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# A new interpolation technique to deal with fluid-porous media interfaces for topology optimization of heat transfer

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## Abstract

This paper proposes a new interpolation technique based on density approach to solve topology optimization problems for heat transfer. Problems are modeled under the assumptions of steady-state laminar flow using the incompressible Navier-Stokes equations coupled to the convection-diffusion equation through the Boussinesq approximation. The governing equations are discretized using finite volume elements and topology optimization is performed using adjoint sensitivity analysis. Material distribution and effective conductivity are interpolated by two sigmoid functions respectively  $h_\tau(\alpha)$  and  $k_\tau(\alpha)$  in order to provide a continuous transition between the solid and the fluid domains. Comparison with standard interpolation function of the literature (RAMP function) shows a smaller transition zone between the fluid and the solid thereby, avoiding some regularization techniques. In order to validate the new method, numerical applications are investigated on some cases from the literature, namely the single pipe and the bend pipe. Lastly, as

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two new parameters are introduced thanks to the interpolation functions, we study their impact on results of the optimization problem. The study shows that the proposed technique is a viable approach for designing geometries and fluid-porous media interfaces are well-defined.

*Keywords:* Topology optimization, Heat transfer, Interpolation function, Sigmoid function, Continuous adjoint method, Finite volume method

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## 1. Introduction

Since its introduction by Bendsoe and Sigmund [1] for solid mechanics problems, topology optimization has become a powerful and increasingly popular tool for designers and engineers for design process. Topology optimization is a material distribution method for finding the optimal structure, for a given problem subject to design constraints. Contrarily to shape optimization where the topology (i.e. the number of boundaries and connectivity) is predetermined, topology optimization allows introduction of new boundaries during the design process.

Topology optimization was pioneered for Stokes flow by Borrvall and Petersson [4]. They introduced a friction term yielding the generalized Stokes equations. Gersborg-Hansen [9] and Olesen et al. [8] extended topology optimization for fluid flow problems to the Navier-Stokes equations.

In topology optimization, the material distribution is parametrized by defining a design variable  $\alpha \in \{0; 1\}$ . This variable is discrete and should either represent solid material ( $\alpha = 1$ ) or fluid ( $\alpha = 0$ ). A common approach to solve the topology optimization problem with this discrete value as optimization parameter, is to change it into a continuous one by introducing a

19 porous media with a continuous permeability variable for each element. This  
20 method, known as the Brinkman penalization, leads to a problem where flow  
21 and (almost) non-flow regions are developed by allowing interpolation be-  
22 tween the lower and upper value of permeability. Generally, authors used  
23 the density interpolation function proposed by Borrvall and Petersson [4]  
24 or a reformulated version of their convex and  $q$ -parametrized interpolation  
25 function. The parameter  $q > 0$  is a penalty parameter that is used to con-  
26 trol the level of 'gray' in the optimal design. However, authors had also  
27 experienced problems with locally optimal solutions. Therefore, they con-  
28 sidered a two-steps solution procedure where the problem was first solved  
29 with a small penalty value of  $q = 0.01$  for example and then the result is  
30 used as initial case for the problem with a penalty value of  $q = 0.1$  [4, 8]  
31 or  $q = 1$  [15]. The mathematical foundation of the interpolation of  $\alpha$  was  
32 further investigated by Evgrafov [14] where the limiting cases of pure fluid  
33 and solid were included. Brinkman approach has since been used for several  
34 problems as transport problem [28], reactive [32] and transient flows [33, 3],  
35 fluid-structure interaction [36] and also flows driven by body forces [37].

36 A variation of the approach is presented by Guest and Prevost [2]. They  
37 proposed to regularize the solid-fluid structure by treating the material phase  
38 as a porous medium where fluid flow is governed by Darcy's law. In their  
39 approach, flows through voids are governed by Stokes flow and, when the  
40 solid phase is impermeable, discrete no-slip condition is simulated by assign-  
41 ing a low permeability to the solid phase. There exists other alternatives  
42 to Brinkman penalization in the literature. The level set approach to topol-  
43 ogy optimization has been applied to fluid flows problems [34, 46, 41], and

44 recently the level set approach was combined with the extended finite ele-  
45 ment method (XFEM) by Kreissl and Maute [5] and by Jenkins et al. [40].  
46 The main drawback of the level set approach is the constraint of remeshing  
47 throughout the optimization process.

48 The second difficulty in topology optimization is to deal with the differ-  
49 ence in thermal conductivity in the solid and fluid domains. Most publica-  
50 tions interpolate the conductivity using the SIMP method (Solid Isotropic  
51 Material with penalization). This method allows to deal with the discrete  
52 nature of conductivity material distribution. So, authors [6, 15, 30, 36] con-  
53 sidered a continuous local thermal conductivity controlled by the design pa-  
54 rameter  $\alpha$  ranging from 0 to 1. Thanks to this function, the optimization al-  
55 gorithm is able to reallocate thermal conductivity material, or creating holes  
56 in its structure to reach the objective function. Moreover, the convex and  
57  $p$ -parametrized function interpolation is similar to the density interpolation  
58 function of Borrvall and Petersson [4] or the RAMP (Rational Approxima-  
59 tion of Material Properties)-style function as introduced by Stolpe [16, 30, 7].  
60 Other methods have also been investigated. Matsumori et al. [19] presented  
61 results with a linear-interpolated design-dependant volumetric heat genera-  
62 tion. Dede [20] used a linear interpolation for thermal conductivity. Thus,  
63 the main issue is to deal with intermediate design variables and nonphysical  
64 flow solutions.

65 There are three main categories of algorithms to solve topology optimiza-  
66 tion problems : gradient-free, gradient-based and hessian-based algorithms.  
67 In topology optimization problems with large number of design variables,  
68 gradient-based algorithms are used to find accurate solutions efficiently. One

69 of the advantages of the interpolation functions described above is the possi-  
70 bility of using gradient-based continuous optimization methods. These meth-  
71 ods are based on derivatives in order to find extrema, and are the so-called  
72 sensitivity analysis. It aims at evaluating the derivative of objective function  
73 with respect of  $\alpha$ . Gradient-based algorithm is widely used by several au-  
74 thors [27, 21, 30, 35, 47]. Moreover, since most of the topology optimization  
75 problems involve a huge number of design variable, specific gradient-based  
76 optimization algorithms must be chosen to handle this difficulty. A famous  
77 algorithm from the literature is the MMA (Method of Moving Asymptotes)  
78 developed initially by Svanberg [43]. In order to reduce the computational  
79 costs, adjoint approach consisting to calculate the sensitivities of the objec-  
80 tive function by an adjoint state has been adopted. Other methods have  
81 been explored to reduce the computation cost: the multigrid preconditioned  
82 conjugate gradients (MGCC) by Amir et al. [22], multi-resolution multi-scale  
83 topology optimization technique by Kim et al. [23], the technique of using  
84 adaptive design variable fields by Guest et al. [26].

85 Moreover, various regularization techniques based on filtering of either  
86 the design variable  $\alpha$  or the sensitivity  $\frac{\partial f}{\partial \alpha}$  [1, 10, 15, 30, 48] exist to ensure  
87 well-posed topology optimization problems. The regularization works by  
88 defining a certain length scale  $r_0$  below which any features in  $\alpha$  or  $\frac{\partial f}{\partial \alpha}$  are  
89 smeared out by the filter; that results in optimized structures with a minimal  
90 feature size  $r_0$  independent of the mesh refinement. As mentioned by some  
91 authors [15, 29], these regularization techniques allow to avoid checkerboard  
92 problems.

93 This paper proposes a new interpolation technique in order to solve a

94 heat transfer topology optimization problem. Design material and effective  
 95 conductivity are interpolated respectively by a function  $h_\tau(\alpha)$  and another  
 96 function  $k_\tau(\alpha)$  in order to provide a continuous transition between the solid  
 97 and the fluid domains. These interpolation functions avoid the use of some  
 98 regularization techniques because the problem can be solved in one-step with-  
 99 out a new value of the convexity parameter. Moreover, these interpolation  
 100 functions allow a smaller transition zone between the fluid and solid regions.  
 101 To prove this claim and get more qualitative results, the size of these tran-  
 102 sition zones is explicitly computed and comparison with standard RAMP  
 103 interpolation is lead. In order to validate the new method, some numerical  
 104 applications are investigated on the single pipe and the bend pipe cases of the  
 105 literature. Lastly, as two new parameters are introduced thanks to the in-  
 106 terpolation functions, we study their impact on the results of a optimization  
 107 problem.

## 108 2. Governing equation

109 The main goal of this paper is to solve topology optimization problems  
 110 for heat transfer in fluid flow. The latter can be written in the general form  
 111 below:

$$\begin{aligned}
 & \text{Minimize} \quad \mathcal{J}(\mathbf{u}, p, \theta) = \int_{\Omega} \mathcal{J}_{\Omega}(\mathbf{u}, p, \theta) d\Omega + \int_{\Gamma} \mathcal{J}_{\Gamma}(\mathbf{u}, p, \theta) d\Gamma \\
 & \text{Subject to} \quad \text{Governing equation for } (\mathbf{u}, p, \theta) \quad , \quad (1) \\
 & \quad \quad \quad \text{Boundary conditions on } \Gamma
 \end{aligned}$$

113 where  $\mathbf{u}$ ,  $p$  and  $\theta$  are respectively the dimensionless velocity, pressure and  
 114 temperature,  $\Omega$  is a bounded open set of  $\mathbb{R}^d$ ,  $d = 2, 3$ , with boundary  $\Gamma =$   
 115  $\partial\Omega$ . The function  $\mathcal{J}_{\Omega}$  and  $\mathcal{J}_{\Gamma}$  are some cost functional modeling a physical

116 effect one wish to minimize. We assume that  $\Gamma$  can be decomposed as  $\Gamma =$   
 117  $\Gamma_1 \cup \Gamma_2 \cup \Gamma_{in} \cup \Gamma_{out}$  where  $\Gamma_{in}$  is the inlet,  $\Gamma_{out}$  the outlet and  $\Gamma_1, \Gamma_2$  are going  
 118 to be considered as walls.

119 For the governing equation, the flows considered in this study are assumed  
 120 Newtonian and incompressible, steady and laminar. The inverse permeability  
 121 field is introduced in the steady-state Navier-Stokes equations as a source  
 122 term  $h_\tau(\alpha)\mathbf{u}$  yielding a Brinkman model with a convection term. The set of  
 123 dimensionless equations governing the conservation of momentum, mass and  
 124 energy for incompressible steady-state fluid flow are the following:

$$\begin{aligned}
 (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + Re^{-1} \Delta \mathbf{u} - h_\tau(\alpha) \mathbf{u} + Ri \theta \vec{e}_y^\top & \text{in } \Omega, \\
 \nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega, \\
 \nabla \cdot (\mathbf{u} \theta) &= \text{div}(Re^{-1} Pr^{-1} k_\tau(\alpha) \nabla \theta) & \text{in } \Omega.
 \end{aligned}
 \tag{2}$$

126 The reduced dimensionless temperature is  $\theta = (T - T_0)/\Delta T$  and the uni-  
 127 form heat flux  $\Phi = \partial_n \theta$  on some part of  $\Gamma_1$  is equal to  $\lambda_f \Delta T/l$ , with  $\lambda_f$   
 128 the thermal conductivity of the fluid. Parameters governing the flow are the  
 129 Reynolds number defined as  $Re = U_0 l/\nu$ , with  $U$  the reference velocity (i.e  
 130 mean velocity here) and  $l$  is the hydraulic diameter, the Richardson number  
 131  $Ri = Gr/Re^2$  where  $Gr$  is the Grashof number defined as  $Gr = g \beta \Delta T l^3/\nu^2$ .  
 132  $h_\tau(\alpha)$  corresponds to the ratio between a kinematic viscosity and a perme-  
 133 ability and  $\alpha$  is the spatially varying design variable field determined by the  
 134 optimization algorithm. Regions with low permeability can be considered as  
 135 solid regions since (at least formally) the velocity of the fluid vanishes in such  
 136 region, and those with very high permeability regions are interpreted as pure  
 137 fluid. The interpolation function for the adimensional thermal conductivity  
 138 is given by  $k_\tau(\alpha)$ .



139 We consider the following set of boundary conditions:

$$\begin{aligned}
 & \mathbf{u} = 0, & \partial_n \theta = -1 & & \text{on } \Gamma_1, \\
 & \mathbf{u} = 0, & \partial_n \theta = 0 & & \text{on } \Gamma_2, \\
 140 & \mathbf{u}_n = 1, \mathbf{u}_t = 0 & \theta = 0 & & \text{on } \Gamma_{in}, \\
 & \partial_n \mathbf{u} = 0, & \partial_n \theta = 0, & p = 0 & \text{on } \Gamma_{out},
 \end{aligned} \tag{3}$$

141 where  $\mathbf{u}_n$  and  $\mathbf{u}_t$  are the normal and tangential components of primal veloc-  
 142 ities, respectively. To summarize, we require a constant horizontal velocity  
 143 and a constant temperature  $\theta_0$  at the inlet, vanishing gradient for both ve-  
 144 locity and temperature of fluid at the outlet. Homogeneous Dirichlet for the  
 145 velocity and Neumann boundary condition are prescribed for the tempera-  
 146 ture on the walls. It is worth noting that boundary conditions on the outlet  
 147 have been considered in [39] and ensure that the fluid does not re-enter in  
 148 the domain.

### 149 3. Interpolation

150 The goal of topology optimization is to end up with binary designs, i.e  
 151 avoid that the design variables take other value than those representing the  
 152 fluid or the solid. The most important thing is to be sure that the inter-  
 153 mediate regions (the transition zones) are unattractive with respect to the  
 154 optimization problem. This is usually carried out by penalizing the inter-  
 155 mediate densities with respect to the material parameters, such as inverse  
 156 permeability and effective conductivity. A standard approach is to use some  
 157 convex interpolation (RAMP) function [4, 30, 15]. In this section, we propose  
 158 to use another interpolation function hence interpolate the inverse permeabil-  
 159 ity and the effective conductivity using sigmoid functions. After giving the

160 definition of our interpolation function, we show that the transition zones  
 161 of the sigmoid are much smaller than those of the standard RAMP function  
 162 therefore motivating the use of such method in topology optimization.

163 *3.1. Interpolation of porosity using a sigmoid function*

164 Inverse permeability is interpolated with a sigmoid function

$$165 \quad h_\tau(\alpha, \alpha_0) = \alpha_{\max} \left( \frac{1}{1 + \exp(-\tau(\alpha - \alpha_0))} - \frac{1}{1 + \exp(\tau\alpha_0)} \right) \quad (4)$$

166 where  $\alpha \in [0, \alpha_{\max}]$  with  $\alpha_{\max}$  being the maximal value  $h_\tau$  can reach. Direct  
 167 computations show that  $h_\tau(0, \alpha_0) = 0$  and that the following point-wise  
 168 convergence holds

$$169 \quad \lim_{\tau \rightarrow +\infty} h_\tau(\alpha, \alpha_0) = \begin{cases} 0 & \text{if } \alpha < \alpha_0, \\ \alpha_0/2 & \text{if } \alpha = \alpha_0, \\ \alpha_{\max} & \text{if } \alpha > \alpha_0. \end{cases} \quad (5)$$

170 This shows that  $h_\tau$  is a smooth regularization of a Heaviside step function.  
 171 From (5), one can see that  $\alpha_0$  can be tuned to control the size of the fluid  
 172 part in the computational domain. Finally, note that the definition of our  
 173 interpolation function has to be changed when  $\alpha_0 = 0$  since we require  $\alpha \geq 0$   
 174 and (4) would lead to a regularization of a step function satisfying  $h_\tau(0, 0) = 0$   
 175 and, for any  $\alpha > 0$ ,  $h_\tau(\alpha, 0) \rightarrow \alpha_{\max}/2$  as  $\tau \rightarrow +\infty$ . In order to cover this  
 176 case, one can use the following interpolation function

$$177 \quad h_\tau(\alpha, 0) = \tilde{h}_\tau(\alpha) = 2 \alpha_{\max} \left( \frac{1}{1 + \exp(-\tau\alpha)} - \frac{1}{2} \right),$$

178 which satisfies  $\tilde{h}_\tau(0) = 0$  and, if  $\alpha > 0$ ,  $\tilde{h}_\tau(\alpha) \rightarrow \alpha_{\max}$  as  $\tau \rightarrow +\infty$ .

179 *3.2. Comparison with standard RAMP interpolation function*

180 The RAMP function has been introduced in [4] and is defined as follows:

$$181 \quad h_{q,R}(s) = \alpha_{\max} + (\alpha_{\min} - \alpha_{\max})(1 - s) \frac{1 + q}{1 - s + q}, \quad (6)$$

182 where  $s \in [0, 1]$  is the design parameter and  $\alpha_{\min}, \alpha_{\max}$  are respectively the  
 183 minimal and maximal value the RAMP function can take. Again, some  
 184 direct computations show that  $h_{q,R}$  is a smooth and convex regularization  
 185 of a Heaviside step function since  $h_{q,R}(0) = \alpha_{\min}$  and, for any  $s > 0$ ,  
 186  $\lim_{q \rightarrow 0} h_{q,R}(s) = \alpha_{\max}$ . In the sequel, we use  $\alpha_{\min} = 0$  since this does not  
 187 add any difficulties nor plus to the discussion. We now compare the perfor-  
 188 mances of our sigmoid (4) with the standard RAMP function (6) by focusing  
 189 on the size of their so-called transition zones. First, note that they are both  
 190 defined on  $\Omega$  but the transition zone only depends on the value of the design  
 191 parameter  $s(x)$  for  $x \in \Omega$ . As a result, we are going to consider that both  
 192 functions have real domain and act on the interval  $[0, 1]$ . Also, to compare  
 193 functions having the same domain and being both smooth regularization of  
 194 step function as  $q \rightarrow 0$ , we rescale the sigmoids (4) thanks to the formula

$$195 \quad h_{q,S}(s, \alpha_0) = h_{1/q}(\alpha_{\max}s, \alpha_0), \quad h_{q,S,0}(s) = \tilde{h}_{1/q}(s\alpha_{\max}), \quad s \in [0, 1].$$

196 By definition, the transition zones are the undesirable values of the design  
 197 variable  $s \in [0, 1]$  for which the velocity of the speed is not enough penalized  
 198 to vanish or the value of the interpolation function is not small enough for  
 199 considering these zones as fluid. As one can see from Figure 1, the sigmoid  
 200 function has much smaller transition zone than the RAMP function.

201 To prove this claim and get more qualitative results we are going to  
 202 compute explicitly the size of these transition zones. For a given small enough

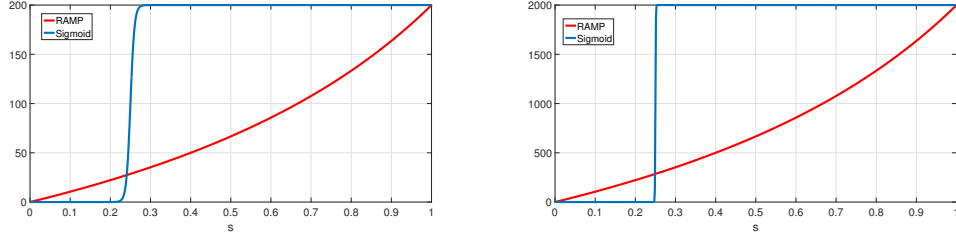


Figure 1: Interpolation functions with  $q = 1$  and  $\alpha_0 = \alpha_{\max}/4$ . Left:  $\alpha_{\max} = 200$ . Right:  $\alpha_{\max} = 2000$ .

203  $\varepsilon > 0$  and a large enough  $M > 0$ , the latter are defined by

204 
$$\mathcal{T}(f, \varepsilon, M) = \{s \in [0, 1] \mid \varepsilon \leq f(s) \leq M\}, f \in \{h_{q,R}, h_{q,S}, h_{q,S,0}\}.$$

205 Since both interpolation function are increasing, they admit an inverse func-  
 206 tion  $f^{-1}$  and the transition zone is thus given by the interval  $\mathcal{T}(f, \varepsilon, M) =$   
 207  $[f^{-1}(\varepsilon), f^{-1}(M)]$  from which we infer that

208 
$$|\mathcal{T}(f, \varepsilon, M)| = f^{-1}(M) - f^{-1}(\varepsilon). \quad (7)$$

209 It then only remains to compute the inverse of the sigmoid and RAMP func-  
 210 tions. This is actually achieved by solving  $f(s) = y$  for a given  $y \in [0, \alpha_{\max}]$ .  
 211 These equations can be solved analytically using only direct computations

212 and give

$$213 \quad s = (h_{q,R})^{-1}(y) = 1 - q \frac{y - \alpha_{\max}}{\alpha_{\max} - y + (1+q)(\alpha_{\min} - \alpha_{\max})} = y \frac{1+q}{\alpha_{\max}q + y} \quad (8)$$

$$215 \quad s = (h_{q,S})^{-1}(y) = \frac{\alpha_0}{\alpha_{\max}} -$$

$$216 \quad \frac{q}{\alpha_{\max}} \begin{cases} -\log \left( \frac{\alpha_{\max} - y(1 + \exp(-\alpha_0/q))}{\alpha_{\max} \exp(-\alpha_0/q) + y(1 + \exp(-\alpha_0/q))} \right) & \text{if } s \leq \alpha_0/\alpha_{\max} \\ \log \left( \frac{\alpha_{\max} - y(1 + \exp(-\alpha_0/q))}{\alpha_{\max} \exp(-\alpha_0/q) + y(1 + \exp(-\alpha_0/q))} \right) & \text{if } s \geq \alpha_0/\alpha_{\max} \end{cases}$$

$$217 \quad s = (h_{q,S,0})^{-1}(y) = -\log \left( \frac{\alpha_{\max} - y}{\alpha_{\max} + y} \right) \frac{q}{\alpha_{\max}}.$$

218 The constraint on  $s$  in the definition of the inverse of  $h_{q,S}$  comes from the fact  
 219 that we have to solve an equation of the form  $\exp((\alpha_0 - s\alpha_{\max})/q) = g(y)$   
 220 for a positive function  $g$ . Using (7) and (8), the size of the transition zones  
 221 are finally

$$222 \quad |\mathcal{T}(h_{q,R}, \varepsilon, M)| := \mathcal{T}_R = q\alpha_{\max} \frac{(1+q)}{(\alpha_{\max}q + M)(\alpha_{\max}q + \varepsilon)} (M - \varepsilon),$$

$$223 \quad |\mathcal{T}(h_{q,S}, \varepsilon, M)| := \mathcal{T}_S = \frac{q}{\alpha_{\max}} \log \left( \frac{\alpha_{\max} - M(1 + \exp(-\alpha_0/q))}{\alpha_{\max} \exp(-\alpha_0/q) + M(1 + \exp(-\alpha_0/q))} \right)$$

$$224 \quad + \frac{q}{\alpha_{\max}} \log \left( \frac{\alpha_{\max} - \varepsilon(1 + \exp(-\alpha_0/q))}{\alpha_{\max} \exp(-\alpha_0/q) + \varepsilon(1 + \exp(-\alpha_0/q))} \right),$$

$$225 \quad |\mathcal{T}(h_{q,S,0}, \varepsilon, M)| := \mathcal{T}_{S,0} = \frac{q}{\alpha_{\max}} \log \left( \frac{(\alpha_{\max} - M)(\alpha_{\max} + \varepsilon)}{(\alpha_{\max} + M)(\alpha_{\max} - \varepsilon)} \right).$$

226 For  $\alpha_0 = 0$ , one can clearly see that  $\mathcal{T}_{S,0} < \mathcal{T}_R$ . To deal with the case  $\alpha_0 > 0$ ,  
 227 note that the parameter  $q$  is small since we wish the interpolation function to  
 228 be close to an ideal step function. Also, we emphasize that  $\alpha_0 > 0$  is going to  
 229 depend on  $\alpha_{\max}$  in order to control the percentage of fluid in  $\Omega$ . Since  $\alpha_{\max}$   
 230 has to be large enough, one gets that  $z = \exp(-\alpha_0/q)$  is a small parameter

231 and we can thus expand  $\mathcal{T}_S$  as  $z \rightarrow 0$ . This gives

$$232 \quad \mathcal{T}_S = \frac{q}{\alpha_{\max}} \log \left( \frac{(\alpha_{\max} - M)(\alpha_{\max} - \varepsilon)}{M\varepsilon} \right) + O(\exp(-\alpha_0/q)),$$

233 from which one can see that  $\mathcal{T}_S < \mathcal{T}_R$ . Thanks to the factor  $q/\alpha_{\max}$ , we can see  
 234 that the intermediate zones of the sigmoid are in fact much smaller than those  
 of the RAMP function. Some numerical calculation of the size of transition

	$\mathcal{T}_R$	$\mathcal{T}_S$	$\mathcal{T}_{S,0}$
$q = 1$	0.6657	0.0380	0.0055
$q = 10^{-4}$	0.1665	$3.8002 \times 10^{-6}$	$5.4881 \times 10^{-7}$
$q = 10^{-5}$	0.0196	$3.8002 \times 10^{-7}$	$5.4881 \times 10^{-8}$

Table 1: Size of the transition zones for  $\alpha_{\max} = 200$ ,  $M = \alpha_{\max}/2$ ,  $\varepsilon = 0.1$  and  $\alpha_0 = \alpha_{\max}/4$ .

	$\mathcal{T}_R$	$\mathcal{T}_S$	$\mathcal{T}_{S,0}$
$q = 1$	0.6666	0.0050	$5.4926 \times 10^{-4}$
$q = 10^{-4}$	0.6665	$4.9517 \times 10^{-7}$	$5.4926 \times 10^{-8}$
$q = 10^{-5}$	0.1666	$4.9517 \times 10^{-8}$	$5.4926 \times 10^{-9}$

Table 2: Size of the transition zones for  $\alpha_{\max} = 2000$ ,  $M = \alpha_{\max}/2$ ,  $\varepsilon = 0.1$  and  $\alpha_0 = \alpha_{\max}/4$ .

235

236 zone confirming this fact can be found in Table 1,2 for  $\alpha_{\max} \in \{200, 2000\}$   
 237 and  $q \in \{1, 10^{-4}, 10^{-5}\}$ .

238 *3.3. Interpolation of the thermal conductivity*

239 The effective conductivity is interpolated using a sigmoid function similar  
 240 to (4):

$$241 \quad k_\tau(\alpha, \alpha_0) = (k_s - k_f) \left( \frac{1}{1 + \exp(-\tau(\alpha - \alpha_0))} - \frac{1}{1 + \exp(\tau\alpha_0)} \right) + k_f \quad (9)$$

242 where  $\alpha \in [0, \alpha_{\max}]$  is the design parameter,  $k_s$  and  $k_f$  are respectively the  
 243 adimensional thermal diffusivity of the solid and of the fluid. Interpolation  
 244 function (9) is again a smooth regularization of a Heaviside step function  
 245 that satisfies  $k_\tau(0, \alpha_0) = k_f$  together with the point-wise convergence

$$246 \quad \lim_{\tau \rightarrow +\infty} k_\tau(\alpha, \alpha_0) = \begin{cases} k_f & \text{if } \alpha < \alpha_0, \\ (k_f + k_s)/2 & \text{if } \alpha = \alpha_0, \\ k_s & \text{if } \alpha > \alpha_0. \end{cases}$$

247 Similarly to  $h_\tau$ , the case  $\alpha_0 = 0$  needs the following slight modification to  
 248 still have, when  $\tau \rightarrow +\infty$ , a fluid-solid step function

$$249 \quad k_\tau(\alpha, 0) = \tilde{k}_\tau(\alpha) = 2(k_s - k_f) \left( \frac{1}{1 + \exp(-\tau\alpha)} - \frac{1}{2} \right) + k_f.$$

250 The previous function then satisfies  $\tilde{k}_\tau(0) = 0$  and, for any  $\alpha > 0$ ,  $\tilde{k}_\tau(\alpha) \rightarrow k_s$   
 251 as  $\tau \rightarrow +\infty$ . A RAMP-like interpolation function is also usually used for  
 252 the thermal conductivity [30, 15]. Computations similar to those of Section  
 253 3.2 can be done to prove that the transition zone of the sigmoid (9) are  
 254 much smaller than those of the RAMP function hence motivating using such  
 255 technique in topology optimization.

256 **4. Gradient computation with the adjoint model**

257 We chose to solve the optimization problem (1) with gradient-based op-  
 258 timization algorithm. The latter requires the computation of the gradient  
 259 of the cost functional with respect to the design variable which needs the  
 260 sensitivities of the solution to the state equation with respect to  $\alpha$ . To do  
 261 this, we are going to use the adjoint method (see e.g. [24, 12, 45]) since it  
 262 allows to compute the gradient of  $\mathcal{J}$  efficiently, only by solving the so-called  
 263 direct problem (the governing equation) and the adjoint model.

264 We adopt here the *differentiate then discretize* approach [25] which means  
 265 that we need to compute the continuous adjoint model associated to (1) as  
 266 done in [17, 18]. The Lagrangian associated to the optimization problem (1)  
 267 can then be defined as follows

$$\begin{aligned}
 \mathcal{L}((\mathbf{u}, \theta, p), (\mathbf{u}^*, \theta^*, p^*), \alpha) &:= \mathcal{J}(\mathbf{u}, \theta, p) - \int_{\Omega} p^* \operatorname{div} \mathbf{u} \, d\Omega \\
 &- \int_{\Omega} \mathbf{u}^* \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - Re^{-1} \Delta \mathbf{u} + h_{\tau}(\alpha, \alpha_0) \mathbf{u} - Ri \theta \vec{e}_y] \, d\Omega \\
 &- \int_{\Omega} \theta^* \cdot [\nabla \cdot (\mathbf{u} \theta) - \operatorname{div}(Re^{-1} Pr^{-1} k_{\tau}(\alpha, \alpha_0) \nabla \theta)] \, d\Omega \\
 &- \int_{\Gamma_1} (\mathbf{u} \cdot \Phi_1 + (\partial_n \theta - 1) \psi_1) \, d\Gamma - \int_{\Gamma_2} (\mathbf{u} \cdot \Phi_2 + \partial_n \theta \psi_2) \, d\Gamma \\
 &- \int_{\Gamma_{\text{in}}} ((\mathbf{u} - \mathbf{u}_{\text{in}}) \cdot \Phi_{\text{in}} + \theta \psi_{\text{in}}) \, d\Gamma - \int_{\Gamma_{\text{out}}} (\partial_n \mathbf{u} \cdot \Phi_{\text{out}} + p q_{\text{out}} + \partial_n \theta \psi_{\text{out}}) \, d\Gamma
 \end{aligned} \tag{10}$$

268 where  $(\mathbf{u}^*, \theta^*, p^*, \Phi_l, \psi_l, q_{\text{out}})$ , for  $l \in \{1, 2, \text{in}, \text{out}\}$ , can be seen as Lagrange  
 269 multiplier (or adjoint variables). We emphasize that the variables  $(\Phi_l, \psi_l, q_{\text{out}})$   
 270 are here to enforce the boundary conditions (3).  
 271

272 The adjoint model is defined thanks to the critical point of  $\mathcal{L}$  with respect  
 273 to the state variables  $(\mathbf{u}, \theta, p)$ . We use below the following notation for the  
 274 derivative of an application  $\mathcal{F} : x \in \mathbf{E} \mapsto \mathcal{F}(x) \in \mathbf{F}$ , where  $\mathbf{E}, \mathbf{F}$  are normed



275 spaces

$$276 \quad \frac{\partial \mathcal{F}}{\partial x}[\delta x] := \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{F}(x + \varepsilon \delta x) - \mathcal{F}(x)}{\varepsilon}. \quad (11)$$

277 First, since  $\mathcal{L}$  is linear with respect to the adjoint variables, it is worth  
 278 noting that we recover the state equation (2)-(3) if, for  $l \in \{1, 2, \text{in}, \text{out}\}$ , we  
 279 solve

$$280 \quad \frac{\partial \mathcal{L}}{\partial(\mathbf{u}^*, \theta^*, p^*, \Phi_l, \psi_l, q_{\text{out}})}[\delta \mathbf{u}^*, \delta \theta^*, \delta p^*, \delta \Phi_l, \delta \psi_l, \delta q_{\text{out}}] = 0.$$

281 Using now (11) to differentiate the Lagrangian (10) with respect to the  
 282 state variables, and integrating by parts to have no terms involving derivative  
 283 of  $(\delta \mathbf{u}, \delta \theta, \delta p)$ , we end up with

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\mathbf{u}, \theta, p)}[\delta \mathbf{u}, \delta \theta, \delta p] &= \int_{\Omega} \frac{\partial \mathcal{J}_{\Omega}}{\partial(\mathbf{u}, \theta, p)}[\delta \mathbf{u}, \delta \theta, \delta p] d\Omega + \int_{\Gamma} \frac{\partial \mathcal{J}_{\Gamma}}{\partial(\mathbf{u}, \theta, p)}[\delta \mathbf{u}, \delta \theta, \delta p] d\Gamma + \\ &\int_{\Omega} \delta \mathbf{u} \cdot (\nabla p^* - h_{\tau}(\alpha, \alpha_0) \mathbf{u}^* - (\nabla \mathbf{u})^T \mathbf{u}^* + \nabla \mathbf{u}^* \mathbf{u} + \theta \nabla \theta^* + Re^{-1} \Delta \mathbf{u}^*) d\Omega \\ &+ \int_{\Omega} \delta \theta (\text{div}(Re^{-1} Pr^{-1} k_{\tau}(\alpha, \alpha_0) \nabla \theta^*) + \mathbf{u} \cdot \nabla \theta^* + Ri \mathbf{u}^* \cdot \mathbf{e}_y) d\Omega \\ &+ \int_{\Omega} \delta p \text{div} \mathbf{u}^* d\Omega - \int_{\Gamma} \delta p (\mathbf{u}^* \cdot \mathbf{n}) d\Gamma \\ &+ \int_{\Gamma} \delta \mathbf{u} \cdot (-n p^* - n \theta \theta^* + Re^{-1} \partial_n \mathbf{u}^* - (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}^*) d\Gamma - \int_{\Gamma} Re^{-1} \partial_n \delta \mathbf{u} \cdot \mathbf{u}^* d\Gamma \\ &+ \int_{\Gamma} \delta \theta ((-\mathbf{u} \cdot \mathbf{n}) \theta^* - Re^{-1} Pr^{-1} k_{\tau}(\alpha, \alpha_0) \partial_n \theta^*) d\Gamma + \int_{\Gamma} \theta^* Re^{-1} Pr^{-1} k_{\tau}(\alpha, \alpha_0) \partial_n \delta \theta d\Gamma \\ &- \int_{\Gamma_1} (\delta \mathbf{u} \cdot \Phi_1 + \partial_n \delta \theta \psi_1) d\Gamma - \int_{\Gamma_2} (\delta \mathbf{u} \cdot \Phi_2 + \partial_n \delta \theta \psi_2) d\Gamma \\ &- \int_{\Gamma_{\text{in}}} (\delta \mathbf{u} \cdot \Phi_{\text{in}} + \delta \theta \psi_{\text{in}}) d\Gamma - \int_{\Gamma_{\text{out}}} (\partial_n \delta \mathbf{u} \cdot \Phi_{\text{out}} + \delta p q_{\text{out}} + \partial_n \delta \theta \psi_{\text{out}}) d\Gamma \end{aligned} \quad (12)$$

284

285 Assuming that  $(\delta \mathbf{u}, \delta \theta, \delta p, \Phi_l, \psi_l, q_{\text{out}}) = 0$  for  $l \in \{1, 2, \text{in}, \text{out}\}$ , that the first  
 286 derivatives of  $\delta \mathbf{u}, \delta \theta$  vanish on  $\Gamma$  and solving  $(\partial \mathcal{L} / \partial(\mathbf{u}, \theta, p))[\delta \mathbf{u}, \delta \theta, \delta p] = 0$

287 yield the adjoint problem:

$$\begin{aligned}
\nabla p^* - h_\tau(\alpha) \mathbf{u}^* + \theta \nabla \theta^* + Re^{-1} \Delta \mathbf{u}^* + \nabla \mathbf{u}^* \cdot \mathbf{u} - (\nabla \mathbf{u})^T \mathbf{u}^* &= -\frac{\partial \mathcal{J}_\Omega}{\partial \mathbf{u}} & \text{in } \Omega \\
\operatorname{div} \mathbf{u}^* &= -\frac{\partial \mathcal{J}_\Omega}{\partial p} & \text{in } \Omega \\
Ri \mathbf{u}^* \cdot \vec{e}_y + \mathbf{u} \cdot \nabla \theta^* + \operatorname{div} (Re^{-1} Pr^{-1} k_\tau(\alpha) \nabla \theta^*) &= -\frac{\partial \mathcal{J}_\Omega}{\partial \theta} & \text{in } \Omega.
\end{aligned} \tag{13}$$

288  
289 For the adjoint boundary condition, we are going to show how to obtain  
290 them on  $\Gamma_1 \cup \Gamma_2$  since the other part of  $\Gamma$  can be done in the same spirit.  
291 On both walls, the primal velocity is fixed to zero (no-slip condition on the  
292 walls) and the heat flux either vanishes or is constant. Since  $\operatorname{div} \mathbf{u} = 0$  and  
293  $\mathbf{u}$  is prescribed or null value, there will be  $\delta \mathbf{u} = 0$ ,  $\operatorname{div} \delta \mathbf{u} = 0$  and  $\partial_n \delta \theta = 0$ .  
294 We recall a formula from [11, Lemma 7] that holds for any vector fields  $\mathbf{w}$   
295 and reads

$$296 \quad \partial_n \mathbf{w} \cdot n = \operatorname{div} \mathbf{w}|_\Gamma - \operatorname{div}_\Gamma \mathbf{w}_t - \kappa \mathbf{w} \cdot n, \tag{14}$$

297 where  $\kappa = \operatorname{div} n$  is the curvature of  $\Gamma$  and  $\operatorname{div}_\Gamma$  is the surface divergence  
298 operator. Since  $\partial_n \delta \mathbf{u} \cdot \mathbf{u}^* = (\partial_n \delta \mathbf{u})_n \cdot u_n^* + (\partial_n \delta \mathbf{u})_t \cdot u_t^*$  and  $\delta \mathbf{u} = 0$  and  
299  $\operatorname{div} \delta \mathbf{u} = 0$ , formula (14) gives that

$$300 \quad \partial_n \delta \mathbf{u} \cdot \mathbf{u}^* = (\partial_n \delta \mathbf{u})_t \cdot u_t^*.$$

301 The boundary conditions on  $\Gamma_1 \cup \Gamma_2$  are then obtained from (12) by consid-  
302 ering, for  $l = 1, 2$ , the following vanishing terms

$$\begin{aligned}
\int_{\Gamma_l} Re^{-1} \partial_n \delta \mathbf{u} \cdot \mathbf{u}^* d\Gamma &= \int_{\Gamma} Re^{-1} (\partial_n \delta \mathbf{u})_t \cdot u_t^* d\Gamma = 0, \\
\int_{\Gamma_l} \delta p \left( \mathbf{u}^* \cdot n - \frac{\partial \mathcal{J}_\Gamma}{\partial p} \right) d\Gamma &= 0, \\
\int_{\Gamma_l} \delta \theta \left( (-\mathbf{u} \cdot n) \theta^* - Re^{-1} Pr^{-1} k_\tau(\alpha, \alpha_0) \partial_n \theta^* + \frac{\partial \mathcal{J}_\Gamma}{\partial \theta} \right) d\Gamma &= 0.
\end{aligned} \tag{15}$$

303 The adjoint boundary conditions on  $\Gamma_{in}$  are obtained as above by taking  
 304  $\delta\theta = 0$ . For the outlet, we use  $\partial_n\delta\theta = 0$ ,  $\partial_n\delta\mathbf{u} = 0$  and  $\delta p = 0$ . Finally, we  
 305 end up with the adjoint boundary condition

$$\begin{aligned}
 \text{On } \Gamma_1 \cup \Gamma_2 : \quad & u_t^* = 0, \quad u_n\theta^* + Re^{-1}Pr^{-1}k_\tau(\alpha, \alpha_0) \partial_n\theta^* = \frac{\partial\mathcal{J}_\Omega}{\partial\theta}, \quad u_n^* = \frac{\partial\mathcal{J}_\Omega}{\partial p}, \\
 \text{On } \Gamma_{in} : \quad & u_n^* = \frac{\partial\mathcal{J}_\Gamma}{\partial p} \quad u_t^* = 0, \quad \theta^* = 0, \\
 \text{On } \Gamma_{out} : \quad & u_n \theta^* + Re^{-1}Pr^{-1}k_\tau(\alpha, \alpha_0) \partial_n\theta^* = \frac{\partial\mathcal{J}_\Gamma}{\partial\theta}, \\
 & n p^* + n \theta \theta^* + Re^{-1}\partial_n u^* + u_n \mathbf{u}^* = \frac{\partial\mathcal{J}_\Gamma}{\partial\mathbf{u}}.
 \end{aligned}
 \tag{16}$$

306

307 Now solving  $(\partial\mathcal{L}/\partial\alpha)[\delta\alpha] = 0$  for all  $\delta\alpha$  yields

$$\begin{aligned}
 \frac{\partial h_{\tau, \alpha_0}}{\partial\alpha} \mathbf{u} \cdot \mathbf{u}^* + Re^{-1}Pr^{-1} \frac{\partial k_{\tau, \alpha_0}}{\partial\alpha} \nabla\theta \cdot \nabla\theta^* &= 0 \quad \text{in } \Omega \\
 \frac{\partial k_{\tau, \alpha_0}}{\partial\alpha} \theta^* &= 0 \quad \text{on } \Gamma_1,
 \end{aligned}
 \tag{17}$$

308

309 which is the optimality condition. Finally, according to the adjoint method  
 310 (see e.g. [24, 12, 25]), the gradient of the cost functional  $\mathcal{J}(\mathbf{u}, \theta, p)$  at some  
 311  $\alpha$  is given by

$$\begin{aligned}
 \frac{\partial\mathcal{J}}{\partial\alpha}(\alpha) &= \frac{\partial h_{\tau, \alpha_0}}{\partial\alpha} \mathbf{u} \cdot \mathbf{u}^* + Re^{-1}Pr^{-1} \frac{\partial k_{\tau, \alpha_0}}{\partial\alpha} \nabla\theta \cdot \nabla\theta^*, \quad \text{in } \Omega, \\
 \frac{\partial\mathcal{J}}{\partial\alpha}(\alpha) &= \frac{\partial k_{\tau, \alpha_0}}{\partial\alpha} \theta^* \quad \text{on } \Gamma_1,
 \end{aligned}
 \tag{18}$$

312

313 where  $(\mathbf{u}, \theta, p)$  satisfy (2-3) and  $(\mathbf{u}^*, \theta^*, p^*)$  satisfy the adjoint problem (13-  
 314 16).

315 **Remark 1.** *The adjoint variables  $(\Phi_l, \psi_l, p_{out})$  used to enforce the boundary*  
 316 *conditions (3) of the primal problem are not needed to solve the adjoint prob-*  
 317 *lem. Nevertheless, they can be determined using (13,15) which cancel many*

318 *terms in (12) and gives*

$$\begin{aligned}
319 \quad \Phi_l &= -n p^* - n \theta \theta^* + Re^{-1} \partial_n \mathbf{u}^* - (\mathbf{u} \cdot \mathbf{n}) \mathbf{u}^* + \frac{\partial \mathcal{J}_\Gamma}{\partial \mathbf{u}} \text{ on } \Gamma_l \\
320 \quad \psi_l &= \theta^* Re^{-1} Pr^{-1} k_\tau(\alpha, \alpha_0) + \frac{\partial \mathcal{J}_\Gamma}{\partial \theta} \text{ on } \Gamma_l, l \in \{1, 2, \text{in}\}, \\
321 \quad \Phi_{\text{out}} &= -Re^{-1} \mathbf{u}^*, \psi_{\text{out}} = \theta^* Re^{-1} Pr^{-1} k_\tau(\alpha, \alpha_0), q_{\text{out}} = -u_n^* - \frac{\partial \mathcal{J}_\Gamma}{\partial p}.
\end{aligned}$$

322 *Note finally that these adjoint variables are also not needed to compute the*  
323 *gradient of the cost functional with respect to the design variable since they*  
324 *does not appear in (18).*

## 325 5. Numerical example

326 This section aims to validate the new interpolation technique. To do so,  
327 we investigate two objective functions. First objective function is related to  
328 the power dissipated by the fluid through the domain  $\Omega$  and can be evaluated  
329 on the basis of total pressure losses as follows:

$$330 \quad \mathcal{J}_1(\mathbf{u}, p, \theta) = - \int_\Gamma p_t \mathbf{u} \cdot \mathbf{n} d\Gamma \quad \text{where } p_t = p + \frac{1}{2} \mathbf{u}^2 \quad (19)$$

331 The second cost function is related to the maximization of the recoverable  
332 thermal power from the domain  $\Omega$  and is given by

$$333 \quad \mathcal{J}_2(\mathbf{u}, p, \theta) = \int_\Gamma \theta \mathbf{u} \cdot \mathbf{n} d\Gamma \quad (20)$$

334 We finally consider the following objective functional

$$\begin{aligned}
\text{minimize:} \quad \mathcal{J}(\mathbf{u}, p, \theta) &= \mathcal{J}_1(\mathbf{u}, p, \theta) - \mathcal{J}_2(\mathbf{u}, p, \theta), \\
&= - \int_\Gamma (p_t + \theta) \mathbf{u} \cdot \mathbf{n} d\Gamma
\end{aligned} \quad (21)$$

335 subject to: Governing equation (2),  
Boundary conditions (3).

336 We notice that the objective function  $\mathcal{J}$  is a linear combination of both  
337 objective functions described above. It's taking place at the inlet and outlet  
338 flow boundary conditions and the combination involves the opposite of  $\mathcal{J}_2$   
339 since the optimization problem aims at minimizing the combinatory function  
340  $\mathcal{J}$ . Using the notation from the general optimization problem (1), we have  
341  $\mathcal{J}_\Omega = 0$  and  $\mathcal{J}_\Gamma(\mathbf{u}, p, \theta) = (p + \theta) \mathbf{u} \cdot \mathbf{n}$ . The adjoint model is then given by  
342 (13)-(16) where all the partial derivative of  $\mathcal{J}_\Omega$  are zero and the following  
343 boundary conditions hold

$$\begin{aligned}
\mathbf{u}^* &= \mathbf{0}, \quad \partial_n \theta^* = 0 && \text{on } \Gamma_1 \cup \Gamma_2, \\
u_t^* &= 0, \quad \theta^* = 0, \quad u_n^* = -u_n && \text{on } \Gamma_{in}, \\
\theta^* u_n + Re^{-1} Pr^{-1} k_\tau(\alpha) \partial_n \theta^* &= -u_n && \text{on } \Gamma_{out}, \\
p^* + \theta^* \theta + Re^{-1} \nabla_n \mathbf{u}^* \cdot \mathbf{n} + u_n^* u_n &= -(\theta + p) && \text{on } \Gamma_{out}.
\end{aligned} \tag{22}$$

345 As said in section 4, we propose to solve the topology optimization problem  
346 with a steepest descent algorithm where the gradient is computed thanks to  
347 the adjoint method. The main flow of the algorithm for the topology opti-  
mization is described in Table 3. The forward problem and the optimization

---

Step 0. Initialization: set all the constants $Re, Ri, Pr$
Step 1. Solve the forward problem (2),(3) problem with the Finite Volume Method
Step 2. Compute objective and constraint values
Step 3. Compute sensitivities by adjoint method
Step 4. Evaluate the optimality condition. If a stopping criterion is met, terminate the calculation
Step 5. Update design variables $\alpha$ with $\alpha_{k+1} = -\nabla \mathcal{J}_{k+1} + \beta_{k+1}^{PR} \alpha_k$ and return to step 1

---

Table 3: Algorithm of topology optimization

348

349 processes are implemented using OpenFOAM [44]. In Step 5, the design

350 variables are evaluated by using the conjugated-gradient descent direction  
351 method associated to Polack-Ribiere method  $\beta_{k+1}^{PR} = \frac{\nabla \mathcal{J}_{k+1}^T (\nabla \mathcal{J}_{k+1} - \nabla \mathcal{J}_k)}{\nabla \mathcal{J}_k^T \nabla \mathcal{J}_k}$ .

352 After optimization process, several results are compared as optimized de-  
353 signs, velocity magnitude distribution, temperature distribution, and lastly,  
354 we focus on fluid-solid interfaces. We also presents the objective function  
355 values for each simulation case. Finally, we investigate to vary the slope  
356 abscissa and the curve of sigmoid functions in order to evaluate their impact  
357 on the optimization problem. That corresponds to consider different values  
358 of  $\alpha_0$  and  $\tau$ . In our study,  $\alpha_0$  will vary in  $\{25, 50, 100\}$  which correspond re-  
359 spectively to  $\{\alpha_{max}/8, \alpha_{max}/4, \alpha_{max}/2\}$  and  $\tau$  will vary in  $\{0.3, 0.7, 1.0\}$ . In  
360 order to perform topology optimization, the  $\alpha$  term may take a finite value  
361 between 0 and  $\alpha_{max} = 200$ .

### 362 5.1. Studied Cases

363 Two representative cases of topology optimization in fluid dynamics are  
364 investigated : the single pipe case and the bend pipe case. Figure 2 shows  
365 schematic illustrations of the two studied cases.

#### 366 5.1.1. Design of a single pipe.

367 We present here a benchmark numerical example used recently by Marck  
368 [15] to illustrate the viability and efficiency of the methodology presented in  
369 this paper. The design domain consists in a cavity of side  $L$ . The compu-  
370 tational domain is discretized using  $100 \times 100$  elements. The inlet flow is  
371 located in the center of the west edge for a length of  $L/5$ . The outlet flow  
372 is lining up with it on the east edge. The rest of west and east edges are  
373 assumed to be adiabatic and will be called  $\Gamma_2$ , whereas the whole north and

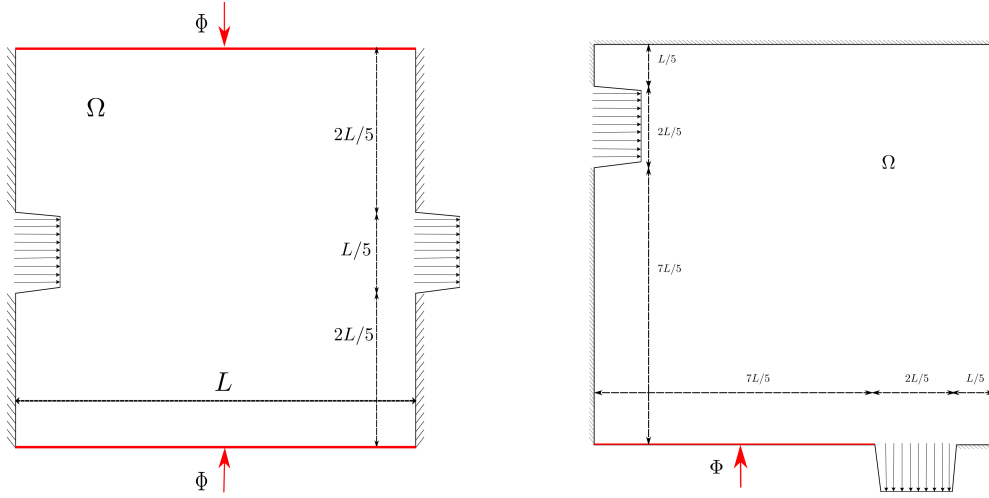


Figure 2: Single pipe and Bend pipe

374 south walls are subjected to a constant flux of temperature, called thereafter  
 375  $\Gamma_1$ .

376 *5.1.2. Design of a bend pipe.*

377 The second example has been studied by several authors as [4] and [15].  
 378 The computational domain is also square-shaped, with a adimensional side  
 379  $L = 1$ , and the design grid is made of  $100 \times 100$  elements. The inlet flow  
 380 is located at  $L/5$  of the west edge. The outlet flow is located on the south  
 381 edge, at  $L/5$  from the east edge. The length of both flow boundary conditions  
 382 is set to  $2L/5$ . The part of the south edge located on the left of the outlet  
 383 is subjected to a constant flux of temperature and will be called  $\Gamma_1$ . The rest  
 384 of the edges are assumed to be adiabatic and will be designated as  $\Gamma_2$ .

385 *5.1.3. Physical parameters.*

386 For the two representative cases studied in this section, the inlet flow  
 387 is prescribed with a constant horizontal velocity equal to  $U_0 = 0.04 \text{ m.s}^{-1}$

388 used in Reynolds number. The inlet temperature is fixed to  $\theta = 0$ . The  
 389 outflow temperature condition is fixed to a zero gradient, as well as the  
 390 velocity gradient. Hot walls are subject to a constant flux of temperature  
 391  $\partial_n \theta = -1$ . The Reynolds number based on the characteristic dimension of  
 392 the inlet length and the average inlet velocity is  $Re = 500$ . The Rayleigh  
 393 number is fixed to  $Ra = 5 \times 10^5$ . Prandtl number is set to 0.71. Grashof  
 394 number has the following expression  $Gr = Ra/Pr$ . Adimensional thermal  
 395 parameters are the following:  $k_s = (9.88 \times 10^{-5} \text{ m}^2 \cdot \text{s}^{-1}) / (2.25 \times 10^{-5} \text{ m}^2 \cdot \text{s}^{-1})$   
 396 [(Aluminum/Air) Diffusivities],  $k_f = 1$ .

## 397 5.2. Results

398 The aim of the problems defined above is to determine the optimal design  
 399 that connects the inlet to the outlet of the cavity and that minimizes the  
 400 objective function  $\mathcal{J}$  subjected to the constraints (2) and (3).

401 First of all, we compute  $\mathcal{J}$  without optimization for the two studied  
 402 cases and we obtained respectively for the single pipe  $J = 9.07 \cdot 10^{-3}$  and for  
 403 the bend pipe  $J = 6.38 \cdot 10^{-3}$ . Compared to values issued from optimization  
 404 process and referenced in Figure 8 and 11, they are reduced between 23% and  
 405 42% for the single pipe case, and between 18% and 19% for the bend pipe.  
 406 So, the new method investigated here allows to both minimize the power  
 407 dissipated by the fluid through the domain  $\Omega$  and maximize the recoverable  
 408 thermal power from the domain to the fluid.

409 Secondly, we plot velocity magnitude for various  $\alpha_0$  and  $\tau$  values. After  
 410 optimization of  $\mathcal{J}$  in the single pipe, we obtain an optimized design in which  
 411 fluid is transported through the single pipe in a straight pipe (see Figure 3). It  
 412 is quite closed to the optimized design obtained by [15] when the mechanical



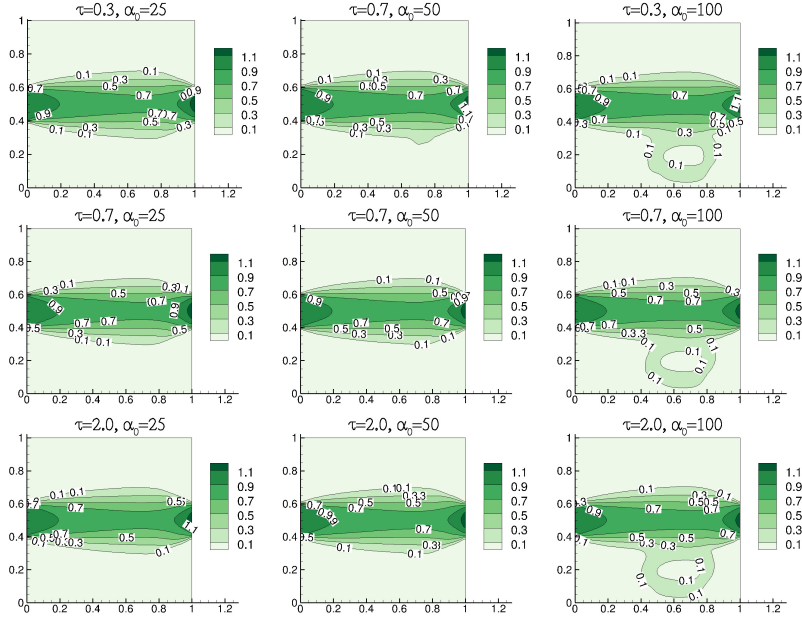


Figure 3: Velocity magnitude distribution in the single pipe

413 power and thermal power are considered with the same weighting coefficient.  
 414 Similarly, fluid circulation forms a half circle whose center would be the left  
 415 bottom corner of the square domain in the optimized bend pipe (see Figure  
 416 4). Contrary to the literature in fluid mechanics, the optimized path obtained  
 417 for the flow is not as straight as possible. It can be explained by the fact that  
 418 we consider here a heat transfer problem. So, the optimized pipe is larger  
 419 and form a real bend because the fluid flow moves away from hot regions.

420 Thereafter, we compared distribution of the parameter  $\alpha$  in the domain  
 421 for the two studied cases (see Figures 6 and 9). We recall that values less  
 422 than  $\alpha_0$  are considered as fluid regions. We can notice that this method  
 423 based on  $\alpha_0$  affected some values of  $\alpha$  to the fluid regions, which would not  
 424 be possible with other methods of the literature. Indeed, generally with the

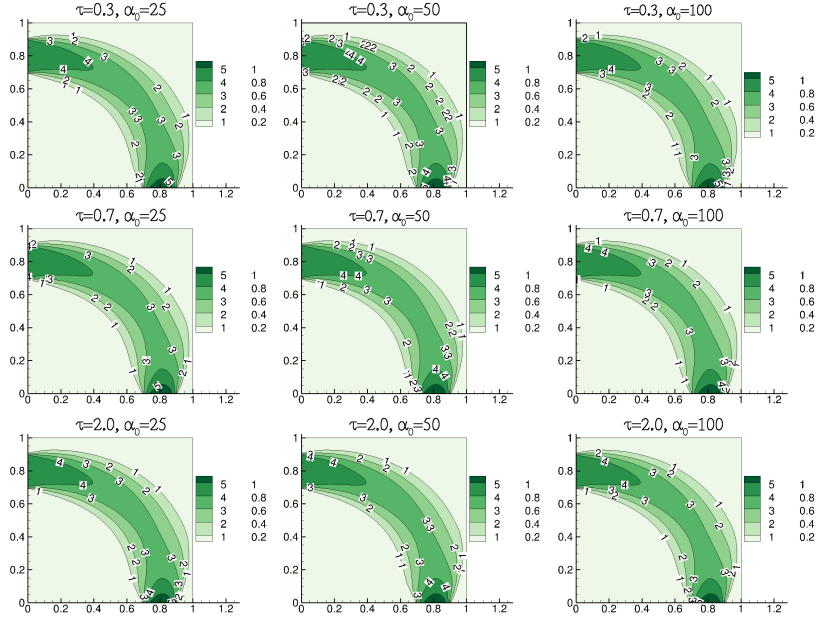


Figure 4: Velocity magnitude distribution in the bend pipe

425 function (6), only value of  $\alpha$  equal to zero is considered as fluid element. As  
 426 can be seen on these figures, intermediate values for the fluid region exist.  
 427 For example, we have different values of  $\alpha$  between 0 and  $\alpha_0$  at right-top  
 428 corner in Figure 9, or below the straight pipe for  $\alpha_0 = 25$  in Figure 6.

429 As expected, the interpolation function  $h_\tau(\alpha)$  suppresses these intermedi-  
 430 ate values noticed previously and affects these volume elements to pure fluid  
 431 volume elements. Indeed, Figures 7 and 10 show the optimized designs for  
 432 various  $\alpha_0$  and  $\tau$  numbers. Compared to  $\alpha$  distribution (Figures 6 and 9),  
 433 they demonstrate that the interpolation function  $h_\tau(\alpha)$  suppresses interme-  
 434 diate values corresponding neither to pure fluid nor to pure solid regions by  
 435 affecting them to solid regions or fluid regions. However, in the bend pipe, a  
 436 porous domain still exists near hot regions. Although fluid does not circulate

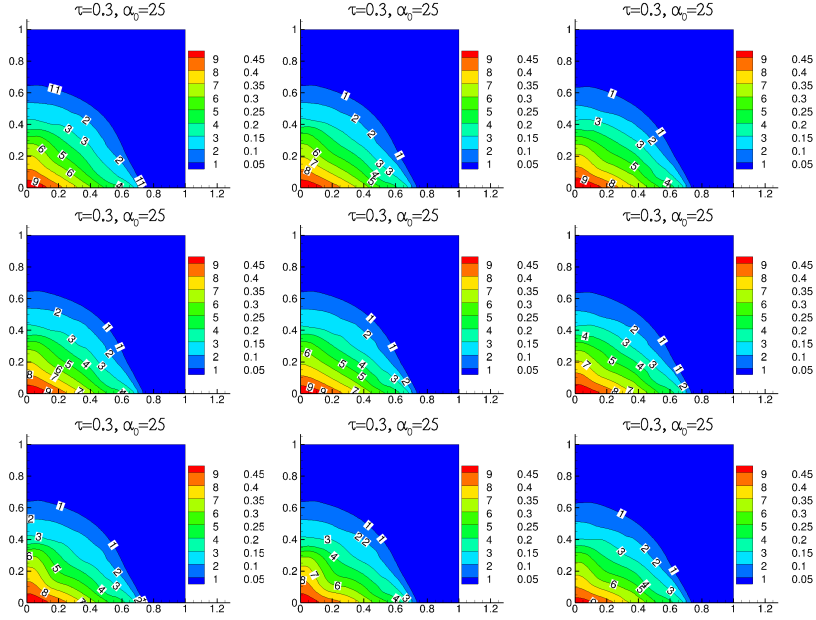


Figure 5: Temperature distribution in the bend pipe

437 in this porous domain, its existence contributes to increasing thermal power  
 438 recovered by the fluid. That is illustrated in Figure 5.

439 At last, it can be seen that thermal conductivity is well distributed, con-  
 440 tinuously, between fluid and solid regions. Figure 8 represents the distri-  
 441 bution of material conductivity after the optimization process in the single  
 442 pipe case. Compared to Figure 7, we observe that frontier between the two  
 443 regions, fluid and solid, is well-established. For the bend pipe case, although  
 444 intermediate regions exist,  $k_\tau$  interpolation function allows to affect to this  
 445 porous domain a conductivity closed to the fluid diffusivity (see Figure 11).  
 446 Therefore, the function  $k_\tau(\alpha)$  provides a continuous transition between the  
 447 solid and the fluid domains.

448 *5.2.1. Influence of  $\alpha_0$ .*

449 It can clearly be seen from Figures 7 and 10 that different designs are ob-  
 450 tained for the different values of  $\alpha_0$ . As detailed in section 3,  $\alpha_0$  is depending  
 451 of  $\alpha_{max}$  in order to control the percentage of fluid in  $\Omega$ . This parameter has  
 452 a significant impact on the quantity of material adding in the domain and  
 453 that can be illustrated by computing the proportion  $Q_t$  of material added in  
 454 the domain  $\Omega$  as follows :

$$455 \quad Q_t = \frac{\int_{\Omega} h_{\tau}(\alpha) d\Omega}{\alpha_{max} V_{tot}} \text{ where } V_{tot} \text{ is the total volume of } \Omega \quad (23)$$

456 Calculation results are referenced in Figures 7 and 10. In the single pipe,  
 457 when  $\alpha_0$  increases, quantity of material diminishes between 27.5% and 28%.  
 458 That means fluid domain is more important and that contributes to the  
 459 increase of  $\mathcal{J}_2$ . So, that will influence the value of  $\mathcal{J}$ . In the bend pipe,  
 460 when  $\alpha_0$  increases, quantity of material also diminishes between 3.6% and  
 461 16%. So, fluid domain is relatively more important than solid domain but  
 462 not enough to significantly impact on the value of  $\mathcal{J}$ , hence, the small effect  
 463 on values of  $\mathcal{J}$  (see Figure 11). So, parameter  $\alpha_0$  has a significant incidence  
 464 on the proportion of fluid domain relatively to solid domain and therefore,  
 465 can influence the value of the functional objective.

466 Futhermore, we observe a recirculation zone for  $\alpha_0 = 100$  in the example  
 467 of the optimized single pipe (see Figure 3). That can also be explained by  
 468 the value of  $\alpha_0$ . Indeed, larger  $\alpha_0$  means that the proportion of fluid elements  
 469 compared to  $\alpha_{max}$  is bigger and thus fluid regions are more important. When  
 470  $\alpha_0 = 100$ , that means  $\alpha_0 = \alpha_{max}/2$ , the algorithm affects some value of  
 471  $\alpha \in [0, 100]$  corresponding to fluid domain and so impacts on the size of fluid  
 472 regions relatively to the solid regions in the domain.

473 *5.2.2. Influence of  $\tau$ .*

474 It can clearly be seen from Figures 7 and 10 that different designs are  
475 obtained for the different values of  $\tau$ . The parameter  $\tau$  in the interpolation  
476 functions  $h_\tau(\alpha)$  and  $k_\tau$  indicates the stiffness of the curve of the functions. It  
477 does not seem to impact significantly the optimization results. However, we  
478 can observe in Figure 11 that the frontier between the fluid and the solid is  
479 sharper when  $\tau = 2.0$ . In Figure 8, for  $\tau = 0.3$ ,  $\tau = 0.7$ , we can notice residual  
480 material in fluid domain while they do not exist anymore when  $\tau = 2.0$ . As  
481 the same, the porous region identified in the bend pipe case between the  
482 main flow and the hot plate is defined mostly with the fluid diffusivity by the  
483 interpolation function. This impact does not influence significantly the result  
484 of the optimization computation when we compared their value in Figure 11  
485 but it improves the thermal transition between fluid and solid regions. So,  
486 parameter  $\tau$  seems to improve the thermal conductivity distribution during  
487 the process optimization.

488 **6. Conclusion**

489 This study shows that the new interpolation technique based on sigmoid  
490 function is a viable method. Design material is well interpolated since inter-  
491 mediate regions between solid and fluid are suppressed hence giving a frontier  
492 between the two regions which is well-defined. Besides, the size of transition  
493 zones between fluid and solid regions is explicitly computed. The result is  
494 that the new interpolation functions reduce their size comparatively to the  
495 RAMP interpolation function. Therefore, the transition between fluid and  
496 solid is improving thanks to this sigmoid function for the effective conduc-

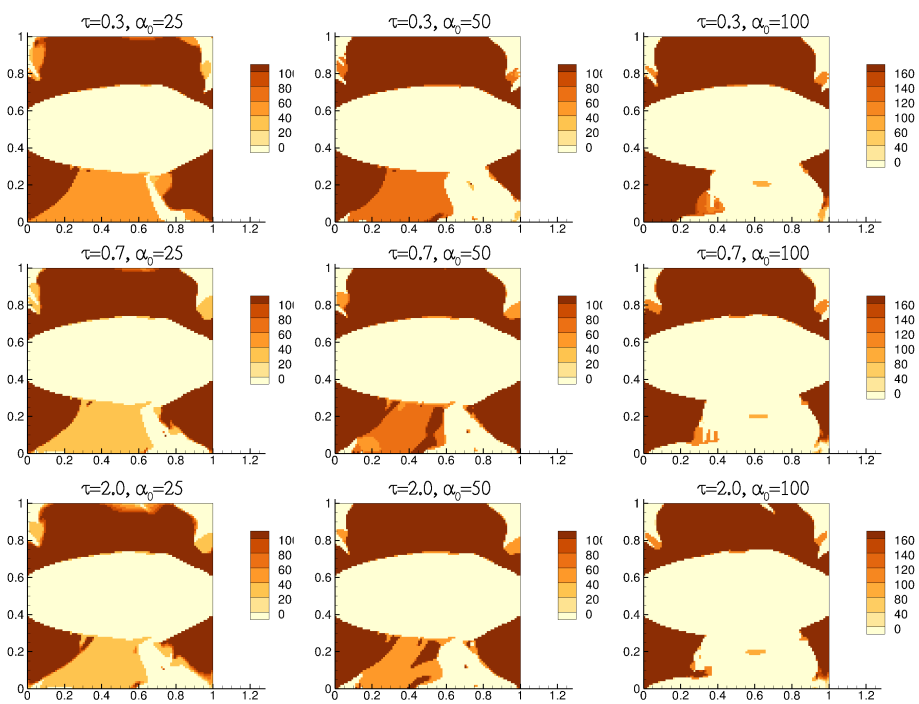


Figure 6:  $\alpha$  distribution in the single pipe

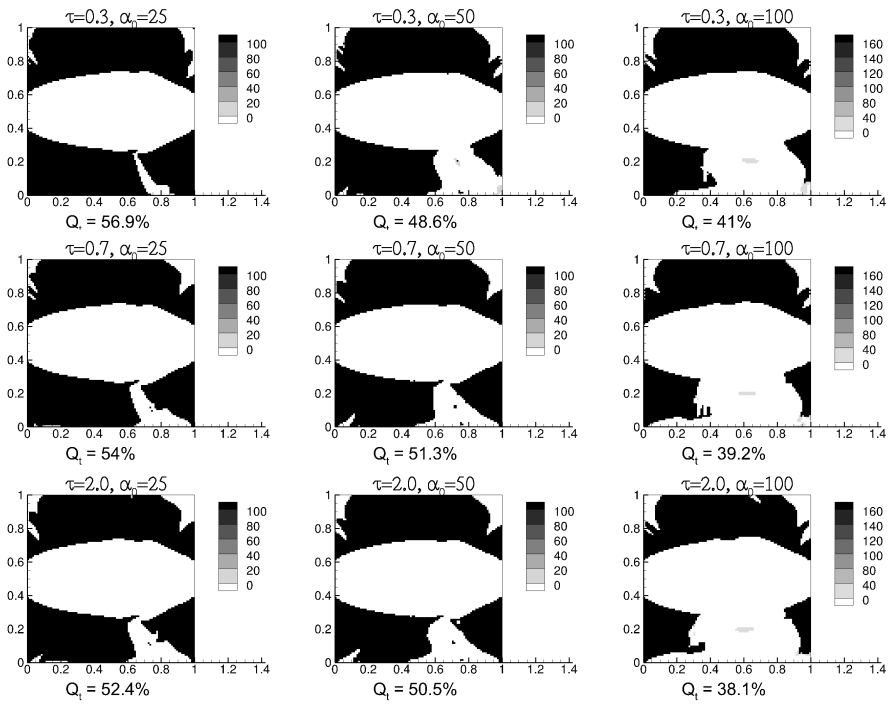


Figure 7:  $h_\tau(\alpha)$  distribution in the single pipe and proportion of material  $Q_t$  added in the domain after optimization

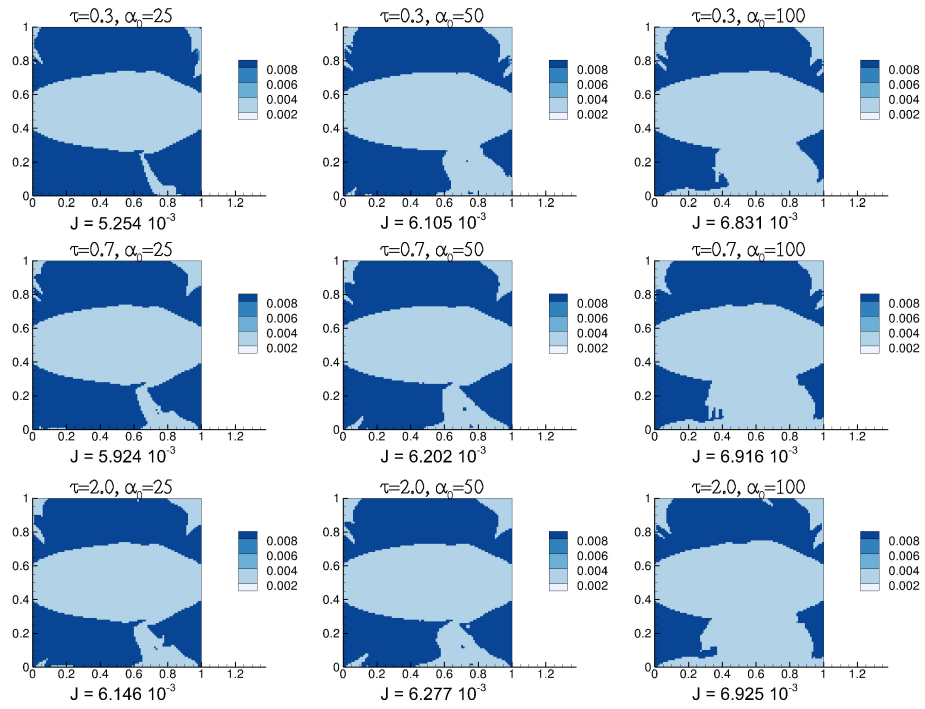


Figure 8:  $k_\tau(\alpha)$  distribution in the single pipe for different values of  $\tau$  and  $\alpha_0$  and value of functional objectif  $\mathcal{J}$  after optimization



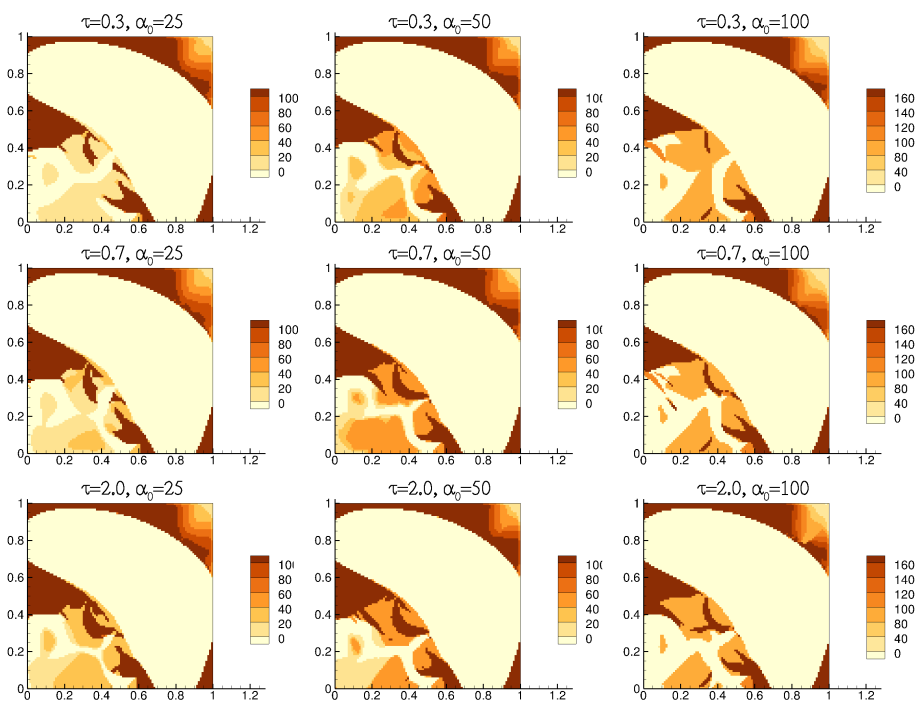


Figure 9:  $\alpha$  distribution in the bend pipe

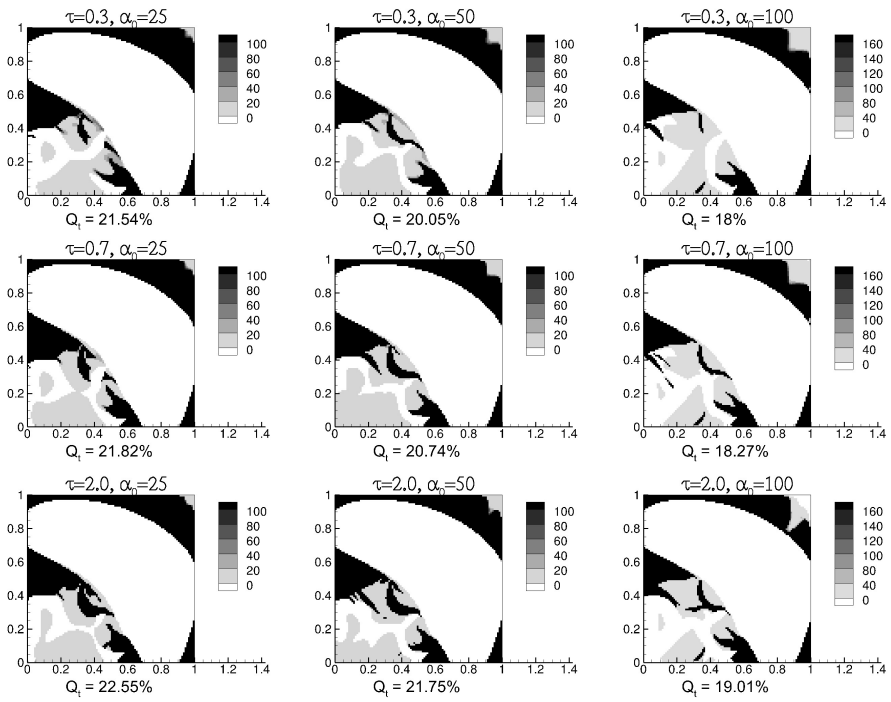


Figure 10:  $h_\tau(\alpha)$  distribution in the bend pipe and proportion of material  $Q_t$  added in the domain after optimization

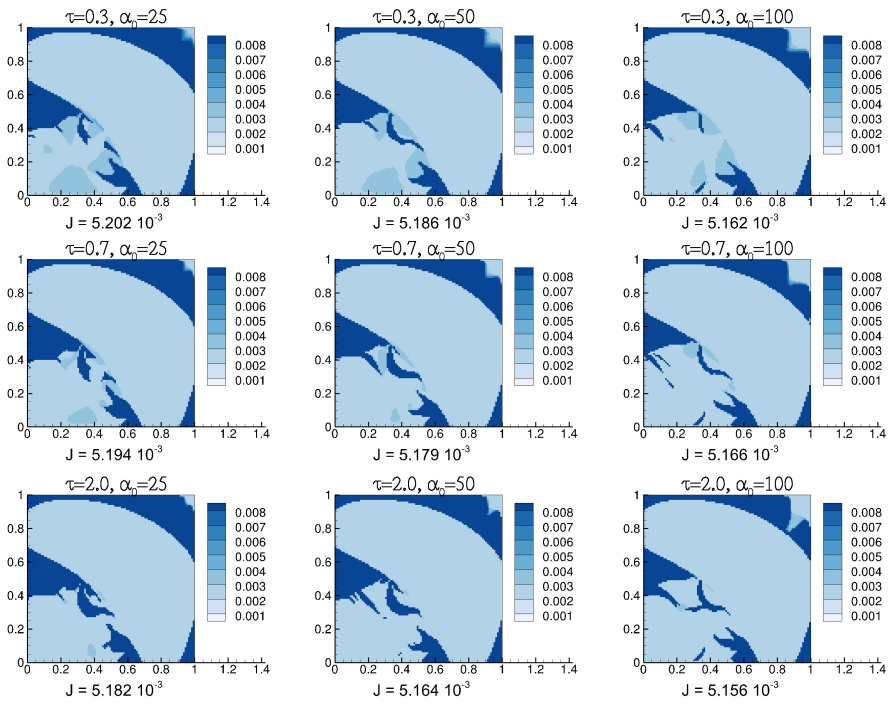


Figure 11:  $k_\tau(\alpha)$  distribution in the bend pipe and value of functional objective  $\mathcal{J}$  after optimization

497 tivity. Computation of the size of transition zones could also be applied to  
498 other interpolation functions in order to evaluate the well-definition of the  
499 frontier between fluid and solid regions. This new interpolation technique do  
500 not require any regularization techniques since no checkerboards have been  
501 identified for the two studied cases. Two new parameters have been intro-  
502 duced : the abscissa slope of the sigmoid function  $\alpha_0$  and the curve of the  
503 sigmoid function  $\tau$ . Their impact has been also investigated.  $\alpha_0$  has a sig-  
504 nificant impact on the quantity of material added in the domain during the  
505 optimization process. The larger  $\alpha_0$  is the bigger will be the fluid domain.  
506 The parameter  $\tau$  has a small impact on the results of optimization. For the  
507 studied cases, the impact was seen after interpolation of thermal conduc-  
508 tivity. Elements of porous domain have been affected to fluid diffusivity or  
509 solid conductivity, and therefore, the thermal transition between material is  
510 improved. So, this study has examined a new interpolation technique to deal  
511 with fluid-porous media interfaces for topology optimization of heat transfer  
512 problems. Further research might explore several type of flow regimes in  
513 optimization problems using these interpolation functions. Another possible  
514 area of future research would be to investigate new cost functions.

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