The Significant Impact of Abstentions on Election Outcomes

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Abstract

Two approaches are considered in an attempt to mollify previous observations regarding the very negative impact that voter abstentions can have on Plurality Rule, Negative Plurality Rule and Borda Rule when voters' preferences are assumed to be statistically independent. The first option is to add a degree of dependence among voters' preferences, and the second is to consider extensions of these three voting rules by using each as the basis for a two-stage elimination election procedure. The end result in both cases is that improved results are indeed observed for both Condorcet Efficiency and the probability that a Borda Paradox is observed for larger values of voter participation rates. The very disturbing result is that both of these options actually tend to make things worse with voter participation rates of 40% or less, and voter participation rates are observed in practice at this level.

Introduction

We consider the impact that voter abstention can have on elections with three candidates $\{A, B, C\}$ when there are n possible voters in the electorate. To begin, we define the preferences of these voters by using $A \succ B$ to denote the fact that any given voter prefers Candidate A to Candidate B. There are six possible linear voter preference rankings on these candidates that are transitive and have no voter indifference between candidates, as shown in Figure 1.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Figure 1. Six possible linear preference rankings on three candidates.

Here, for example, n_1 represents the number of possible voters with the preference ranking $A \succ B \succ C$, with $A \succ C$ being required by transitivity. The total number of possible voters is n with $n = \sum_{i=1}^{6} n_i$, and an *actual voting situation* defines any combination of n_i terms that sum to n.

Much of what follows is based on the notion of using the preference rankings of all possible voters in an actual voting situation to perform majority rule comparisons on pairs of candidates, such that Candidate A beats B by *Pairwise Majority Rule* $[AMB]$ if $A \succ B$ more times than $B \succ A$ in the preference rankings of the possible voters. Candidate A is defined as the *Actual Condorcet Winner* (*ACW*) if both AMB and AMC, with:

$$n_1 + n_2 + n_4 > n_3 + n_5 + n_6 \quad [AMB] \tag{1}$$
$$n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \quad [AMC]. \tag{2}$$

It is well known that an ACW does not necessarily exist [Condorcet (1785)], but such a candidate would clearly be a very good choice for selection as the winner of an election whenever there is one, since a majority of the possible voters would oppose the choice of either of the other two candidates. The *Actual Condorcet Loser* (*ACL*) is then defined in the obvious manner, and such a candidate would be a terrible choice for selection as the winner.

We proceed to consider election outcomes when abstentions are allowed, so that some of the possible voters can choose not to participate for any reason. Let n_i^* denote the number of voters with the associated preference ranking in Figure 1 who actually choose to participate in the election, with $0 \le n_i^* \le n_i$, for $i = 1, 2, 3, 4, 5, 6$. The total number of voters who participate is defined by $n^* = \sum_{i=1}^{6} n_i^*$, so that the voter participation rate is $\frac{n^*}{n}$. An *observed voting situation* is then defined by any combination of n_i^* terms that sum to n^*.

Candidate A is the *Observed Condorcet Winner* (*OCW*) based on the preference rankings of the participating voters if AM^*B and AM^*C, with:

$$n_1^* + n_2^* + n_4^* > n_3^* + n_5^* + n_6^* \quad [AM^*B] \tag{3}$$
$$n_1^* + n_2^* + n_3^* > n_4^* + n_5^* + n_6^* \quad [AM^*C]. \tag{4}$$

The *Observed Condorcet Loser* (*OCL*) is also defined in the obvious manner.

Gehrlein and Lepelley (2017a) perform an analysis to conclude that the likelihood that some very bad election outcomes might be observed is significantly increased when voter participation rates are low. For example, depending upon which voters choose to abstain in any particular case, the ACW and the OCW do not necessarily have to coincide, and this current study begins by considering the probability that the ACW and OCW will be the same candidate as a function of the voter participation rate.

This overall analysis is then significantly extended to determine just how serious the impact of abstentions might be on a number of other very negative election outcomes. The second phase of our study evaluates some commonly considered voting rules on the basis of their Condorcet Efficiency, which measures the conditional probability that each of these rules will elect the ACW given that one exists, based on the available results from observed voting situations after abstention takes place. The voting rules that we initially consider are the single-stage voting rules: Plurality Rule (*PR*), Negative Plurality Rule (*NPR*) and Borda Rule (*BR*). The potentially extreme negative impact of voter abstention is obvious in this case, since all voting rules effectively become random choosers for the winner of an election, with Condorcet Efficiencies

of 33.3%, when voter participation rates are near zero! The point of interest is how quickly voting rules recover to achieve acceptable Condorcet Efficiency values as voter participation rates increase from near-zero.

After the initial Condorcet Efficiency component of this analysis is completed, the probability that these voting rules will perform in a very poor manner by electing the ACL is considered as a function of the voter participation rate. We then proceed to extend this same overall evaluation to consider common two-stage voting rules: Plurality Elimination Rule (*PER*), Negative Plurality Elimination Rule (*NPER*) and Borda Elimination Rule (*BER*). As in the case of Condorcet Efficiency values, the likelihood that voting rules will select the ACL also approaches 33.3% for near-zero voter participation rates. We limit our attention that the number of possible voters is very large as $n \to \infty$, and the overall results of our analysis typically present a somewhat pessimistic outlook for the performance of voting rules for scenarios in which voter participation rates are as low as those that can be observed in actual elections.

1 Probability of ACW and OCW Coincidence

Candidate *A* will be both the ACW and OCW whenever the actual and observed voting situations have preference rankings for voters that are simultaneously consistent with (1), (2), (3) and (4). The coincidence probability for the ACW and OCW is obviously driven both by the probability that various actual voting situations are observed and by the mechanism that determines the subset of possible voters who choose to participate in the election. Two standard assumptions from the literature are used as a basis for models to consider these two components, and each will be seen to have its own interpretation of how the voter participation rate α_{VR} is defined. The first of these models is $IC(\alpha_{VR})$ which is based on the *Impartial Culture Condition* (*IC*) that assumes complete independence between voters' preferences. The second model is $IAC(\alpha_{VR})$ which is based on the *Impartial Anonymous Culture Condition* (*IAC*) that inherently assumes some degree of dependence among voters' preferences. Probability representations for the coincidence of the ACW and OCW will be considered in turn with these two models.

1.1 ACW and OCW Coincidence Results with IC

When $IC(\alpha_{VR})$ is considered, we focus on the probability that a randomly selected voter from the set of possible voters will have each of the associated linear preference rankings in Figure 1, where p_i is the probability that the associated i^{th} raking will be observed on a random selection with $\sum_{i=1}^{6} p_i = 1$. The basis of the IC requires that $p_i = \frac{1}{6}$ for all $i = 1,2,...,6$, so that each possible voter is equally likely to have any of the six linear rankings. Then, Gehrlein and Fishburn (1978) developed an extension of IC, such that $IC(\alpha_{VR})$ further assumes that each possible voter will independently have a probability α_{VR} of participating in the election.

The basic IC assumption was used by Guilbaud (1952) to develop a representation for the limiting probability $P_{ACW}(IC, \infty)$ that an ACW exists to begin with as $n \to \infty$, and

$$P_{ACW}(IC, \infty) = \frac{3}{4} + \frac{3}{2\pi} Sin^{-1}\left(\frac{1}{3}\right) \approx .91226. \tag{5}$$

The development of a representation for the limiting probability that the ACW and OCW coincide begins with the definitions of four variables $\{X_1, X_2, X_3, X_4\}$ that have different values that are based on the linear preference ranking that is associated with a randomly selected voter and on whether, or not, that voter participates in the election. These variables are defined in Table 1.

Table 1. Definitions of X_1, X_2, X_3 and X_4.

Ranking	$X_1\,[AMB]$	$X_2\,[AMC]$	$X_3\,[AM^*B]$	$X_4\,[AM^*C]$
$A \succ B \succ C$ (p_1)	+1	+1	+1	+1
$A \succ C \succ B$ (p_2)	+1	+1	+1	+1
$B \succ A \succ C$ (p_3)	−1	+1	−1	+1
$C \succ A \succ B$ (p_4)	+1	−1	+1	−1
$B \succ C \succ A$ (p_5)	−1	−1	−1	−1
$C \succ B \succ A$ (p_6)	−1	−1	−1	−1
Abstention	---	---	0	0

The entries from Table 1 show that the value of X_1 for the linear preference ranking of a randomly selected voter is +1 whenever $A \succ B$ and it is -1 if $B \succ A$, so that AMB for n voters if the average value of X_1 has $\bar{X}_1 > 0$. In the same fashion AMC if $\bar{X}_2 > 0$, so Candidate A is the ACW whenever both $\bar{X}_1 > 0$ and $\bar{X}_2 > 0$. It then follows from the same logic that A will be the OCW for participating voters when both $\bar{X}_3 > 0$ and $\bar{X}_4 > 0$.

The expected values of these four variables with $IC(\alpha_{VR})$ are given by:

$$E(X_1) = +1p_1 + 1p_2 - 1p_3 + 1p_4 - 1p_5 - 1p_6$$

$$E(X_2) = +1p_1 + 1p_2 + 1p_3 - 1p_4 - 1p_5 - 1p_6$$

$$E(X_3) = +1p_1\alpha_{VR} + 1p_2\alpha_{VR} - 1p_3\alpha_{VR} + 1p_4\alpha_{VR} - 1p_5\alpha_{VR} - 1p_6\alpha_{VR} + 0(1 - \alpha_{VR})$$

$$E(X_4) = +1p_1\alpha_{VR} + 1p_2\alpha_{VR} + 1p_3\alpha_{VR} - 1p_4\alpha_{VR} - 1p_5\alpha_{VR} - 1p_6\alpha_{VR} + 0(1 - \alpha_{VR}).$$

With the restriction that $p_i = \frac{1}{6}$ for all $i = 1,2,\ldots,6$; $E(X_i) = 0$ for $i = \{1,2,3,4\}$, so that all $E(\bar{X}_i) = 0$ also. The probability that Candidate A is both the ACW and OCW is therefore given by the joint probability that $\bar{X}_i > E(\bar{X}_i)$, or $\bar{X}_i\sqrt{n} > E(\bar{X}_i\sqrt{n})$, for $i = \{1,2,3,4\}$. The Central Limit Theorem requires that the distribution of the $\bar{X}_i\sqrt{n}$ variables is multivariate normal as $n \rightarrow \infty$, where the correlation matrix for the $\bar{X}_i\sqrt{n}$ terms is obtained directly from the correlations between the original X_i variables.

The variance and covariance terms of the X_i variables in Table 1 with $IC(\alpha_{VR})$ are given by:

$$E(X_1^2) = E(X_2^2) = 1 \text{ and } E(X_3^2) = E(X_4^2) = \alpha_{VR}$$

$$E(X_1X_2) = (+1)(+1)p_1 + (+1)(+1)p_2 + (-1)(+1)p_3 + (+1)(-1)p_4 + (-1)(-1)p_5 + (-1)(-1)p_6 = \frac{1}{3}$$

$$E(X_1X_3) = (+1)(+1)p_1\alpha_{VR} + (+1)(+1)p_2\alpha_{VR} + (-1)(-1)p_3\alpha_{VR} + (+1)(+1)p_4\alpha_{VR} + (-1)(-1)p_5\alpha_{VR} + (-1)(-1)p_6\alpha_{VR} + 0(1-\alpha_{VR}) = \alpha_{VR}$$

$$E(X_1X_4) = (+1)(+1)p_1\alpha_{VR} + (+1)(+1)p_2\alpha_{VR} + (-1)(+1)p_3\alpha_{VR} + (+1)(-1)p_4\alpha_{VR} + (-1)(-1)p_5\alpha_{VR} + (-1)(-1)p_6\alpha_{VR} + 0(1-\alpha_{VR}) = \frac{\alpha_{VR}}{3}$$

$$E(X_2X_3) = (+1)(+1)p_1\alpha_{VR} + (+1)(+1)p_2\alpha_{VR} + (+1)(-1)p_3\alpha_{VR} + (-1)(+1)p_4\alpha_{VR} + (-1)(-1)p_5\alpha_{VR} + (-1)(-1)p_6\alpha_{VR} + 0(1-\alpha_{VR}) = \frac{\alpha_{VR}}{3}$$

$$E(X_2X_4) = (+1)(+1)p_1\alpha_{VR} + (+1)(+1)p_2\alpha_{VR} + (+1)(+1)p_3\alpha_{VR} + (-1)(-1)p_4\alpha_{VR} + (-1)(-1)p_5\alpha_{VR} + (-1)(-1)p_6\alpha_{VR} + 0(1-\alpha_{VR}) = \alpha_{VR}$$

$$E(X_3X_4) = (+1)(+1)p_1\alpha_{VR} + (+1)(+1)p_2\alpha_{VR} + (-1)(+1)p_3\alpha_{VR} + (+1)(-1)p_4\alpha_{VR} + (-1)(-1)p_5\alpha_{VR} + (-1)(-1)p_6\alpha_{VR} + 0(1-\alpha_{VR}) = \frac{\alpha_{VR}}{3}.$$

The resulting correlation matrix for these four variables is then given by $\boldsymbol{R_1}$, with:

$$\boldsymbol{R_1} = \begin{bmatrix} 1 & \frac{1}{3} & \sqrt{\alpha_{VR}} & \frac{\sqrt{\alpha_{VR}}}{3} \\ & 1 & \frac{\sqrt{\alpha_{VR}}}{3} & \sqrt{\alpha_{VR}} \\ & & 1 & \frac{1}{3} \\ & & & 1 \end{bmatrix}.$$

The probability that any $\bar{X}_i\sqrt{n}$ takes on a specific value, including $E(\bar{X}_i\sqrt{n})$, is zero in any continuous distribution. So, the limiting probability that Candidate A is both the ACW and OCW is given by the multivariate normal positive orthant probability $\Phi_4(\boldsymbol{R_1})$ that $\bar{X}_i\sqrt{n} \geq E(\bar{X}_i\sqrt{n})$, for $i = \{1,2,3,4\}$. The symmetry of $IC(\alpha_{VR})$ with respect to the three candidates leads to the conclusion that the conditional limiting probability $P_{ACW}^{OCW}(IC(\alpha_{VR}),\infty)$ that the ACW and OCW coincide, given that an ACW exists, is then directly obtained from

$$P_{ACW}^{OCW}(IC(\alpha_{VR}),\infty) = \frac{3\Phi_4(\boldsymbol{R_1})}{P_{ACW}(IC,\infty)}. \tag{6}$$

Closed form representations for these positive orthant probabilities only exist for special cases, and $\boldsymbol{R_1}$ is one such case from Cheng (1969). Using that result with (5) and (6) leads to

$$P_{ACW}^{OCW}(IC(\alpha_{VR}),\infty) = \frac{\begin{bmatrix} \frac{3}{16} + \frac{3}{4\pi}\left\{Sin^{-1}\left(\frac{1}{3}\right) + Sin^{-1}\left(\sqrt{\alpha_{VR}}\right) + Sin^{-1}\left(\frac{\sqrt{\alpha_{VR}}}{3}\right)\right\} \\ + \frac{3}{4\pi^2}\left\{\left(Sin^{-1}\left(\frac{1}{3}\right)\right)^2 + \left(Sin^{-1}\left(\sqrt{\alpha_{VR}}\right)\right)^2 - \left(Sin^{-1}\left(\frac{\sqrt{\alpha_{VR}}}{3}\right)\right)^2\right\} \end{bmatrix}}{\frac{3}{4} + \frac{3}{2\pi}Sin^{-1}\left(\frac{1}{3}\right)}. \tag{7}$$

This representation corresponds to the result in Gehrlein and Fishburn (1978), and the derivation has been developed in detail here as an example, since the same basic procedure will be used later to develop additional new probability representations for other election outcomes.

There is no election output to evaluate without any voter participation when $\alpha_{VR} = 0$, so Table 2 lists values of $P_{ACW}^{OCW}(IC(\alpha_{VR}), \infty)$ from (7) for $\alpha_{VR} \to 0$ and for each $\alpha_{VR} = .1(.1)1.0$.

Table 2. Limiting ACW and OCW Coincidence Probabilities.

α_{VR}	$IC(\alpha_{VR})$	$IAC(\alpha_{VR})$
≈ 0	.3041	$5/16 = .3125$
.1	.4236	.3482
.2	.4806	.3949
.3	.5290	.4564
.4	.5742	.5357
.5	.6186	.6310
.6	.6640	.7291
.7	.7126	.8160
.8	.7675	.8887
.9	.8365	.9492
1.0	1.0000	1.0000

Some results from the $P_{ACW}^{OCW}(IC(\alpha_{VR}), \infty)$ values in Table 2 are predictable. In particular, the coincidence probability is one when all voters participate with $\alpha_{VR} = 1$, and the coincidence probability decreases as this participation rate declines. When almost all voters independently abstain as $\alpha_{VR} \to 0$, the candidate that becomes the OCW is effectively selected at random if there is an OCW, so the coincidence probability goes to $\frac{1}{3}P_{ACW}(IC, \infty)$. What is surprising is the very steep rate of decline in $P_{ACW}^{OCW}(IC(\alpha_{VR}), \infty)$ values as α_{VR} decreases, with only about a 62% chance of coincidence when $\alpha_{VR} = .5$. Low voter participation rates can clearly have a huge negative impact on election outcomes with the independent voter $IC(\alpha_{VR})$ scenario.

1.2 ACW and OCW Coincidence Results with IAC

It is well known that the introduction of a degree of dependence among voters' preferences generally reduces the probability that most paradoxical election outcomes will be observed, relative to the case of complete independence with IC [see for example Gehrlein and Lepelley (2011)]. Gehrlein and Lepelley (2017b) investigated this impact on the conditional probability for AOW and OCW coincidence with the assumption of $IAC(\alpha_{VR})$, such that all actual and observed voting situations are equally likely to be observed, given that the voter participation rate is fixed at $\alpha_{VR} = \frac{n^*}{n}$.

To begin developing this representation for the coincidence probability of ACW and OCW with $IAC(\alpha_{VR})$, we note that this requires that (1), (2), (3) and (4) must hold, along with

$$n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \tag{8}$$
$$n_1^* + n_2^* + n_3^* + n_4^* + n_5^* + n_6^* = n^* \tag{9}$$
$$0 \le n_i^* \le n_i, \text{ for } i = 1, 2, 3, 4, 5, 6 \tag{10}$$

We would start by obtaining a representation for the total number of voting situations for which the restrictions (1), (2), (3), (4), (8), (9) and (10) simultaneously apply as a function of n and n^*,

and then divide that by the total number of voting situations for which (1), (2), (8), (9) and (10) simultaneously apply as a function of n and n^*. The final representation could then be expressed as a function of n and α_{VR}.

In the limit as $n \to \infty$, this process of developing representations to count the number of voting situations that meet the specified restrictions reduces to computing volumes of subspaces. All observed voting situations with the same value of α_{VR} are then assumed to be equally likely to be observed with $IAC(\alpha_{VR})$, but it is *not* assumed that all α_{VR} are equally likely to be observed. This general procedure has been used many times to develop limiting IAC-based probability representations in the literature, and we rely throughout this study on a particular method for doing this that uses the multi-parameter version of Barvinok's Algorithm that is described in detail in Lepelley et al. (2008). The resulting representation for the limiting conditional probability that the ACW and OCW coincide with $IAC(\alpha_{VR})$ is denoted $P_{OCW}^{ACW}(IAC(\alpha_{VR}), \infty)$, and it is given by:

$$P_{OCW}^{ACW}(IAC(\alpha_{VR}), \infty) = \frac{444\alpha_{VR}^5 - 4376\alpha_{VR}^4 + 11817\alpha_{VR}^3 - 15576\alpha_{VR}^2 + 10080\alpha_{VR} - 2520}{128(42\alpha_{VR}^5 - 274\alpha_{VR}^4 + 603\alpha_{VR}^3 - 624\alpha_{VR}^2 + 315\alpha_{VR} - 63)}, \quad 0 \le \alpha_{VR} \le \frac{1}{2} \quad (11)$$

$$= \frac{10812\alpha_{VR}^5 - 364\alpha_{VR}^4 - 3947\alpha_{VR}^3 + 1761\alpha_{VR}^2 - 185\alpha_{VR} - 13}{128(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \quad \frac{1}{2} \le \alpha_{VR} \le 1.$$

Computed values of $P_{OCW}^{ACW}(IAC(\alpha_{VR}), \infty)$ from (11) are listed in Table 2 for $\alpha_{VR} \to 0$ and for each $\alpha_{VR} = .1(.1)1.0$, to show some very surprising results. The introduction of a degree of dependence with $IAC(\alpha_{VR})$ can indeed increase the probability of ACW and OCW coincidence relative to the case of complete voter independence with $IC(\alpha_{VR})$. But, this only occurs for $\alpha_{VR} \ge .5$, and the coincidence probability is actually reduced by introducing dependence for most cases with $\alpha_{VR} \le .5$! When a large proportion of voters choose not to participate in an election, it is now quite evident that very bad things really can happen with the resulting election outcomes and adding a degree of dependence can make things even worse, so we proceed to investigate just how bad things can get in such cases.

2 Actual Condorcet Efficiency of Single-Stage Voting Rules

As mentioned above, the Condorcet Efficiency of a voting rule measures the conditional probability that the voting rule will elect the ACW, based on the results of an observed voting situation. A single-stage voting rule determines the winner of an election in a single step from the information on the voters' ballots. A weighted scoring rule $WSR(\lambda)$ requires voters to rank the three candidates and a weight of one is assigned to each voter's most preferred candidate, a weight of zero to the least preferred candidate and a weight of λ to the middle-ranked candidate. The candidate who receives the greatest total score from the observed voting situation is then declared as the winner. Such a voting rule reduces to PR when $\lambda = 0$, so that voters must only report their most preferred candidate on a ballot. When $\lambda = 1$, voters must only report their two more-preferred candidates, which defines NPR since it is the same as having each voter cast a ballot against their least-preferred candidate. Voters obviously do not really have to report a ranking on the candidates when either PR or NPR is employed, but they must do so for BR which uses $\lambda = 1/2$.

2.1 Actual Single-Stage Rule Efficiency with IC

The development of a representation for the Condorcet Efficiency $CE_{WSR(\lambda)}(IC(\alpha_{VR}), \infty)$ of $WSR(\lambda)$ as $n \to \infty$ with the assumption of $IC(\alpha_{VR})$ begins by defining four variables that are based on the likelihood that given voter preference rankings are observed, as shown in Table 3.

Table 3. Definitions of Y_1, Y_2, Y_3 and Y_4.

Ranking	$Y_1 [AMB]$	$Y_2[AMC]$	$Y_3[AWB]$	$Y_4[AWC]$
$A > B > C\ (p_1)$	+1	+1	$1 - \lambda$	+1
$A > C > B\ (p_2)$	+1	+1	+1	$1 - \lambda$
$B > A > C\ (p_3)$	−1	+1	$\lambda - 1$	$+\lambda$
$C > A > B\ (p_4)$	+1	−1	$+\lambda$	$\lambda - 1$
$B > C > A\ (p_5)$	−1	−1	−1	$-\lambda$
$C > B > A\ (p_6)$	−1	−1	$-\lambda$	−1
Abstention	---	---	0	0

Following the logic of development of the representation for $P_{ACW}^{OCW}(IC(\alpha_{VR}), \infty)$, Y_1 and Y_2 are identical to X_1 and X_2 respectively, so that Candidate A is the ACW if $\overline{Y_1} > 0$ and $\overline{Y_2} > 0$. Variables Y_3 and Y_4 define the difference in scores obtained by Candidate A over each of B and C respectively in the possible voter preference rankings, so that A will be the winner by $WSR(\lambda)$ in an observed voting situation whenever both $\overline{Y_3} > 0$ and $\overline{Y_4} > 0$.

By utilizing the methodology that led to (6), it follows that

$$CE_{WSR(\lambda)}(IC(\alpha_{VR}), \infty) = \frac{3\Phi_4(\boldsymbol{R_2})}{P_{ACW}(IC, \infty)}. \tag{12}$$

Based on the representation in (12), computations with the Y_i definitions from Table 3 lead to:

$$\boldsymbol{R_2} = \begin{bmatrix} 1 & \frac{1}{3} & \sqrt{\frac{2}{3z}} & \sqrt{\frac{1}{6z}} \\ & 1 & \sqrt{\frac{1}{6z}} & \sqrt{\frac{2}{3z}} \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{bmatrix}, \text{ with } z = \frac{1 - \lambda(1 - \lambda)}{\alpha_{VR}}.$$

Since z is symmetric about $\lambda = 1/2$ for all α_{VR}, it follows that PR and NPR must have the same limiting Condorcet Efficiency as $n \to \infty$ for any given α_{VR}. The form of $\boldsymbol{R_2}$ unfortunately does not meet the conditions for any special case with a closed form representation for $\Phi_4(\boldsymbol{R_2})$, as we observed above for $\Phi_4(\boldsymbol{R_1})$ in the development of (7). Gehrlein and Fishburn (1979) therefore used a representation from Gehrlein (1979) to obtain values of $\Phi_4(\boldsymbol{R_2})$ by using numerical integration over a single variable. Resulting values of $CE_{WSR(\lambda)}(IC(\alpha_{VR}), \infty)$ for PR, NPR and BR from (12) are listed in Table 4 for the case that $\alpha_{VR} \to 0$ and for each $\alpha_{VR} = .1(.1)1.0$.

Table 4. Computed Limiting Values for Condorcet Efficiency with $IC(\alpha_{VR})$.

α_{VR}	Voting Rule	
	PR & NPR	BR
≈ 0	1/3 = .3333	1/3 = .3333
.1	.4399	.4576
.2	.4882	.5151
.3	.5277	.5630
.4	.5630	.6068
.5	.5961	.6490
.6	.6280	.6911
.7	.6595	.7345
.8	.6911	.7810
.9	.7234	.8337
1.0	.7572	.9012

We see the expected result that Condorcet Efficiency values approach 1/3 to make all voting rules equivalent to random selection procedures when $\alpha_{VR} \to 0$. We also note that BR dominates PR and NPR for all non-zero participation rates. The efficiencies decrease rapidly as voter participation rates decrease, and when this rate is 40% or less, all voting rules have Condorcet Efficiency less than 61%. This level of efficiency is quite disappointing, since voter participation rates as low as 40% are definitely observed in practice. For example, voter participation rates in the 2014 US elections [see McDonald (2018) or other sources] range from 28.7% in Indiana to 58.7% in Maine, with an overall national participation rate of only 36.7%. While these participation rates do increase for elections during years when a president is being chosen, they are even lower during the primary elections to select final candidates. So, we certainly hope that the introduction of a degree of dependence among voters' preferences will improve this situation.

2.2 Actual Single-Stage Rule Efficiency with IAC

The impact that the inclusion of some degree of dependence among voters' preferences might have on this dreary expected result is considered by using the assumption of $IAC(\alpha_{VR})$ to obtain representations for $CE_{Rule}(IAC(\alpha_{VR}), \infty)$. It is not feasible to obtain such a representation for a generalized $WSR(\lambda)$ with a reasonable effort, as we did with $CE_{WSR(\lambda)}(IC(\alpha_{VR}), \infty)$, but we can obtain a representation for each specified voting rule $Rule \in \{PR, NPR, BR\}$.

Based on the definitions of PR, NPR and BR that were given above, the following restrictions on observed voting situations apply:

Candidate *A* is the PR winner if

$$n_1^* + n_2^* > n_3^* + n_5^* \ [APB] \tag{13}$$
$$n_1^* + n_2^* > n_4^* + n_6^* \ [APC]. \tag{14}$$

Candidate *A* is the NPR winner if

$$n_5^* + n_6^* < n_2^* + n_4^* \ [ANB] \tag{15}$$
$$n_5^* + n_6^* < n_1^* + n_3^* \ [ANC]. \tag{16}$$

Candidate A is the BR winner if

$$2(n_1^* + n_2^*) + n_3^* + n_4^* > 2(n_3^* + n_5^*) + n_1^* + n_6^* \quad [ABB] \tag{17}$$
$$2(n_1^* + n_2^*) + n_3^* + n_4^* > 2(n_4^* + n_6^*) + n_2^* + n_5^* \quad [ABC]. \tag{18}$$

The same procedure that was used to obtain the limiting representation for $P_{OCW}^{ACW}(IAC(\alpha_{VR}), \infty)$ in (11) from the restrictions in (1), (2), (3), (4), (8), (9) and (10) is used to obtain representations for each $CE_{Rule}(IAC(\alpha_{VR}), \infty)$. This is done by replacing (3) and (4) with (13) and (14) for PR to obtain:

$$CE_{PR}(IAC(\alpha_{VR}), \infty) = \tag{19}$$

$$\frac{1517527\alpha_{VR}^5 - 13088244\alpha_{VR}^4 + 33868125\alpha_{VR}^3 - 43056360\alpha_{VR}^2 + 27051003\alpha_{VR} - 6613488}{31492828(42\alpha_{VR}^5 - 274\alpha_{VR}^4 + 603\alpha_{VR}^3 - 624\alpha_{VR}^2 + 315\alpha_{VR} - 63)}, 0 \le \alpha_{VR} \le \frac{1}{2}$$

$$= \frac{\begin{bmatrix} 18667651456\alpha_{VR}^{10} - 94540776960\alpha_{VR}^9 + 187428504960\alpha_{VR}^8 - 169349253120\alpha_{VR}^7 \\ +37682184960\alpha_{VR}^6 + 57987062784\alpha_{VR}^5 - 55553299200\alpha_{VR}^4 + 21238744320\alpha_{VR}^3 \\ -3741344640\alpha_{VR}^2 + 158644980\alpha_{VR} + 21907179 \end{bmatrix}}{201553920(\alpha_{VR}-1)^5(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \frac{1}{2} \le \alpha_{VR} \le \frac{3}{4}$$

$$= \frac{-(13661\alpha_{VR}^5 - 72995\alpha_{VR}^4 + 68740\alpha_{VR}^3 - 26460\alpha_{VR}^2 + 3735\alpha_{VR} - 9)}{128(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \frac{3}{4} \le \alpha_{VR} \le 1.$$

A limiting representation is obtained for the Condorcet Efficiency of NPR by replacing (3) and (4) with (15) and (16):

$$CE_{NPR}(IAC(\alpha_{VR}), \infty) = \tag{20}$$

$$\frac{-(33037\alpha_{VR}^5 + 3096510\alpha_{VR}^4 - 10408905\alpha_{VR}^3 + 14029200\alpha_{VR}^2 - 8726130\alpha_{VR} + 2066715)}{98415(42\alpha_{VR}^5 - 274\alpha_{VR}^4 + 603\alpha_{VR}^3 - 624\alpha_{VR}^2 + 315\alpha_{VR} - 63)}, 0 \le \alpha_{VR} \le \frac{1}{2}$$

$$= \frac{\begin{bmatrix} 677320000\alpha_{VR}^{10} - 5117808000\alpha_{VR}^9 + 16793438400\alpha_{VR}^8 - 31338161280\alpha_{VR}^7 \\ +36609330240\alpha_{VR}^6 - 27803103552\alpha_{VR}^5 + 13822189920\alpha_{VR}^4 - 4419227160\alpha_{VR}^3 \\ +862705890\alpha_{VR}^2 - 90246555\alpha_{VR} + 3562623 \end{bmatrix}}{6298560(1-\alpha_{VR})^5(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \frac{1}{2} \le \alpha_{VR} \le \frac{3}{4}$$

$$= \frac{459\alpha_{VR}^5 + 365\alpha_{VR}^4 - 400\alpha_{VR}^3 + 270\alpha_{VR}^2 - 120\alpha_{VR} + 21}{15(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \frac{3}{4} \le \alpha_{VR} \le 1.$$

A limiting representation for the Condorcet Efficiency of BR is obtained by replacing (3) and (4) with (17) and (18):

$$CE_{BR}(IAC(\alpha_{VR}), \infty) = \tag{21}$$

$$\frac{931991\alpha_{VR}^5 - 11823930\alpha_{VR}^4 + 33929685\alpha_{VR}^3 - 45596520\alpha_{VR}^2 + 29546370\alpha_{VR} - 7348320}{349920(42\alpha_{VR}^5 - 274\alpha_{VR}^4 + 603\alpha_{VR}^3 - 624\alpha_{VR}^2 + 315\alpha_{VR} - 63)}, 0 \le \alpha_{VR} \le \frac{1}{2}$$

$$= \frac{-\begin{bmatrix} 971901664\alpha_{VR}^{10} - 4826652480\alpha_{VR}^9 + 9189897120\alpha_{VR}^8 - 7389999360\alpha_{VR}^7 \\ +1632960\alpha_{VR}^6 + 4616051328\alpha_{VR}^5 - 3672527040\alpha_{VR}^4 + 1323475200\alpha_{VR}^3 \\ -223482240\alpha_{VR}^2 + 8281440\alpha_{VR} + 1422603 \end{bmatrix}}{11197440(1-\alpha_{VR})^5(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \frac{1}{2} \le \alpha_{VR} \le \frac{3}{4}$$

$$= \frac{-(187353\alpha_{VR}^5 - 987625\alpha_{VR}^4 + 527980\alpha_{VR}^3 + 221880\alpha_{VR}^2 - 253805\alpha_{VR} + 56249)}{4320(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \frac{3}{4} \le \alpha_{VR} \le 1.$$

The representations in (19), (20) and (21) are then used to obtain the $CE_{Rule}(IAC(\alpha_{VR}), \infty)$ values with each $Rule$ = PR, NPR and BR for values of $\alpha_{VR} \to 0$ and for each $\alpha_{VR} = .1(.1)1$. The results are listed in Table 5.

Table 5. Computed Limiting Values for Condorcet Efficiency with $IAC(\alpha_{VR})$.

α_{VR}	Voting Rule		
	PR	NPR	BR
≈ 0	1/3 = .3333	1/3 = .3333	1/3 = .3333
.1	.3679	.3621	.3704
.2	.4131	.3975	.4184
.3	.4723	.4411	.4807
.4	.5481	.4928	.5600
.5	.6378	.5475	.6533
.6	.7267	.5912	.7457
.7	.7988	.6153	.8212
.8	.8476	.6255	.8737
.9	.8738	.6289	.9026
1.0	119/135 = 0.8815	41/45 = .6296	17/27 = .9111

The results in Table 5 show that the addition of dependence improved efficiency results for PR and BR with voter participation of 50% or more. However, the addition of dependence had a negative impact on NPR efficiency at all non-zero levels of participation. The overall efficiency results for $IAC(\alpha_{VR})$ in Table 5 are even more discouraging with lower participation rates for all voting rules than the results for $IC(\alpha_{VR})$ from Table 4. BR still outperforms PR and NPR for all non-zero participation rates, but the addition of a degree of dependence among voters' preferences has led to even lower efficiencies for the voting rules when α_{VR} is small. The efficiencies of all voting rules are now less than 56% when participation rates are 40% or less. Adding dependence among voters' preferences has made things worse when participation rates are less than 50%!

3 Borda Paradox Probabilities

The occurrence of a *Borda Paradox* is much more disconcerting than having a voting rule fail to elect the ACW. The worst possible outcome occurs when a Borda Paradox is observed, such that a voting rule elects the ACL. Given the potentially significant negative impact that voter abstentions have been seen to have on the Condorcet Efficiency of voting rules, it is natural to wonder how serious the impact of voter abstentions might be for observing examples of this even more dramatic Borda Paradox.

3.1 Borda Paradox Probabilities for Single-Stage Rules with IC

It is easy to develop a representation for the limiting probability $BP_{WSR(\lambda)}(IC(\alpha_{VR}), \infty)$ that a Borda Paradox is observed when using $WSR(\lambda)$ as $n \to \infty$ with the assumption of $IC(\alpha_{VR})$ by

mirroring the development of the representation for $CE_{WSR(\lambda)}(IC(\alpha_{VR}), \infty)$ in (12). All that has to be done is to reverse the signs of the Y_1 and Y_2 variable values in Table 3 to make Candidate A the ACL and the winner by $WSR(\lambda)$. It obviously follows from the logic that led to (12) and the definition of $\boldsymbol{R_2}$ that

$$BP_{WSR(\lambda)}(IC(\alpha_{VR}), \infty) = \frac{3\Phi_4(\boldsymbol{R_3})}{P_{ACW}(IC, \infty)}, \tag{22}$$

where

$$\boldsymbol{R_3} = \begin{bmatrix} 1 & \frac{1}{3} & -\sqrt{\frac{2}{3z}} & -\sqrt{\frac{1}{6z}} \\ & 1 & -\sqrt{\frac{1}{6z}} & -\sqrt{\frac{2}{3z}} \\ & & 1 & \frac{1}{2} \\ & & & 1 \end{bmatrix}, \text{ with } z = \frac{1-\lambda(1-\lambda)}{\alpha_{VR}}.$$

Since z remains symmetric about $\lambda = 1/2$ for all α_{VR}, it follows as in the case of Condorcet Efficiencies, that PR and NPR have the same limiting probability of exhibiting a Borda Paradox for any specified value of α_{VR}. The form of $\boldsymbol{R_3}$ is not a special case for which $\Phi_4(\boldsymbol{R_3})$ has a closed form representation, so the representation from Gehrlein (1979) is used to obtain values of $\Phi_4(\boldsymbol{R_3})$. Computed $BP_{WSR(\lambda)}(IC(\alpha_{VR}), \infty)$ values from (22) are listed in Table 6 for $\alpha_{VR} \to 0$ and for each $\alpha_{VR} = .1(.1)1.0$

Table 6. Computed Limiting Borda Paradox Probabilities with $IC(\alpha_{VR})$.

	Voting Rule	
α_{VR}	PR & NPR	BR
≈ 0	1/3 = .3333	1/3 = .3333
.1	.2355	.2208
.2	.1965	.1761
.3	.1670	.1423
.4	.1423	.1140
.5	.1207	.0892
.6	.1012	.0669
.7	.0834	.0466
.8	.0669	.0280
.9	.0515	.0115
1.0	.0371	0

The expected result in Table 6 is that the Borda Paradox Probabilities approach 1/3 as $\alpha_{VR} \to 0$, since all voting rules are equivalent to random selection procedures in this particular case, and that BR has a lower Borda Paradox probability than PR and NPR for all non-zero participation rates. A well-known result is that BR cannot elect the OCL [Fishburn and Gehrlein (1976)], so that the ACL cannot be elected by BR if all voters participate. However, this result is not true when some voters abstain so that the OCL and ACL do not necessarily coincide. The probability of observing a Borda Paradox consistently increases as voter participation rates decrease, and

when this rate is 40% or less, all voting rules have a Borda Paradox probability of greater than 11%! That is a shockingly large probability for such a very negative outcome, given that voter participation rates as low as 40% are actually observed in practice.

3.2 Borda Paradox Probabilities for Single-Stage Rules with IAC

We consider the impact that the introduction of some degree of dependence among voters' preferences will have on the probability that Borda's Paradox will be observed by extending this analysis with the assumption of $IAC(\alpha_{VR})$. This begins by noting that Candidate A is the ACL whenever:

$$n_3 + n_5 + n_6 > n_1 + n_2 + n_4 \quad [BMA] \tag{23}$$
$$n_4 + n_5 + n_6 > n_1 + n_2 + n_3 \quad [CMA]. \tag{24}$$

The representation for $CE_{PR}(IAC(\alpha_{VR}), \infty)$ in (19) was obtained by considering actual and observed voting situations for which (1), (2), (13), (14), (8), (9) and (10) held simultaneously. Here, (1) and (2) respectively required that AMB and AMC to make Candidate A the ACW. This process is then repeated by replacing (1) and (2) with (23) and (24) to obtain a representation for $BP_{PR}(IAC(\alpha_{VR}), \infty)$, with

$$BP_{PR}(IAC(\alpha_{VR}), \infty) = \tag{25}$$

$$\frac{58215686\alpha_{VR}^5 - 284362350\alpha_{VR}^4 + 519458670\alpha_{VR}^3 - 455939280\alpha_{VR}^2 + 195419385\alpha_{VR} - 33067440}{1574640(42\alpha_{VR}^5 - 274\alpha_{VR}^4 + 603\alpha_{VR}^3 - 624\alpha_{VR}^2 + 315\alpha_{VR} - 63)},$$
$$0 \leq \alpha_{VR} \leq \tfrac{1}{2}.$$

$$\frac{\left[\begin{array}{l}1687165696\alpha_{VR}^{10} - 10095594240\alpha_{VR}^9 + 23635307520\alpha_{VR}^8 - 22382853120\alpha_{VR}^7 - 8582837760\alpha_{VR}^6 + \\ 45924060672\alpha_{VR}^5 - 53966062080\alpha_{VR}^4 + 33709893120\alpha_{VR}^3 - 12060167760\alpha_{VR}^2 + 2317476420\alpha_{VR} - 186417693\end{array}\right]}{201553920(\alpha_{VR}-1)^5 (42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR}+1)},$$
$$\tfrac{1}{2} \leq \alpha_{VR} \leq \tfrac{3}{4}$$

$$\frac{-\left(1989\alpha_{VR}^5 - 9340\alpha_{VR}^4 + 14390\alpha_{VR}^3 - 9900\alpha_{VR}^2 + 2925\alpha_{VR} - 288\right)}{120(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR}+1)}, \quad \tfrac{3}{4} \leq \alpha_{VR} \leq 1.$$

This process is then repeated for both NPR and BR, and the resulting limiting representations for observing Borda's Paradox are given by:

$$BP_{NPR}(IAC(\alpha_{VR}), \infty) = \tag{26}$$

$$\frac{2571059\alpha_{VR}^5 - 14546670\alpha_{VR}^4 + 28993545\alpha_{VR}^3 - 26888760\alpha_{VR}^2 + 11941020\alpha_{VR} - 2066715}{98415(42\alpha_{VR}^5 - 274\alpha_{VR}^4 + 603\alpha_{VR}^3 - 624\alpha_{VR}^2 + 315\alpha_{VR} - 63)}, \ 0 \leq \alpha_{VR} \leq \tfrac{1}{2}$$

$$\frac{\left[\begin{array}{l}198413056\alpha_{VR}^{10} - 1667619840\alpha_{VR}^9 + 6862786560\alpha_{VR}^8 - 17302947840\alpha_{VR}^7 + 28555571520\alpha_{VR}^6 \\ -31612472640\alpha_{VR}^5 + 23544017280\alpha_{VR}^4 - 11583051840\alpha_{VR}^3 + 3587226750\alpha_{VR}^2 - 629659170\alpha_{VR} + 47731275\end{array}\right]}{25194240(1-\alpha_{VR})^5(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR}+1)},$$
$$\text{for } \tfrac{1}{2} \leq \alpha_{VR} \leq \tfrac{3}{4}$$

$$\frac{340\alpha_{VR}^5 + 20\alpha_{VR}^4 - 1060\alpha_{VR}^3 + 1260\alpha_{VR}^2 - 525\alpha_{VR} + 84}{60\left(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR}+1\right)}, \text{ for } \tfrac{3}{4} \leq \alpha_{VR} \leq 1.$$

$$BP_{BR}(IAC(\alpha_{VR}), \infty) = \qquad (27)$$

$$\frac{2603797\alpha_{VR}^5 - 12826098\alpha_{VR}^4 + 23528961\alpha_{VR}^3 - 20631672\alpha_{VR}^2 + 8787366\alpha_{VR} - 1469664}{69984\left(42\alpha_{VR}^5 - 274\alpha_{VR}^4 + 603\alpha_{VR}^3 - 624\alpha_{VR}^2 + 315\alpha_{VR} - 63\right)}, 0 \le \alpha_{VR} \le \frac{1}{2}$$

$$\frac{\left[\begin{array}{c}37038752\alpha_{VR}^{10} - 252012096\alpha_{VR}^9 + 721014048\alpha_{VR}^8 - 1093927680\alpha_{VR}^7 + 868408128\alpha_{VR}^6 - \\ 191382912\alpha_{VR}^5 - 275970240\alpha_{VR}^4 + 284850432\alpha_{VR}^3 - 121223952\alpha_{VR}^2 + 25345872\alpha_{VR} - 2140425\end{array}\right]}{2239488(\alpha_{VR} - 1)^5\left(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1\right)}, \frac{1}{2} \le \alpha_{VR} \le \frac{3}{4}$$

$$\frac{(1 - \alpha_{VR})^3\left(99387\alpha_{VR}^2 - 94358\alpha_{VR} + 24131\right)}{864\left(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1\right)}, \frac{3}{4} \le \alpha_{VR} \le 1.$$

Associated values of $BP_{PR}(IAC(\alpha_{VR}), \infty)$ from (25), $BP_{NPR}(IAC(\alpha_{VR}), \infty)$ from (26) and $BP_{BR}(IAC(\alpha_{VR}), \infty)$ from (27) are listed in Table 7 for $\alpha_{VR} \to 0$ and for each $\alpha_{VR} = .1(.1)1.0$.

Table 7. Computed Limiting Borda Paradox Probabilities with $IAC(\alpha_{VR})$.

α_{VR}	Voting Rule		
	PR	NPR	BR
≈ 0	$1/3 = .3333$	$1/3 = .3333$	$1/3 = .3333$
.1	.3006	.3046	.2980
.2	.2626	.2691	.2564
.3	.2186	.2257	.2077
.4	.1693	.1745	.1526
.5	.1192	.1207	.0962
.6	.0778	.0773	.0497
.7	.0508	.0512	.0202
.8	.0366	.0383	.0055
.9	.0309	.0328	.0006
1.0	$4/135 = .0296$	$17/540 = .0315$	0

A comparison of the results in Tables 6 and 7 shows the same general outcome that was observed when the Condorcet Efficiency of voting rules was analyzed earlier with $IC(\alpha_{VR})$ and $IAC(\alpha_{VR})$. That is, the addition of a degree of dependence among voters' preferences with $IAC(\alpha_{VR})$ actually makes things worse for lower levels of voter participation. We now see that the probability of observing a Borda Paradox is greater than 15% for all levels of voter participation at 40% or less! The results for BR are uniformly the best of these options for all positive levels of voter participation, but BR still performs very poorly for lower levels of voter participation. If this very poor overall performance of single-stage voting rules at lower levels of voter participation is to be improved, the likely approach would be to consider the use of some other forms of voting rules.

4 Two-stage Voting Rules

Two-stage voting rules are sequential elimination procedures that take place in two steps. In the first stage, a voting rule is used to determine the candidate that is ranked as the worst of the three candidates. That candidate is eliminated from consideration in the second stage, where majority rule is used to determine the ultimate winner from the remaining two candidates. All that changes with the two-stage rules is the voting rule that is used to eliminate the loser in the first

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stage. We consider Plurality Elimination Rule (*PER*), Negative Plurality Elimination Rule (*NPER*) and Borda Elimination Rule (*BER*). It is definitely of interest to determine if these two-stage voting rules can exhibit better performance than the single-stage rules on the basis of Condorcet Efficiency and the probability that Borda's Paradox is observed with lower levels of voter participation rates.

4.1 Condorcet Efficiency of Two-stage Voting Rules with IC

The development of the representation for $CE_{WSR(\lambda)}(IC(\alpha_{VR}), \infty)$ in (12) was based on the definitions of the four Y_i variables in Table 3 had $\bar{Y}_i > 0$ for $i = 1,2,3,4$. When we consider instead the more complex two-stage elimination rules $WSRE(\lambda)$, the resulting development of a representation for $CE_{WSRE(\lambda)}(IC(\alpha_{VR}), \infty)$ requires the use of five variables that are denoted as Z_i for $i = 1,2,3,4,5$ in Table 8.

Table 8. Definitions of Z_1, Z_2, Z_3, Z_4 and Z_5.

Ranking	$Z_1 [AMB]$	$Z_2[AMC]$	$Z_3[AWB]$	$Z_4[CWB]$	$Z_5[AM^*C]$
$A \succ B \succ C\ (p_1)$	+1	+1	$1 - \lambda$	$-\lambda$	+1
$A \succ C \succ B\ (p_2)$	+1	+1	+1	λ	+1
$B \succ A \succ C\ (p_3)$	−1	+1	$\lambda - 1$	−1	+1
$C \succ A \succ B\ (p_4)$	+1	−1	λ	+1	−1
$B \succ C \succ A\ (p_5)$	−1	−1	−1	$\lambda - 1$	−1
$C \succ B \succ A\ (p_6)$	−1	−1	$-\lambda$	$1 - \lambda$	−1
Abstention	---	---	0	0	0

By comparing Table 8 to Table 3, we see that Z_1 and Z_2 are identical to Y_1 and Y_2, so that Candidate *A* will be the ACW if $\bar{Z}_1 > 0$ and $\bar{Z}_2 > 0$. It is also true that Z_3 is identical to Y_3, so that *A* beats *B* by $WSR(\lambda)$ if $\bar{Z}_3 > 0$. Variable Z_4 defines the relative margin by which *C* beats *B* in each ranking under $WSR(\lambda)$. So, *B* will be the lowest ranked candidate by $WSR(\lambda)$ when $\bar{Z}_3 > 0$ and $\bar{Z}_4 > 0$, and it therefore will be eliminated under $WSRE(\lambda)$. Variable Z_5 accounts for the fact that the ACW Candidate *A* will then be the majority rule winner over *C* for participating voters in the second stage of $WSRE(\lambda)$ if $\bar{Z}_5 > 0$. Candidate *A* will therefore be the ACW and the winner by $WSRE(\lambda)$ when $\bar{Z}_i > 0$ for $i = 1,2,3,4,5$.

The correlation matrix between these five variables is found to be given by $\boldsymbol{R_5}$:

$$\boldsymbol{R_5} = \begin{bmatrix} 1 & \frac{1}{3} & \sqrt{\frac{2}{3z}} & \sqrt{\frac{1}{6z}} & \frac{\sqrt{\alpha_{VR}}}{3} \\ & 1 & \sqrt{\frac{1}{6z}} & -\sqrt{\frac{1}{6z}} & \sqrt{\alpha_{VR}} \\ & & 1 & \frac{1}{2} & \sqrt{\frac{1}{6z\alpha_{VR}}} \\ & & & 1 & -\sqrt{\frac{1}{6z\alpha_{VR}}} \\ & & & & 1 \end{bmatrix}, \text{ with } z = \frac{1 - \lambda(1-\lambda)}{\alpha_{VR}}.$$

There are three candidates that could be the ACW and there are two remaining candidates that could be eliminated in the second stage of $WSRE(\lambda)$, so a representation for limiting Condorcet Efficiency is obtained from

$$CE_{WSRE(\lambda)}(IC(\alpha_{VR}), \infty) = \frac{6\Phi_5(R_5)}{P_{ACW}(IC, \infty)}. \tag{28}$$

As we have observed before, z is symmetric about $z = 1/2$, so the Condorcet Efficiency values of PER and NPER are identical for $IC(\alpha_{VR})$ for any given α_{VR}. The process of obtaining values of multivariate-normal positive orthant probabilities becomes much more complicated for the case of five variables, but they can still be obtained by using numerical techniques over a series of integrals on a single variable, as described in Gehrlein (2017). That procedure is used here to obtain values of $\Phi_5(R_5)$ for use in (28), to obtain values of $CE_{WSRE(\lambda)}(IC(\alpha_{VR}), \infty)$ for PER, NPER and BER that are listed in Table 9 for $\alpha_{VR} \to 0$ and for each $\alpha_{VR} = .1(.1)1.0$.

Table 9. Computed Limiting Values for Two-Stage Efficiencies with $IC(\alpha_{VR})$.

	Voting Rule	
α_{VR}	PER & NPER	BER
≈ 0	1/3 = .3333	1/3 = .3333
.1	.4516	.4537
.2	.5075	.5109
.3	.5546	.5593
.4	.5982	.6043
.5	.6407	.6484
.6	.6836	.6934
.7	.7288	.7411
.8	.7786	.7944
.9	.8387	.8597
1.0	.9629	1

The results in Table 9 produce some interesting outcomes. First of all, it is well known from Daunou (1803) that BR cannot rank the OCW in last place, so that it cannot be eliminated in the first stage with BER and the OCW must then win in the second stage. The Actual Condorcet Efficiency of BER is therefore 100% when all voters participate so that the OCW and ACW must coincide. However, this is not true when some voters abstain and the OCW and ACW do not necessarily coincide.

When we compare the Condorcet Efficiency results for the two-stage rules with $IC(\alpha_{VR})$ in Table 9 to their counterpart single-stage rules in Table 4. BER dominates PER and NPER for all non-zero voter participation rates, just as BR dominated PR and NPR. However, the degree of this domination is significantly dampened with the two-stage rules, since BER has a less than 1% advantage over PER and NPER for voter participation rates of 60% or less. It seems logical to expect increased Condorcet Efficiency results with two-stage rules, and this is true for PER and NPER. But, it is a very surprising outcome to observe that BER efficiencies are actually

marginally smaller than BR efficiencies for $.1 \leq \alpha_{VR} \leq .5$ and the Condorcet Efficiency values for two-stage rules remain less than 61% for voter participation rates that are 40% or less.

4.2 Condorcet Efficiency of Two-stage Voting Rules with IAC

The same type of two-stage voting rule Condorcet Efficiency analysis is extended with the assumption of $IAC(\alpha_{VR})$ and the resulting representations for PER, NPER and BER are shown respectively in (29), (30) and (31).

$$CE_{PER}(IAC(\alpha_{VR}), \infty) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad (29)$$

$$\frac{3823792093\alpha_{VR}^5 - 32968107660\alpha_{VR}^4 + 85500757710\alpha_{VR}^3 - 109197460800\alpha_{VR}^2 + 68946837120\alpha_{VR} - 16930529280}{806215680(42\alpha_{VR}^5 - 274\alpha_{VR}^4 + 603\alpha_{VR}^3 - 624\alpha_{VR}^2 + 315\alpha_{VR} - 63)},$$
$$\text{for } 0 \leq \alpha_{VR} \leq \frac{1}{2}$$

$$\frac{\left[\begin{array}{l} 69127262173\alpha_{VR}^{10} - 348198438540\alpha_{VR}^9 + 678673944270\alpha_{VR}^8 - 579624310080\alpha_{VR}^7 + 63656046720\alpha_{VR}^6 \\ +284644523520\alpha_{VR}^5 - 244169976960\alpha_{VR}^4 + 91152760320\alpha_{VR}^3 - 16099119360\alpha_{VR}^2 + 755827200\alpha_{VR} + 81881280 \end{array}\right]}{806215680(\alpha_{VR}-1)^5(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)},$$
$$\text{for } \frac{1}{2} \leq \alpha_{VR} \leq \frac{2}{3}$$

$$\frac{\left[\begin{array}{l} 759280879\alpha_{VR}^{10} - 2333165430\alpha_{VR}^9 - 1845638460\alpha_{VR}^8 + 18995483340\alpha_{VR}^7 - 39014909955\alpha_{VR}^6 \\ +42548701734\alpha_{VR}^5 - 28497933135\alpha_{VR}^4 + 12208971240\alpha_{VR}^3 - 3302020080\alpha_{VR}^2 + 517269240\alpha_{VR} - 36039573 \end{array}\right]}{12597120(\alpha_{VR}-1)^5(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)},$$
$$\text{for } \frac{2}{3} \leq \alpha_{VR} \leq \frac{3}{4}$$

$$\frac{91279\alpha_{VR}^5 + 132885\alpha_{VR}^4 - 153065\alpha_{VR}^3 + 47235\alpha_{VR}^2 + 525\alpha_{VR} - 1707}{1920(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \text{ for } \frac{3}{4} \leq \alpha_{VR} \leq 1.$$

$$CE_{NPER}(IAC(\alpha_{VR}), \infty) = \qquad\qquad\qquad\qquad\qquad\qquad (30)$$

$$\frac{26697173\alpha_{VR}^5 - 245571270\alpha_{VR}^4 + 650959335\alpha_{VR}^3 - 842387040\alpha_{VR}^2 + 536121180\alpha_{VR} - 132269760}{6298560(42\alpha_{VR}^5 - 274\alpha_{VR}^4 + 603\alpha_{VR}^3 - 624\alpha_{VR}^2 + 315\alpha_{VR} - 63)}, \text{ for } 0 \leq \alpha_{VR} \leq \frac{1}{2}$$

$$\frac{\left[\begin{array}{l} 19679445664\alpha_{VR}^{10} - 1016681630400\alpha_{VR}^9 + 209258419680\alpha_{VR}^8 - 207447920640\alpha_{VR}^7 + 79851744000\alpha_{VR}^6 \\ +26877215232\alpha_{VR}^5 - 39945467520\alpha_{VR}^4 + 15947953920\alpha_{VR}^3 - 2579260320\alpha_{VR}^2 + 9054180\alpha_{VR} + 30488967 \end{array}\right]}{201553920(\alpha_{VR}-1)^5\left(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1\right)},$$
$$\text{for } \frac{1}{2} \leq \alpha_{VR} \leq \frac{3}{4}$$

$$\frac{65013\alpha_{VR}^5 - 16605\alpha_{VR}^4 + 54380\alpha_{VR}^3 - 72240\alpha_{VR}^2 + 33285\alpha_{VR} - 5145}{960(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \text{ for } \frac{3}{4} \leq \alpha_{VR} \leq 1.$$

$$CE_{BER}(IAC(\alpha_{VR}), \infty) = \qquad\qquad\qquad\qquad\qquad\qquad (31)$$

$$\frac{6379861\alpha_{VR}^5 - 55971360\alpha_{VR}^4 + 146241045\alpha_{VR}^3 - 187945920\alpha_{VR}^2 + 119257110\alpha_{VR} - 29393280}{1399680(42\alpha_{VR}^5 - 274\alpha_{VR}^4 + 603\alpha_{VR}^3 - 624\alpha_{VR}^2 + 315\alpha_{VR} - 63)}, \text{ for } 0 \leq \alpha_{VR} \leq \frac{1}{2}$$

$$\frac{\left[\begin{array}{l} 119753941\alpha_{VR}^{10} - 603246240\alpha_{VR}^9 + 1176055605\alpha_{VR}^8 - 1004659200\alpha_{VR}^7 + 110071710\alpha_{VR}^6 \\ +494174520\alpha_{VR}^5 - 423906210\alpha_{VR}^4 + 158251320\alpha_{VR}^3 - 27949860\alpha_{VR}^2 + 1312200\alpha_{VR} + 142155 \end{array}\right]}{1399680(\alpha_{VR}-1)^5(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}. \text{ for } \frac{1}{2} \leq \alpha_{VR} \leq \frac{3}{5}$$

$$\frac{\left[\begin{array}{l} 187666713\alpha_{VR}^{10} - 993390820\alpha_{VR}^9 + 2139714765\alpha_{VR}^8 - 2358795600\alpha_{VR}^7 + 1333918530\alpha_{VR}^6 \\ -285155640\alpha_{VR}^5 - 45978030\alpha_{VR}^4 + 15921360\alpha_{VR}^3 + 10661625\alpha_{VR}^2 - 5205060\alpha_{VR} + 642249 \end{array}\right]}{1866240(\alpha_{VR}-1)^5(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \text{ for } \frac{3}{5} \leq \alpha_{VR} \leq \frac{3}{4}$$

$$\frac{170049\alpha_{VR}^5 + 22465\alpha_{VR}^4 + 42320\alpha_{VR}^3 - 144960\alpha_{VR}^2 + 88055\alpha_{VR} - 16649}{2560(42\alpha_{VR}^5 + 64\alpha_{VR}^4 - 73\alpha_{VR}^3 + 39\alpha_{VR}^2 - 10\alpha_{VR} + 1)}, \text{ for } \frac{3}{4} \le \alpha_{VR} \le 1.$$

Computed values of $CE_{Rule}(IAC(\alpha_{VR}), \infty)$ for PER, NPER and BER from (29), (30) and (31) are listed in Table 10 for $\alpha_{VR} \to 0$ and for each $\alpha_{VR} = .1(.1)1.0$.

Table 10. Computed Limiting Values for Two-Stage Efficiencies with $IAC(\alpha_{VR})$.

	Voting Rule		
α_{VR}	PER	NPER	BER
≈ 0	1/3 = .3333	1/3 = .3333	1/3 = .3333
.1	.3687	.3693	.3692
.2	.4148	.4163	.4162
.3	.4755	.4779	.4779
.4	.5534	.5571	.5574
.5	.6465	.6514	.6527
.6	.7407	.7470	.7501
.7	.8213	.8286	.8354
.8	.8844	.8914	.9045
.9	.9323	.9372	.9584
1.0	523/540=.9685	131/135=.9704	1

A comparison of the Condorcet Efficiency results with $IAC(\alpha_{VR})$ in Table 10 to their counterpart single-stage rules in Table 5 produces very similar results to those observed immediately above with $IC(\alpha_{VR})$. BER typically dominates PER and NPER for all non-zero voter participation rates, but NPER has a very small advantage over BER for α_{VR} values of .1 and .2. All differences in efficiencies are significantly dampened with the two-stage rules, with a less than 1% difference between the efficiencies for voter participation rates of 50% or less. Increased Condorcet Efficiency results are observed for PER and NPER over the range of non-zero voter participation rates, but BER efficiencies are marginally smaller than BR efficiencies for $.1 \le \alpha_{VR} \le .5$. Condorcet Efficiency values for two-stage rules are now less than 56% for voter participation rates that are 40% or less, to make things worse than we observed for the independent voter case with $IC(\alpha_{VR})$ with two-stage voting rules in Table 9.

4.3 Borda Paradox Probabilities for Two-stage Voting Rules with IC

The development of a representation for the limiting probability that Borda's Paradox will be observed for two-stage voting rules with $IC(\alpha_{VR})$ directly follows the process that led to the representation for $CE_{WSRE(\lambda)}(IC(\alpha_{VR}), \infty)$ in (28). The only difference is that the signs for the variables Z_1 and Z_2 in Table 8 are reversed to make Candidate A the ACL, which is then elected as the ultimate winner by $WSRE(\lambda)$. It follows directly that

$$CE_{WSRE(\lambda)}(IC(\alpha_{VR}), \infty) = \frac{6\Phi_5(\mathbf{R_6})}{P_{ACW}(IC,\infty)}, \text{ where} \tag{32}$$

$$R_6 = \begin{bmatrix} 1 & \frac{1}{3} & -\sqrt{\frac{2}{3z}} & -\sqrt{\frac{1}{6z}} & -\frac{\sqrt{\alpha_{VR}}}{3} \\ & 1 & -\sqrt{\frac{1}{6z}} & \sqrt{\frac{1}{6z}} & -\sqrt{\alpha_{VR}} \\ & & 1 & \frac{1}{2} & \sqrt{\frac{1}{6z\alpha_{VR}}} \\ & & & 1 & -\sqrt{\frac{1}{6z\alpha_{VR}}} \\ & & & & 1 \end{bmatrix} \text{, with } z = \frac{1-\lambda(1-\lambda)}{\alpha_{VR}}.$$

Table 11 displays computed values of $BP_{WSRE(\lambda)}(IC(\alpha_{VR}), \infty)$ from (32) for each PER, NPER and BER as $\alpha_{VR} \to 0$ and for each $\alpha_{VR} = .1(.1)1.0$.

Table 11. Limiting Borda Paradox Probability for Two-stage Rules with $IC(\alpha_{VR})$.

	Voting Rule	
α_{VR}	PER & NPER	BER
≈ 0	1/3 = .3333	1/3 = .3333
.1	.2291	.2278
.2	.1883	.1866
.3	.1575	.1556
.4	.1318	.1297
.5	.1091	.1070
.6	.0885	.0863
.7	.0691	.0668
.8	.0503	.0481
.9	.0310	.0288
1.0	0	0

It follows from the definition of two-stage elimination rules that the OCL cannot win with any of these rules, so that Borda's Paradox cannot be observed if all voters participate so that the ACL and OCL coincide. However, this observation is not true when some voters abstain, so that the OCW and ACW might not be the same.

By comparing the results of Table 11 to Table 6, very different behaviors are observed for the two-stage voting rules when they are compared to their single-stage rule counterparts. The Borda Paradox probabilities are reduced at all levels of non-zero voter participation for PER and NPER compared to PR and NPR. However, the BER Borda Paradox probabilities all increase from their corresponding BR values for all $0 < \alpha_{VR} < 1$. BER still has lower Borda Paradox probabilities than PER and NPER over the range of voter participation rates, but the differences are less than 0.3% in all cases. The particularly disturbing result from Table 11 is that when the voter participation rates is 40% or less, all two-stage voting rules now have a Borda Paradox probability of greater than about 13%, compared to the 11% that was observed in Table 6 for the single-stage rules!

4.4 Borda Paradox Probabilities for Two-stage Voting Rules with IAC

Representations are also obtained for the limiting probabilities $BP_{PER}(IAC(\alpha_{VR}),\infty)$, $BP_{NPER}(IAC(\alpha_{VR}),\infty)$ and $BP_{BER}(IAC(\alpha_{VR}),\infty)$ by using the same process that lead respectively to the Condorcet Efficiency results in (29), (30) and (31). These resulting limiting probability representations are given by:

$$BP_{PER}(IAC(\alpha_{VR}),\infty) = \tag{33}$$

$$\frac{\left[\begin{array}{c}30368797609\alpha_{VR}^5-147521223660\alpha_{VR}^4+268423789950\alpha_{VR}^3-234843935040\alpha_{VR}^2\\+100358455680\alpha_{VR}-16930529280\end{array}\right]}{806215680(42\alpha_{VR}^5-274\alpha_{VR}^4+603\alpha_{VR}^3-624\alpha_{VR}^2+315\alpha_{VR}-63)}, 0 \le \alpha_{VR} \le \frac{1}{2}$$

$$\frac{\left[\begin{array}{c}8600974249\alpha_{VR}^{10}-56418851820\alpha_{VR}^9+150514746750\alpha_{VR}^8-197354905920\alpha_{VR}^7+95067665280\alpha_{VR}^6\\+81478172160\alpha_{VR}^5-161633646720\alpha_{VR}^4+115641561600\alpha_{VR}^3-43875768960\alpha_{VR}^2+8692012800\alpha_{VR}-711737280\end{array}\right]}{806215680(\alpha_{VR}-1)^5(42\alpha_{VR}^5+64\alpha_{VR}^4-73\alpha_{VR}^3+39\alpha_{VR}^2-10\alpha_{VR}+1)}, \frac{1}{2} \le \alpha_{VR} \le \frac{2}{3}$$

$$\frac{\left[\begin{array}{c}1484308489\alpha_{VR}^{10}-10365275280\alpha_{VR}^9+32356702665\alpha_{VR}^8-59403130380\alpha_{VR}^7+70958056995\alpha_{VR}^6\\-57586120074\alpha_{VR}^5+32164285545\alpha_{VR}^4-12237314760\alpha_{VR}^3+3052242810\alpha_{VR}^2-454677300\alpha_{VR}+30921993\end{array}\right]}{12597120(\alpha_{VR}-1)^5(42\alpha_{VR}^5+64\alpha_{VR}^4-73\alpha_{VR}^3+39\alpha_{VR}^2-10\alpha_{VR}+1)}, \frac{2}{3} \le \alpha_{VR} \le \frac{3}{4}$$

$$\frac{(1-\alpha_{VR})(25239\alpha_{VR}^4-36326\alpha_{VR}^3+28314\alpha_{VR}^2-6546\alpha_{VR}-6)}{1920\left(42\alpha_{VR}^5+64\alpha_{VR}^4-73\alpha_{VR}^3+39\alpha_{VR}^2-10\alpha_{VR}+1\right)}, \frac{3}{4} \le \alpha_{VR} \le 1.$$

$$BP_{NPER}(IAC(\alpha_{VR}),\infty) = \tag{34}$$

$$\frac{238551113\alpha_{VR}^5-1159474830\alpha_{VR}^4+2109732075\alpha_{VR}^3-1844208000\alpha_{VR}^2+786576420\alpha_{VR}-132269760}{6298560(42\alpha_{VR}^5-274\alpha_{VR}^4+603\alpha_{VR}^3-624\alpha_{VR}^2+315\alpha_{VR}-63)}, 0 \le \alpha_{VR} \le \frac{1}{2}$$

$$\frac{\left[\begin{array}{c}388210400\alpha_{VR}^{10}-3308366400\alpha_{VR}^9+14344659360\alpha_{VR}^8-39847541760\alpha_{VR}^7+74031874560\alpha_{VR}^6-92271384576\alpha_{VR}^5\\+76451921280\alpha_{VR}^4-41116999680\alpha_{VR}^3+13647404880\alpha_{VR}^2-2517455700\alpha_{VR}+197676369\end{array}\right]}{201553920(1-\alpha_{VR})^5(42\alpha_{VR}^5+64\alpha_{VR}^4-73\alpha_{VR}^3+39\alpha_{VR}^2-10\alpha_{VR}+1)}, \frac{1}{2} \le \alpha_{VR} \le \frac{3}{4}$$

$$\frac{(1-\alpha_{VR})(24471\alpha_{VR}^4-66184\alpha_{VR}^3+66276\alpha_{VR}^2-25284\alpha_{VR}+3171)}{960\left(42\alpha_{VR}^5+64\alpha_{VR}^4-73\alpha_{VR}^3+39\alpha_{VR}^2-10\alpha_{VR}+1\right)}, \frac{3}{4} \le \alpha_{VR} \le 1.$$

$$BP_{BER}(IAC(\alpha_{VR}),\infty) = \tag{35}$$

$$\frac{10663043\alpha_{VR}^5-51663168\alpha_{VR}^4+93817521\alpha_{VR}^3-81924048\alpha_{VR}^2+34935138\alpha_{VR}-5878656}{279936(42\alpha_{VR}^5-274\alpha_{VR}^4+603\alpha_{VR}^3-624\alpha_{VR}^2+315\alpha_{VR}-63)}, 0 \le \alpha_{VR} \le \frac{1}{2}$$

$$\frac{\left[\begin{array}{c}3104771\alpha_{VR}^{10}-20030400\alpha_{VR}^9+52876881\alpha_{VR}^8-68907024\alpha_{VR}^7+33098058\alpha_{VR}^6\\+28291032\alpha_{VR}^5-56122794\alpha_{VR}^4+40153320\alpha_{VR}^3-15234642\alpha_{VR}^2+3018060\alpha_{VR}-247131\end{array}\right]}{279936(\alpha_{VR}-1)^5(42\alpha_{VR}^5+64\alpha_{VR}^4-73\alpha_{VR}^3+39\alpha_{VR}^2-10\alpha_{VR}+1)}, \frac{1}{2} \le \alpha_{VR} \le \frac{3}{5}$$

$$\frac{\left[\begin{array}{c}794736\alpha_{VR}^{10}-7818275\alpha_{VR}^9+33936246\alpha_{VR}^8-86024484\alpha_{VR}^7+140947128\alpha_{VR}^6\\-155365938\alpha_{VR}^5+116001396\alpha_{VR}^4-57518100\alpha_{VR}^3+17990262\alpha_{VR}^2-3186459\alpha_{VR}+243486\end{array}\right]}{186624(1-\alpha_{VR})^5(42\alpha_{VR}^5+64\alpha_{VR}^4-73\alpha_{VR}^3+39\alpha_{VR}^2-10\alpha_{VR}+1)}, \frac{3}{5} \le \alpha_{VR} \le \frac{3}{4}$$

$$\frac{(\alpha_{VR}-1)(528\alpha_{VR}^4-581\alpha_{VR}^3-1524\alpha_{VR}^2+855\alpha_{VR}-118)}{256(42\alpha_{VR}^5+64\alpha_{VR}^4-73\alpha_{VR}^3+39\alpha_{VR}^2-10\alpha_{VR}+1)}, \frac{3}{4} \le \alpha_{VR} \le 1.$$

Computed values for the Borda Paradox probabilities in (33), (34) and (35) for each PER, NPER and BER respectively are listed in Table 12 as $\alpha_{VR} \to 0$ and for each $\alpha_{VR} = .1(.1)1.0$.

Table 12. Computed Limiting Borda Paradox Probabilities with $IAC(\alpha_{VR})$.

	Voting Rule		
α_{VR}	PER	NPER	BER
≈ 0	1/3 = .3333	1/3 = .3333	1/3 = .3333
.1	.3000	.2993	.2995
.2	.2613	.2596	.2601
.3	.2166	.2137	.2147
.4	.1665	.1620	.1638
.5	.1155	.1089	.1118
.6	.0727	.0641	.0681
.7	.0430	.0334	.0378
.8	.0234	.0152	.0186
.9	.0100	.0053	.0069
1.0	0	0	0

When we compare the results of Table 12 to those in Table 7, the PRE and NPRE Borda Paradox probabilities are smaller than the counterpart PR and NPR probabilities over the range of voter participation rates with $0 < \alpha_{VR} < 1$, while the BRE probabilities increase compared to BR. The end result is that the Borda Paradox probabilities in Table 12 display only a small degree of variability across the voting rules. The comparison of the results in Table 12 to those in Table 11 indicate that the addition of voter dependence with $IAC(\alpha_{VR})$ again causes an increase in the probability of observing a Borda Paradox for lower levels of voter participation with α_{VR} values of 40% or less.

5 Conclusion

An initial analysis was performed to consider the probability that the ACW and OCW would coincide for the limiting case of voters with independent voters' preferences when voters have the option to abstain. This indicated that the probability of non-coincidence becomes relatively high as voter participation rates decline. Rather pessimistic results were also found for the Condorcet Efficiency and the probability that a Borda Paradox is observed with PR, NPR and BR under the same independent voter scenario. Two options were considered to improve this negative result. The first, was to add a degree of dependence among voters' preferences. The second was to use more complex versions of these three voting rules by using each as the basis for a two-stage elimination election procedure. The end result is that both of these options tended to make things better for both Condorcet Efficiency and the probability that a Borda Paradox is observed for larger values of voter participation rates. However, both options tended to make things worse for voter participation rates of 40% or less. The results indicate that low levels of voter participation that are consistent with some actual observed elections result in very poor performance for standard voting rules when voters have independent preferences. And, simple attempts to try to improve this dreary observation typically act to make things worse when voter participation rates are 40% or less.

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