) = f(x)\} \in \mathcal{F}$ .
  - Then  $\tilde{f}(x) = \Gamma(\text{can}(f(x))) = f(x)$  because  $\Gamma \circ \text{can} = \text{Id}_{\mathbb{R}}$

We worked on several structures : abelian ordered group with positive order unit, real vector spaces with positive order unit, or unital  $C^*$ -algebras.

Given an abelian ordered group  $G$  (resp. a real ordered vector space  $E$ ) with a positive order unit  $e$ , a *pure state* on  $G$  (resp. on  $E$ ) is an extreme point of the convex set of  $e$ -states on  $G$  (resp. on  $E$ ).

## Question

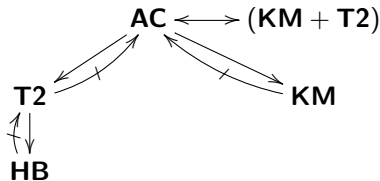
Which consequence of axiom of choice do we need to prove the existence of states or pure states on ordered groups or ordered vector spaces with order unit ?



Consider the two other following weak forms of the Axiom of Choice :

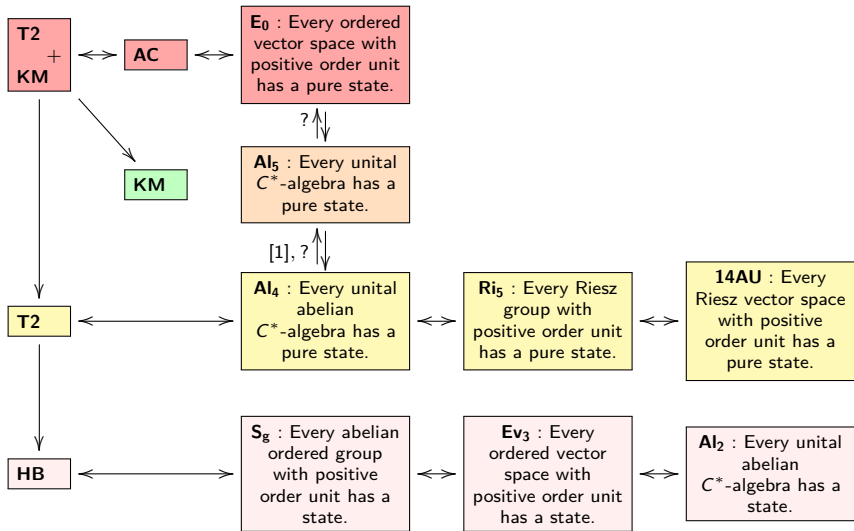
- **KM** (*Krein-Milman axiom*) : Let  $K$  be a non-empty compact convex subset of a topological locally convex Hausdorff real vector space  $X$ . Then  $K$  has an extreme point.
- **T2** (*Tychonov's axiom*) : For every family  $(X_i)_{i \in I}$  of compact Hausdorff spaces, the product  $\prod_{i \in I} X_i$  is compact.

We have the following diagram :



We obtained the following results :

# Diagram : states and axioms of choice



Thank you for listening.



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