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Abstract. Topology optimization for fluid flow aims at finding the location of a porous medium minimizing a cost functional under constraints given by the Navier-Stokes equations. The location of the porous media is usually taken into account by adding a penalization term $\alpha\mathbf{u}$, where α is a kinematic viscosity divided by a permeability and $\mathbf{u}$ is the velocity of the fluid. The fluid part is obtained when $\alpha = 0$ while the porous (solid) part is defined for large enough α since this formally yields $\mathbf{u} = \mathbf{0}$. The main drawback of this method is that only solid that does not let the fluid to enter, that is *perfect* solid, can be considered. In this paper, we propose to use the porosity of the media as optimization parameter hence to minimize some cost function by finding the location of a porous media. The latter is taken into account through a singular perturbation of the Navier-Stokes equations for which we prove that its weak-limit corresponds to an interface fluid-porous medium problem modeled by the Navier-Stokes-Darcy equations. This model is then used as constraint for a topology optimization problem. We give necessary condition for such problem to have at least an optimal solution and derive first order necessary optimality condition. This paper ends with some numerical simulations, for Stokes flow, to show the interest of this approach.

Key words. Navier-Stokes-Darcy model, Porous media, PDE-constrained optimization, Topology optimization.

AMS subject classifications. 35Q30, 76S05, 49J20, 76D55

1. Introduction. Many practical problem in applied science and engineering aim at finding the location of a solid to improve a desired physical behavior. Such kind of problem, usually referred as topology optimization, can be written as a constrained optimization problem where the parameter we look for is the location of the solid, the constraint are given by the governing equation of the fluid and the function one wants to minimize describe a physical effect.

Several techniques have been proposed to use the location of the porous media as optimization parameter. For instance, the topological asymptotic expansion [2, 3] consider the solid as holes in the computational domain. The topological gradient is then the first order term of the asymptotic expansion of the cost functional as the size of the hole vanishes. One of the advantage of this method is that the solid is clearly taken into account in the optimization process. However, this method requires the computation of an asymptotic expansion which can be a hard task that depend on the shape of the hole, the dimension of space and the problem under study. For these reasons, the penalization method that has been introduced in [12] has been favored over the past few years.

The method introduced by T. Borrvall and J. Petersson [12] is based on a penalization model [8] that define the location of the porous media with a friction term $\alpha(\rho)\mathbf{u}$ where $\mathbf{u}$ is the velocity of the fluid and $\alpha(\rho) = \psi(\rho^{-1} - 1)$ is (physically) an inverse of a permeability by a kinematic viscosity with ψ a dimensioned coefficient. The fluid part of the optimal design is then defined when $\rho = 1$ while $\rho = 0$ gives the solid part since, at least formally, the velocity of the fluid goes to zero in these zones. As pointed out by A. Evgrarov [22], the parameter ρ belong to a non-convex nor weakly* closed set thus both the mathematical study and the numerical approximation of the topology optimization problem is difficult. It has then been proposed (see e.g. [12, 45]) to work with an interpolation of α hence to use $\rho \in [0, 1]$ as design parameter. This interpolation technique has then been used extensively in several work [45, 12, 23, 22, 4, 27, 37] (see also the review paper [18]) and extended to other physical modeling like heat transfert in fluid [40, 36, 41] or natural convection [1].

The major drawback of the technique introduced by T. Borrvall & J. Petersson is that the penalization model is relevant only for *perfect* solid that is a material that does not allow the fluid to enter. In this paper,

[1]Affiliation: Physique et Ingénierie Mathématique pour l'Énergie et l'Environnement (PIMENT), Université de la Réunion, 2 rue Joseph Wetzell, 97490 Sainte-Clotilde.
E-mail: {Alain.Bastide, Pierre – Henri.Cocquet, Delphine.Ramalingom}@univ-reunion.fr.

we want to propose an alternative method to solve topology optimization problem that allows the optimal design to be characterized by a porous medium and not only a perfect solid. Following the same idea as the penalization method for which we have a solid as α goes to infinity, we would like to use the porosity α as optimization parameter and get the Navier-Stokes equations when $\alpha = 0$ while, for positive α, we wish to have the Darcy law for porous medium with porosity α. Our new penalized model is going to be given by a singular perturbation of the Navier-Stokes problem whose weak limit is the Navier-Stokes-Darcy model. The techniques used to build our penalized model are based on some vanishing-viscosity method similar to those developed in [26, 25] for advection-diffusion problem.

This paper is organized as follow: We first derive the singular perturbation model used latter as constraint in our topology optimization problem. We show next that its weak-limit is a fluid-porous media interface problem given by a Navier-Stokes-Darcy problem. We then use this penalization model as constraint for a topology optimization problem and give necessary condition for the existence of optimal design as well as first order necessary optimality condition. This paper ends with some numerical simulation for Stokes flow to show the potential application of the approach developed in the present paper.

2. The penalized Navier-Stokes-Darcy model. Let Ω be a bounded open set of $\mathbb{R}^d$, $d = 3$ [2], with Lipschitz boundary $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ ($\Gamma_1 \cap \Gamma_2 = \emptyset$) whose outward unitary normal is denoted by $\mathbf{n}$. We assume that Ω can be written as $\Omega = \Omega_{\mathrm{p}} \cup \Omega_f$, where the subscript f denotes the fluid part and p the porous part of the computational domain. We assume that one has the incompressible Navier-Stokes equation in Ω_f and the Darcy law for porous medium with porosity α in Ω_{p}. This reads

$$\left\{ \begin{array}{rll} -\nu\Delta\mathbf{u}_f + \frac{1}{\rho_f}\nabla p_f + (\mathbf{u}_f\cdot\nabla)\mathbf{u}_f & = \mathbf{f} & \text{in } \Omega_f, \\ \operatorname{div}\mathbf{u}_f & = 0 & \text{in } \Omega_f \\ \alpha\mathbf{u}_{\mathrm{p}} + \frac{1}{\rho_f}\nabla p_{\mathrm{p}} & = \mathbf{f} & \text{in } \Omega_{\mathrm{p}}, \\ \operatorname{div}\mathbf{u}_{\mathrm{p}} & = 0 & \text{in } \Omega_{\mathrm{p}}, \end{array} \right. \tag{2.1}$$

with suitable interface condition on $(\partial\Omega_f \cap \partial\Omega_{\mathrm{p}}) \setminus \Gamma$ and some boundary condition on Γ that will be specified latter. In (2.1), ρ_f is the constant fluid volumetric mass and is equal to 1 in the sequel of the paper.

Our first goal is to obtain Equation (2.1) as the limit of a model involving a single equation giving the Navier-Stokes equation for $\alpha = 0$ and, at the limit, the Darcy law when $\alpha > 0$. The latter can be achieved using function of the form $\exp(-\tau\alpha)$ which converges, as $\tau \to +\infty$, toward the indicator function of the set $\{x \in \Omega \mid \alpha(x) = 0\} := \Omega_f$. Therefore, multiplying the second order term and the non-linear term of Equation (2.1) by $\exp(-\tau\alpha)$ is going, at least formally, to do the job. However, this method yields a problem whose mathematical analysis can be difficult and this is the reason why we propose below another approach based on the following formula

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u}\times(\nabla\times\mathbf{u}), \tag{2.2}$$

Using (2.2), one can write the incompressible Navier-Stokes equations as follow

$$-\nu\Delta\mathbf{u} + \nabla p + \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u}\times(\nabla\times\mathbf{u}) = \mathbf{f} \text{ in } \Omega_f.$$

We then consider the following penalized incompressible Navier-Stokes system with Darcy law for porous medium

$$\left\{ \begin{array}{rll} -\operatorname{div}\left(\nu e^{-\tau\alpha}\nabla\mathbf{u} - p\mathbb{I}_3 - \frac{1}{2}e^{-2\tau\alpha}|\mathbf{u}|^2\mathbb{I}_3\right) - e^{-2\tau\alpha}\mathbf{u}\times(\nabla\times\mathbf{u}) + \alpha\mathbf{u} & = f & \text{in } \Omega, \\ \operatorname{div}(\mathbf{u}) & = 0 & \text{in } \Omega, \\ \mathbf{u} & = 0 & \text{on } \Gamma_1, \\ \nu e^{-\tau\alpha}\partial_{\mathbf{n}}\mathbf{u} - (p + \frac{1}{2}e^{-2\tau\alpha}|\mathbf{u}|^2)\mathbf{n} & = \varphi & \text{on } \Gamma_2, \end{array} \right. \tag{2.3}$$

[2]The case $\Omega \subset \mathbb{R}^2$ is discussed along the paper and does not yield additional difficulties.

where div denotes alternatively the divergence of a tensor or a vector field, $\alpha \in L^{\infty}(\Omega)$ is a kinematic viscosity divided by a permeability, $f \in L^2(\Omega)^3$ is a density of forces and $\varphi \in L^2(\Gamma_2, \mathbb{R}^3)$ is a surface source term. We now derive formally the limiting problem as $\tau \to +\infty$. Note that we recover the Navier-Stokes equations on Ω_f since α vanishes. Also taking the formal limit as $\tau \to +\infty$ of the solution to Problem (2.3), we get that it satisfies

$$\operatorname{div} \mathbf{u} = 0, \ \alpha \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega_{\mathrm{p}},$$

which is the Darcy law for porous media. Problem (2.3) is therefore a penalization model for an interface fluid-porous media problem.

Remark 2.1 (Two-dimensional case). *In a two-dimensional setting, one only has to use that*

$$(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{2}\nabla |\mathbf{u}|^2 = \begin{pmatrix} \mathbf{u}_2 \partial_2 \mathbf{u}_1 - \mathbf{u}_2 \partial_1 \mathbf{u}_2 \\ \mathbf{u}_1 \partial_1 \mathbf{u}_2 - \mathbf{u}_1 \partial_2 \mathbf{u}_1 \end{pmatrix}.$$

Therefore, we simply replace $\nabla \times$ by the above formula in Problem (2.3) to get the penalized Navier-Stokes-Darcy equation in $\mathbb{R}^2$.

We start by studying the existence of solution to Problem (2.3). Let $\mathbb{X}$ be the subspace of $H^1(\Omega)^3$ defined by

$$\mathbb{X} := \left\{ \psi \in H^1(\Omega)^3 \mid \psi = 0 \text{ on } \Gamma_1 \right\}.$$

It is a Hilbert space for the norm $\|.\|_{\mathbb{X}} := \|.\|_{H^1(\Omega)^d}$. We recall the following Poincaré inequality that holds if $|\Gamma_1| > 0$

$$\exists C_{\mathrm{P}} > 0 \text{ such that } \forall \varphi = (\varphi_j) \in \mathbb{X}, \ \|\varphi_j\|_{L^2(\Omega)} \le C_{\mathrm{P}} \|\nabla \varphi_j\|_{L^2(\Omega)^3}, \tag{2.4}$$

where the constant C_{P} only depends on Ω. In the sequel, we use the following notation

$$\forall \mathbf{u} \in \mathbb{X}, \ \|\nabla \mathbf{u}\|^2_{L^2(\Omega)^d} := \sum_{j=1}^{d} \|\nabla \mathbf{u}_j\|^2_{L^2(\Omega)^d},$$

which is actually the norm of the Jacobian matrix of the vector field $x \in \Omega \mapsto \mathbf{u}(x)$.

A variational formulation for Problem (2.3) is given below

$$\begin{aligned} &\textit{Find } (\mathbf{u}, p) \in \mathbb{X} \times L^2(\Omega) \textit{ such that} \\ &\forall \mathbf{v} \in \mathbb{X}, \ a(\mathbf{u}, \mathbf{v}) + N(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \int_{\Omega} \mathbf{f}(x) \cdot \mathbf{v}(x) dx + \int_{\Gamma_2} \varphi \cdot \mathbf{v} d\sigma, \\ &\forall q \in L^2(\Omega), \ b(\mathbf{u}, q) = 0, \end{aligned} \tag{2.5}$$

where, using subscript j to denote the components of a vector, the multilinear forms are defined as follow

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \nu \sum_{j=1}^{d} \int_{\Omega} \mathrm{e}^{-\tau\alpha} \nabla \mathbf{u}_j \cdot \nabla \mathbf{v}_j dx + \int_{\Omega} \alpha \mathbf{u} \cdot \mathbf{v} dx, \\ N(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= -\int_{\Omega} \frac{\mathrm{e}^{-2\tau\alpha}}{2} \mathbf{u} \cdot \mathbf{v} \operatorname{div}(\mathbf{w}) + \mathrm{e}^{-2\tau\alpha} \mathbf{u} \times (\nabla \times \mathbf{v}) \cdot \mathbf{w} dx, \\ b(\mathbf{v}, p) &= -\int_{\Omega} \operatorname{div} \mathbf{v} p \, dx. \end{aligned} \tag{2.6}$$

The equivalence between the weak-formulation (2.5) and Problem (2.3) is classical and can be found in [10]. We show below the continuity of the trilinear form associated to the non-linear term N.

LEMMA 2.2. *The trilinear form N is continuous on $\mathbb{X}^3$*

$$|N(\mathbf{u},\mathbf{v},\mathbf{w})| \leq C_{\mathrm{N}} \|\nabla \mathbf{u}\|_{L^2(\Omega)^3} \|\nabla \mathbf{v}\|_{L^2(\Omega)^3} \|\nabla \mathbf{w}\|_{L^2(\Omega)^3},$$

where $C_{\mathrm{N}} > 0$ depends only on Ω.

Proof. We recall the following compact Sobolev embedding theorem (see e.g [24, 44])

$$H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega). \tag{2.7}$$

Let $(\mathbf{u},\mathbf{v},\mathbf{w}) \in \mathbb{X}^3$. Using (2.7), that $\mathrm{e}^{-\tau\alpha} \leq 1$ and Holder inequality then give

$$\begin{aligned}|N(\mathbf{u},\mathbf{v},\mathbf{w})| &\leq 2\|\mathbf{u}\|_{L^4(\Omega)^3} \|\nabla \mathbf{v}\|_{L^2(\Omega)^3} \|\mathbf{w}\|_{L^4(\Omega)^3} \\ &\leq C_{\mathrm{N}} \|\nabla \mathbf{u}\|_{L^2(\Omega)^3} \|\nabla \mathbf{v}\|_{L^2(\Omega)^3} \|\nabla \mathbf{w}\|_{L^2(\Omega)^3}\end{aligned}$$

which proves that N is continuous. $\square$

We first study the existence and uniqueness of $(\mathbf{u}_\tau, p_\tau)$ satisfying Problem (2.3) for finite $0 < \tau$.

THEOREM 2.3. *Suppose that the following set of assumptions hold*

1. *$\alpha \in L^\infty(\Omega)$ is almost everywhere positive.*
2. *The domain Ω can be written as $\Omega = \Omega_{\mathrm{p}} \cup \Omega_f$ where $\Omega_f = \{x \in \Omega \mid \alpha(x) = 0\}$ satisfies $\overline{\Omega_f} \cap \Gamma_1 \neq \emptyset$.*
3. *There exists $\Omega_{\mathrm{p}} \subset \Omega$ and $\alpha_{\min} > 0$ such that*

$$a.e\ x \in \Omega_{\mathrm{p}},\ \alpha_{\min} \leq \alpha(x).$$

Then for any $0 < \tau \leq M$, there exists at least one (u_τ, p_τ) satisfying Problem (2.5) and the following a priori estimates

$$\begin{aligned}\|\mathbf{u}_\tau\|_{H^1(\Omega)^3} &\leq C_1 \frac{\mathrm{e}^{M\|\alpha\|_{L^\infty(\Omega)}}}{\nu} \left(\|\mathbf{f}\|_{L^2(\Omega)^3} + \|\varphi\|_{L^2(\Gamma_2)^3}\right), \\ \|p_\tau\|_{L^2(\Omega)} &\leq \beta^{-1} C_2 \left(1 + \|\alpha\|_{L^\infty(\Omega)} \frac{C_1 \mathrm{e}^{M\|\alpha\|_{L^\infty(\Omega)}}}{\nu}\right) \left(\|\mathbf{f}\|_{L^2(\Omega)^d} + \|\varphi\|_{L^2(\Gamma_2)^3}\right) \\ &\quad + \beta^{-1} \frac{C_{\mathrm{N}} C_1^2 \mathrm{e}^{M\|\alpha\|_{L^\infty(\Omega)}}}{\nu^2} \left(\|\mathbf{f}\|_{L^2(\Omega)^3} + \|\varphi\|_{L^2(\Gamma_2)^3}\right)^2,\end{aligned}$$

where C_1, C_2 are positive constant that does not depend on ν nor on M.

Proof. We first prove the upper bound for the solution to Problem (2.3). Note that the non-linear term satisfy

$$\begin{aligned}N(\mathbf{u},\mathbf{u},\mathbf{u}) &= -\int_\Omega \mathrm{e}^{-2\tau\alpha} \left(|\mathbf{u}|^2 \mathrm{div}(\mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{u}) \cdot \mathbf{u}\right) dx \\ &= \int_\Omega \mathrm{e}^{-2\tau\alpha} (\nabla \times \mathbf{u}) \cdot (\mathbf{u} \times \mathbf{u}) dx = 0.\end{aligned}$$

Taking $\mathbf{v} = \mathbf{u}$ in the weak formulation (2.5), using Cauchy-Schwarz, Poincaré (2.4) and trace inequality and that $\mathrm{e}^{-\tau\alpha} \geq \mathrm{e}^{-M\|\alpha\|_{L^\infty(\Omega)}}$ then show that

$$\|\mathbf{u}\|_{H^1(\Omega)^3} \leq C_1 \frac{\mathrm{e}^{M\|\alpha\|_{L^\infty(\Omega)}}}{\nu} \left(\|\mathbf{f}\|_{L^2(\Omega)^3} + \|\varphi\|_{L^2(\Gamma_2)^3}\right), \tag{2.8}$$

where C_1 is a constant that does not depend on ν nor M. To get the estimate on the pressure, one needs the next *inf-sup* condition whose proof can be found in [11] Theorem 2.1, p. 338.

$$\exists \beta > 0 \text{ such that } \forall q \in L^2(\Omega),\ \sup_{\mathbf{v} \in \mathbb{X}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbb{X}}} \geq \beta \|q\|_{L^2(\Omega)}. \tag{2.9}$$

Since $p \in L^2(\Omega)$ satisfies Problem (2.5), we obtain

$$b(\mathbf{v}, p) = \int_\Omega \mathbf{f}(x) \cdot \mathbf{v}(x) dx - \int_{\Gamma_2} \varphi \mathbf{v} \cdot \mathbf{n} d\sigma - a(\mathbf{u}, \mathbf{v}) - N(\mathbf{u}, \mathbf{u}, \mathbf{v}), \ \forall \mathbf{v} \in \mathbb{X}.$$

Using the *inf-sup* condition (2.9) together with Lemma 2.2 then give

$$\begin{aligned} \|p_\tau\|_{L^2(\Omega)} &\leq \beta^{-1} \sup_{\mathbf{v} \in \mathbb{X}} \frac{\int_\Omega \mathbf{f}(x) \cdot \mathbf{v}(x) dx - \int_{\Gamma_2} \varphi \mathbf{v} \cdot \mathbf{n} d\sigma - a(\mathbf{u}_\tau, \mathbf{v}) - N(\mathbf{u}_\tau, \mathbf{u}_\tau, \mathbf{v})}{\|\mathbf{v}\|_{\mathbb{X}}}, \\ &\leq \beta^{-1} \left(\|\mathbf{f}\|_{L^2(\Omega)^3} + \|\varphi\|_{L^2(\Gamma_2)^3} + (1 + \|\alpha\|_{L^\infty(\Omega)}) \|\mathbf{u}_\tau\|_{H^1(\Omega)^3} \right) \\ &\quad + \beta^{-1} C_{\mathrm{N}} \|\mathbf{u}_\tau\|^2_{H^1(\Omega)^3} \end{aligned}$$

which yields the desired result thanks to (2.8).

We now prove the existence of at least one solution to Problem (2.3). First we introduce the kernel of the linear form $q \in L^2(\Omega) \mapsto b(., q) \in \mathbb{X}'$ namely

$$\mathbb{V} := \{ \mathbf{v} \in \mathbb{X} \mid \mathrm{div}(\mathbf{v}) = 0 \text{ in } \Omega \}.$$

It is worth noting that $\mathbf{u} \in \mathbb{V}$. We introduce the non-linear mapping $\Phi : \mathbb{V} \to \mathbb{V}'$ defined by

$$\langle \Phi(\mathbf{u}), \mathbf{v} \rangle := a(\mathbf{u}, \mathbf{v}) + N(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \int_\Omega \mathbf{f}(x) \cdot \mathbf{v}(x) dx - \int_{\Gamma_2} \varphi \cdot \mathbf{v} d\sigma.$$

We then need to prove that there exists $\mathbf{u} \in \mathbb{V}$ such that $\Phi(\mathbf{u}) = 0$ to get the existence of $\mathbf{u}$ satisfying Problem (2.5). Since $N(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0$, we infer

$$\langle \Phi(\mathbf{u}), \mathbf{u} \rangle \geq C_1 \|\mathbf{u}\|^2_{H^1(\Omega)^3} - C_2 (\|\mathbf{f}\|_{L^2(\Omega)^3} + \|\varphi\|_{L^2(\Gamma_2)^3}) \|\mathbf{u}\|_{H^1(\Omega)^3},$$

where $C_j > 0$ does not depend on ν. One thus gets that $\langle \Phi(\mathbf{u}), \mathbf{u} \rangle$ is non-negative on the sphere with radius $r_0 := C_2 (\|\mathbf{f}\|_{L^2(\Omega)^3} + \|\varphi\|_{L^2(\Gamma_2)^3}) / C_1$. Since $\mathbb{V}$ is separable (as a closed subspace of the Banach space $H^1(\Omega)^3$), there exists an increasing sequence of finite dimensional subspaces $\mathbb{V}_n$ such that $\cup_n \mathbb{V}_n$ is dense in $\mathbb{V}$. From Brower's fixed point Theorem, there exists $\mathbf{u}_n \in \mathbb{V}_n$ such that $\|\mathbf{u}_n\|_{H^1(\Omega)^3} \leq r_0$ and

$$\forall \mathbf{v}_n \in \mathbb{V}_n, \ \langle \Phi(\mathbf{u}_n), \mathbf{v}_n \rangle = 0.$$

The sequence $\mathbb{V}_n$ being increasing, one has

$$\forall m \geq n, \ \forall \mathbf{v}_m \in \mathbb{V}_m, \ \langle \Phi(\mathbf{u}_n), \mathbf{v}_m \rangle = 0.$$

Since $(\mathbf{u}_n)_n \subset \mathbb{V}$ is bounded in $\mathbb{X}$ and the embedding $\mathbb{X} \subset L^4(\Omega)^3$ is compact (see (2.7)), there exists a subsequence (still denoted $(\mathbf{u}_n)_n$) that converges weakly in $\mathbb{X}$ and strongly in $L^4(\Omega)^3$ to some $\mathbf{u} \in \mathbb{N}$. The definition of the weak convergence then yield

$$\lim_{n \to \infty} a(\mathbf{u}_n, \mathbf{v}_m) = a(\mathbf{u}, \mathbf{v}_m).$$

Since $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in $\mathbb{X}$ and $\mathbf{u}_n \to \mathbf{u}$ strongly in $L^4(\Omega)^3$, one gets that $\mathbf{u}_n \cdot \mathbf{v}_m \to \mathbf{u} \cdot \mathbf{v}_m$ strongly in $L^2(\Omega)^3$ and thus

$$\lim_{n \to \infty} N(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}_m) = N(\mathbf{u}, \mathbf{u}, \mathbf{v}_m).$$

From the above limits, we infer

$$\forall m \in \mathbb{N}, \ \forall \mathbf{v}_m \in \mathcal{V}_n, \ \langle \Phi(\mathbf{u}), \mathbf{v}_m \rangle = 0.$$

The density of $\cup_n \mathbb{V}_n$ into $\mathbb{V}$ then gives $\langle \Phi(\mathbf{u}), \mathbf{v}\rangle = 0$ for any $\mathbf{v} \in \mathbb{V}$ and thus the existence of a velocity field for Problem (2.3) is proven. The existence of $p \in L^2(\Omega)$ then follow from the *inf-sup* condition (2.9) and Lemma 4.1 p.38 from [28]. □

REMARK 2.4. *The existence proof also works for $\mathbf{f} \in \mathbb{X}^*$ and $\varphi \in H^{-1/2}(\Gamma_2, \mathbb{R}^3)$ and one only has to change the norm in the upper bound of the solution.*

We do not detail here the proof of the unicity since this can be done using standard technique from [44, 24, 10] and we are mainly interested in existence when doing PDE-constrained optimization problems. Nevertheless, this demand ν to be large enough.

3. Derivation of the limiting problem. We are going to characterize the limit of $(\mathbf{u}_\tau, p_\tau)$ satisfying Problem (2.3) as $\tau \to +\infty$. We assume from now that the data satisfies $\mathbf{f} \in L^2(\Omega)^3$ and $\varphi = 0$. We require $\mathbf{f} \in L^2(\Omega)^3$ because this regularity is useful to give sense to the interface condition as shown in the proof of Theorem 3.2. We also consider vanishing φ because this greatly simplify the derivation of some bound proved in the sequel. Finally, we assume that the fluid part $\Omega_f \subset \overline{\Omega}$ and the porous part $\Omega_{\rm p} \subset \overline{\Omega}$ are bounded open set with Lipschitz boundary since we are going to write (2.3) as an interface problem.

Using that $\Omega = \Omega_f \cup \Omega_{\rm p}$ and omitting the τ-dependance to lighten the expression, Problem (2.3) can be written as the following transmission problem

$$\left\{\begin{array}{rl}
-\mathrm{div}\left(\nu \mathrm{e}^{-\tau\alpha}\nabla \mathbf{u}_{\rm p} - p_{\tau,\rm p}\mathbb{I}_3 - \frac{1}{2}\mathrm{e}^{-2\tau\alpha}|\mathbf{u}_{\rm p}|^2\mathbb{I}_3\right) - \mathrm{e}^{-2\tau\alpha}\mathbf{u}_{\rm p}\times(\nabla\times\mathbf{u}_{\rm p}) + \alpha\mathbf{u}_{\rm p} & = \mathbf{f} \text{ in } \Omega_{\rm p},\\
\mathrm{div}(\mathbf{u}_{\rm p}) & = 0 \text{ in } \Omega_{\rm p},\\
-\mathrm{div}\left(\nu\nabla\mathbf{u}_f - p_f\mathbb{I}_3 - \frac{1}{2}|\mathbf{u}_f|^2\mathbb{I}_3\right) - \mathbf{u}_f\times(\nabla\times\mathbf{u}_f) & = \mathbf{f} \text{ in } \Omega_f,\\
\mathrm{div}(\mathbf{u}_f) & = 0 \text{ in } \Omega_f,\\
\mathbf{u}_f & = 0 \text{ on } \Gamma_1\cap\partial\Omega_f,\\
\mathbf{u}_{\rm p} & = 0 \text{ on } \Gamma_1\cap\partial\Omega_{\rm p},\\
\nu\partial_{\mathbf{n}}\mathbf{u}_f - (p_f + \frac{1}{2}|\mathbf{u}_f|^2)\mathbf{n} & = 0 \text{ on } \Gamma_2\cap\partial\Omega_f,\\
\mathrm{e}^{-\tau\alpha}\nu\partial_{\mathbf{n}}\mathbf{u}_{\rm p} - (p_{\rm p} + \frac{1}{2}\mathrm{e}^{-2\tau\alpha}|\mathbf{u}_{\rm p}|^2)\mathbf{n} & = 0 \text{ on } \Gamma_2\cap\partial\Omega_{\rm p},\\
\mathbf{u}_f - \mathbf{u}_{\rm p} & = 0 \text{ on } \Gamma_{p,f},\\
T_\tau[(\mathbf{u}_{\rm p}, p_{\rm p}), (\mathbf{u}_f, p_f)] & = 0 \text{ on } \Gamma_{p,f},
\end{array}\right. \tag{3.1}$$

where $\Gamma_{p,f} = (\partial\Omega_{\rm p}\cap\partial\Omega_f)\setminus\Gamma$ is the fluid-porous media interface. The application T_τ defines the flux transmission conditions. Chosing $\mathbf{n}_{\rm p\mathit{f}}$ as unitary normal to $\Gamma_{{\rm p},f}$, for instance inward to $\Omega_{\rm p}$ and outward to Ω_f, the latter is given by

$$\begin{array}{rcl}
T_\tau[(\mathbf{u}_{\rm p}, p_{\rm p}), (\mathbf{u}_f, p_{\rm p})] & = & (\nu\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_{\rm p} - p_{\rm p}\mathbb{I}_3 - \frac{1}{2}\mathrm{e}^{-2\tau\alpha}|\mathbf{u}_{\rm p}|^2\mathbb{I}_3)\mathbf{n}_{{\rm p}f}\\
& - & \left(\nu\nabla\mathbf{u}_f - p_f\mathbb{I}_3 - \frac{1}{2}|\mathbf{u}_f|^2\mathbb{I}_3\right)\mathbf{n}_{{\rm p}f}.
\end{array} \tag{3.2}$$

Note that Theorem 2.3 ensures the existence of $(\mathbf{u}_{s,\tau}, p_{\tau,\rm p}, \mathbf{u}_{f,\tau}, p_{\tau,f}) \in H^1(\Omega_{\rm p})^3\times L^2(\Omega_{\rm p})\times H^1(\Omega_f)^3\times L^2(\Omega_f)^3$ that satisfy the transmission Problem (3.1-3.2). Before giving the variational formulation of Problem (3.1-3.2), we introduce the following functional spaces

$$\begin{array}{l}
\mathbb{X}_f := \left\{\psi \in H^1(\Omega_f)^3 \mid \psi = 0 \text{ on } \Gamma_1\cap\partial\Omega_f\right\}.\\
\mathbb{X}_{\rm p} := \left\{\psi \in H^1(\Omega_{\rm p})^3 \mid \psi = 0 \text{ on } \Gamma_1\cap\partial\Omega_{\rm p}\right\}.
\end{array}$$

The variational formulation then reads

$$\begin{array}{l}
\textit{Find } (\mathbf{u}_{\tau,f}, p_{\tau,f}, u_{\rm p}, p_{\tau,\rm p}) \in \mathbb{X}_f\times L^2(\Omega_f)\times\mathbb{X}_{\rm p}\times L^2(\Omega_{\rm p}) \textit{ such that}\\
\mathbf{u}_f - \mathbf{u}_{\rm p} = 0 \text{ on } \Gamma_{p,f} \text{ and}\\
a_f(\mathbf{u}_{\tau,f}, \mathbf{v}_f) + N_f(\mathbf{u}_{\tau,f}, \mathbf{u}_{\tau,f}, \mathbf{v}_f) + b_f(\mathbf{v}_f, p_{\tau,f}) + a_{\rm p}(\mathbf{u}_{\tau,\rm p}, \mathbf{v}_{\rm p}) + N_{\rm p}(\mathbf{u}_{\tau,\rm p}, \mathbf{u}_{\tau,\rm p}, \mathbf{v}_{\rm p})\\
+b(\mathbf{v}_{\rm p}, p_{\tau,\rm p}) - \displaystyle\int_{\Gamma_{{\rm p},f}}\left(\nu\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_{\tau,\rm p} - p_{\tau,\rm p}\mathbb{I}_3 - \frac{1}{2}\mathrm{e}^{-2\tau\alpha}|\mathbf{u}_{\tau,\rm p}|^2\mathbb{I}_3\right)\mathbf{n}_{{\rm p}f}\cdot(\mathbf{v}_{\rm p} - \mathbf{v}_f)d\sigma\\
= \displaystyle\int_{\Omega_f}\mathbf{f}(x)\cdot\mathbf{v}_f(x)dx + \int_{\Omega_{\rm p}}\mathbf{f}(x)\cdot\mathbf{v}_{\rm p}(x)dx\\
b_f(\mathbf{u}_{\tau,f}, q_f) + b_{\rm p}(\mathbf{u}_{\tau,\rm p}, q_{\rm p}) = 0,\\
\forall(\mathbf{v}_f, \mathbf{v}_{\rm p}) \in \mathbb{X}_f\times\mathbb{X}_{\rm p}\ \forall(q_f, q_{\rm p}) \in L^2(\Omega_f)\times L^2(\Omega_{\rm p}).
\end{array} \tag{3.3}$$

Above the $a_f, a_{\rm p}, N_f, N_{\rm p}, b_f, b_{\rm p}$ are defined as in (2.6) with Ω replaced by Ω_f and $\Omega_{\rm p}$ according to their subscript.

3.1. First estimates. We now prove some estimates that are sharper than those from Theorem 2.3 and are going to be useful to pass to the limit in Problem (3.1-3.2).

THEOREM 3.1. *Assume that the assumptions of Theorem 2.3 hold and that $\alpha|_{\Omega_{\rm p}} \in \mathrm{Lip}(\Omega_{\rm p})$. Then the solution to Problem (3.1) satisfy the following bounds*

$$\begin{aligned}
&\|\mathbf{u}_{f,\tau}\|_{L^2(\Omega_f)^3} + \|\mathbf{u}_{{\rm p},\tau}\|_{L^2(\Omega_{\rm p})^3} \le \frac{C}{\gamma}\|\mathbf{f}\|_{L^2(\Omega)^3},\\
&\|\nabla\mathbf{u}_{f,\tau}\|_{L^2(\Omega_f)^3} \le \frac{C}{\gamma}\|\mathbf{f}\|_{L^2(\Omega)^3},\\
&\left\|\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_{{\rm p},\tau}\right\|_{L^2(\Omega_{\rm p})^3} \le \mathrm{e}^{-\tau\alpha_{\min}/2}\frac{C}{\sqrt{\nu\gamma}}\|\mathbf{f}\|_{L^2(\Omega)^3},\\
&\|p_{{\rm p},\tau}\|_{L^2(\Omega_{\rm p})} + \|p_{f,\tau}\|_{L^2(\Omega_f)^3} \le B,
\end{aligned}$$

where $\gamma = \min(C_{\rm P}^{-1}\nu, \alpha_{\min})$, $C>0$ is a generic constant that does not depend on τ nor on ν and $B>0$ is a constant that does not depend on τ.

Proof. We omit the τ dependence to lighten the overall expressions. We now choose $(\mathbf{v}_f, \mathbf{v}_{\rm p}) \in \mathbb{X}_f \times \mathbb{X}_{\rm p}$ defined as $\mathbf{v}_f = \mathbf{u}_f$, $\mathbf{v}_{\rm p} = \mathbf{u}_{\rm p}$ in the variational formulation (3.3). Using then that $N_t(\mathbf{w}_t, \mathbf{w}_t, \mathbf{w}_t) = 0$ for any $\mathbf{w}_t \in \mathbb{X}_f$ with subscript $t \in \{f, {\rm p}\}$ referring to the fluid or the porous part of Ω, one gets

$$\begin{aligned}
&\sum_{j=1}^{3}\left(\int_{\Omega_{\rm p}}\nu\mathrm{e}^{-\tau\alpha}|\nabla\mathbf{u}_{s,j}|^2dx + \int_{\Omega_f}\nu|\nabla\mathbf{u}_{f,j}|^2dx\right) + \int_{\Omega_{\rm p}}\alpha|\mathbf{u}_{\rm p}|^2dx\\
&= \int_{\Omega_{\rm p}}\mathbf{f}\cdot\mathbf{u}_{\rm p}dx + \int_{\Omega_f}\mathbf{f}\cdot\mathbf{u}_f dx
\end{aligned}$$

Using Cauchy-Schwarz, trace and Poincaré inequalities (2.4), we obtain

$$\alpha_{\min}\|\mathbf{u}_{\rm p}\|^2_{L^2(\Omega_{\rm p})^3} + C_{\rm P}^{-1}\nu\|\mathbf{u}_f\|^2_{L^2(\Omega_f)^3} \le \|\mathbf{f}\|_{L^2(\Omega)^3}\left(\|\mathbf{u}_{\rm p}\|_{L^2(\Omega_{\rm p})^3} + \|\mathbf{u}_{\rm p}\|_{L^2(\Omega_{\rm p})^3}\right).$$

Since $2(a^2+b^2) \ge (a+b)^2$ for any $a, b \in \mathbb{R}^+$, we get the existence of a constant $C>0$ that does not depend on ν nor τ such that

$$\|\mathbf{u}_f\|_{L^2(\Omega_f)^3} + \|\mathbf{u}_{\rm p}\|_{L^2(\Omega_{\rm p})^3} \le \frac{C}{\min(C_{\rm P}^{-1}\nu, \alpha_{\min})}\|\mathbf{f}\|_{L^2(\Omega)^3}, \tag{3.4}$$

which yields the first estimate. The second estimates is easily obtained thanks to Poincaré inequality (2.4). For the last estimate, we infer from a Cauchy-Schwarz inequality and (3.4) that

$$\sum_{j=1}^{3}\int_{\Omega_{\rm p}}\nu\mathrm{e}^{-\tau\alpha}|\nabla\mathbf{u}_{s,j}|^2dx \le \frac{C}{\min(C_{\rm P}^{-1}\nu, \alpha_{\min})}(\|\mathbf{f}\|_{L^2(\Omega)^3})^2. \tag{3.5}$$

Multiplying (3.5) by $\mathrm{e}^{-\tau\alpha_{\min}}$ and using that $\alpha_{\min} \le \alpha(x)$ for any $x \in \Omega_{\rm p}$, we obtain

$$\begin{aligned}
\nu\left\|\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_{\rm p}\right\|^2_{L^2(\Omega_{\rm p})^3} &= \nu\sum_{j=1}^{3}\int_{\Omega_{\rm p}}\mathrm{e}^{-2\tau\alpha}|\nabla\mathbf{u}_{{\rm p},j}|^2dx\\
&\le \nu\mathrm{e}^{-\tau\alpha_{\min}}\int_{\Omega_{\rm p}}\mathrm{e}^{-\tau\alpha}|\nabla\mathbf{u}_{{\rm p},j}|^2dx \qquad (3.6)\\
&\le \nu\frac{C\mathrm{e}^{-\tau\alpha_{\min}}}{\nu\min(C_{\rm P}^{-1}\nu, \alpha_{\min})}\|\mathbf{f}\|^2_{L^2(\Omega)^3}
\end{aligned}$$

and the bounds for the velocity are therefore complete. To get the bound for the pressure $(p_{\mathrm{p}}, p_f) \in L^2(\Omega_{\mathrm{p}}) \times L^2(\Omega_f)$, we use the following formula which comes from the weak formulation (2.5) together with the transmission condition (3.2)

$$\begin{aligned}
b(\mathbf{v}, p) &= -\int_{\Omega_{\mathrm{p}}} p_{\mathrm{p}} \mathrm{div}\mathbf{v}\, dx - \int_{\Omega_f} p_f \mathrm{div}\mathbf{v}\, dx \\
&= \int_{\Omega} \mathbf{f}(x)\cdot\mathbf{v}(x) dx - a_{\mathrm{p}}(\mathbf{u}_{\mathrm{p}}, \mathbf{v}_{\mathrm{p}}) - N_{\mathrm{p}}(\mathbf{u}_{\mathrm{p}}, \mathbf{u}_{\mathrm{p}}, \mathbf{v}_{\mathrm{p}}) \\
&\quad - a_f(\mathbf{u}_f, \mathbf{v}_f) - N_f(\mathbf{u}_f, \mathbf{u}_f, \mathbf{v}_f),
\end{aligned}$$

where $\mathbf{v} \in \mathbb{X}$ and $(\mathbf{v}_{\mathrm{p}}, \mathbf{v}_f) = (\mathbf{v}|_{\Omega_{\mathrm{p}}}, \mathbf{v}|_{\Omega_f})$. Following the proof of Lemma 2.2 and using that $\mathrm{e}^{-2\tau\alpha} = (\mathrm{e}^{-\tau\alpha})^2$ we obtain the next refined estimates

$$\begin{aligned}
|N_{\mathrm{p}}(\mathbf{u}_{\mathrm{p}}, \mathbf{u}_{\mathrm{p}}, \mathbf{v}_{\mathrm{p}})| &\le C_{\mathrm{N}} \left\|(\mathrm{e}^{-\tau\alpha}\mathbf{u}_{\mathrm{p}})\right\|^2_{H^1(\Omega_{\mathrm{p}})^3} \|\mathbf{v}_{\mathrm{p}}\|_{H^1(\Omega_{\mathrm{p}})^3}\,, \\
|a_{\mathrm{p}}(\mathbf{u}_{\mathrm{p}}, \mathbf{v}_{\mathrm{p}})| &\le \left\|\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_{\mathrm{p}}\right\|_{L^2(\Omega_{\mathrm{p}})^3} \|\nabla\mathbf{v}_{\mathrm{p}}\|_{L^2(\Omega_{\mathrm{p}})} \\
&\qquad + \|\alpha\|_{L^\infty(\Omega)} \|\mathbf{u}_{\mathrm{p}}\|_{L^2(\Omega_{\mathrm{p}})^3}) \|\nabla\mathbf{v}_{\mathrm{p}}\|_{L^2(\Omega_{\mathrm{p}})^3}\,.
\end{aligned} \tag{3.7}$$

Since $\alpha|_{\Omega_{\mathrm{p}}} \in \mathrm{Lip}(\Omega_{\mathrm{p}})$, it is almost everywhere differentiable with (essentially) bounded derivatives thanks to Rademacher Theorem. One thus have $\nabla\mathrm{e}^{-\tau\alpha} = -\tau\mathrm{e}^{-\tau\alpha}\nabla\alpha$ on Ω_{p}. Using now that $0 \le \mathrm{e}^{-\tau\alpha(x)} \le \mathrm{e}^{-\tau\lambda_{\min}}$ for almost every $x \in \Omega$, we obtain

$$\begin{aligned}
\left\|(\mathrm{e}^{-\tau\alpha}\mathbf{u}_{\mathrm{p}})\right\|_{H^1(\Omega_{\mathrm{p}})^3} &= \left\|\nabla(\mathrm{e}^{-\tau\alpha}\mathbf{u}_{\mathrm{p}})\right\|_{L^2(\Omega_{\mathrm{p}})^3} + \left\|\mathrm{e}^{-\tau\alpha}\mathbf{u}_{\mathrm{p}}\right\|_{L^2(\Omega_{\mathrm{p}})^3} \\
&\le \left\|\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_{\mathrm{p}}\right\|_{L^2(\Omega_{\mathrm{p}})^3} + \left(1 + \tau\|\nabla\alpha\|_{L^\infty(\Omega_{\mathrm{p}})}\right)\left\|\mathrm{e}^{-\tau\alpha}\mathbf{u}_{\mathrm{p}}\right\|_{L^2(\Omega_{\mathrm{p}})^3} \\
&\le \left\|\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_{\mathrm{p}}\right\|_{L^2(\Omega_{\mathrm{p}})^3} + \mathrm{e}^{-\tau\lambda_{\min}}\left(1 + \tau\|\nabla\alpha\|_{L^\infty(\Omega_{\mathrm{p}})}\right)\|\mathbf{u}_{\mathrm{p}}\|_{L^2(\Omega_{\mathrm{p}})^3}\,.
\end{aligned} \tag{3.8}$$

Using now bounds (3.6-3.4) and that $\tau > 0$ is large enough, we infer

$$\begin{aligned}
|N_{\mathrm{p}}(\mathbf{u}_{\mathrm{p}}, \mathbf{u}_{\mathrm{p}}, \mathbf{v}_{\mathrm{p}})| &\le C\left(\mathrm{e}^{-\tau\lambda_{\min}/2}\left(1 + \tau\|\nabla\alpha\|_{L^\infty(\Omega_{\mathrm{p}})}\right)\right)^2 \left(\|\mathbf{f}\|_{L^2(\Omega)^3}\right)^2 \|\mathbf{v}_{\mathrm{p}}\|_{H^1(\Omega_{\mathrm{p}})^3} \\
&\le C\tau^2\mathrm{e}^{-\tau\lambda_{\min}}\left(\|\mathbf{f}\|_{L^2(\Omega)^3}\right)^2 \|\mathbf{v}_{\mathrm{p}}\|_{H^1(\Omega_{\mathrm{p}})^3}
\end{aligned} \tag{3.9}$$

where $C > 0$ is a generic constant that does not depend on τ nor on $\mathbf{f}$. The upper bound (3.9), the *inf-sup* condition (2.9) together with (3.6-3.4-3.7) then give the existence of $B > 0$ that does not depend on τ such that

$$\|p_{\mathrm{p}}\|_{L^2(\Omega_{\mathrm{p}})} + \|p_f\|_{L^2(\Omega_f)} = \|p\|_{L^2(\Omega)} \le \beta^{-1} \sup_{\mathbf{v}\in\mathbb{X}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbb{X}}} \le B,$$

where $(p|_{\Omega_{\mathrm{p}}}, p|_{\Omega_f}) = (p_{\mathrm{p}}, p_f)$. □

3.2. Characterization of the limiting problem for $(\mathbf{u}_\tau, p_\tau)$. From Theorem 3.1, we infer that the sequence $((\mathbf{u}_{\mathrm{p},\tau}, p_{\mathrm{p},\tau}, \mathbf{u}_{f,\tau}, p_{f,\tau}))_\tau$ is bounded, uniformly with respect to τ in $L^2(\Omega_{\mathrm{p}})^3 \times L^2(\Omega_{\mathrm{p}}) \times H^1(\Omega_f)^3 \times L^2(\Omega_f)$ and thus there exists subsequence and $(\mathbf{u}_{\mathrm{p}}, p_{\mathrm{p}}, \mathbf{u}_f, p_f) \in L^2(\Omega_{\mathrm{p}})^3 \times L^2(\Omega_{\mathrm{p}}) \times H^1(\Omega_f) \times L^2(\Omega_f)$ such that

$$\begin{aligned}
&\mathbf{u}_{s,\tau} \to \mathbf{u}_{\mathrm{p}} \text{ weakly in } L^2(\Omega_{\mathrm{p}})^3, \\
&\mathbf{u}_{f,\tau} \to \mathbf{u}_f \text{ weakly in } H^1(\Omega_f)^3 \text{ and strongly in } L^2(\Omega_f)^3, \\
&p_{\mathrm{p},\tau} \to p_{\mathrm{p}} \text{ weakly in } L^2(\Omega_{\mathrm{p}}) \text{ and } p_{f,\tau} \to p_f \text{ weakly in } L^2(\Omega_f), \\
&\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_{s,\tau} \to 0 \text{ strongly in } L^2(\Omega_{\mathrm{p}}).
\end{aligned} \tag{3.10}$$

The limiting problem satisfied by $(\mathbf{u}_\mathrm{p}, p_\mathrm{p}, \mathbf{u}_f, p_f)$ is given in the next result.

THEOREM 3.2. *The distributions* $(\mathbf{u}_\mathrm{p}, p_\mathrm{p}, \mathbf{u}_f, p_f) \in L^2(\Omega_\mathrm{p})^3 \times L^2(\Omega_f) \times H^1(\Omega_f)^3 \times L^2(\Omega_f)$ *defined as the weak limits of* $(\mathbf{u}_\tau, p_\tau)$ *satisfying Problem (3.1,3.2) satisfy the following equation*

$$\left\{\begin{array}{rll} \nabla p_\mathrm{p} + \alpha \mathbf{u}_\mathrm{p} & = \mathbf{f} & \text{in } \mathcal{D}'(\Omega_\mathrm{p}, \mathbb{R}^3), \\ \mathrm{div}(\mathbf{u}_\mathrm{p}) & = 0 & \text{in } \mathcal{D}'(\Omega_\mathrm{p}), \\ -\mathrm{div}\left(\nu \nabla \mathbf{u}_f - p_f \mathbb{I}_3\right) + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f & = \mathbf{f} & \text{in } \mathcal{D}'(\Omega_f, \mathbb{R}^3), \\ \mathrm{div}(\mathbf{u}_f) & = 0 & \text{in } \mathcal{D}'(\Omega_f), \\ \mathbf{u}_f & = 0 & \text{in } H^{1/2}(\Gamma_1 \cap \partial\Omega_f, \mathbb{R}^3), \\ \mathbf{n} \cdot \mathbf{u}_\mathrm{p} & = 0 & \text{in } H^{-1/2}(\Gamma_1 \cap \partial\Omega_\mathrm{p}) \\ \mathbf{u}_f \cdot \mathbf{n}_{\mathrm{p}f} & = \mathbf{u}_\mathrm{p} \cdot \mathbf{n}_{\mathrm{p}f} & \text{in } H^{-1/2}(\Gamma_{\mathrm{p},f}), \\ \nu \partial_\mathbf{n} \mathbf{u}_f - (p_f + \frac{1}{2}|\mathbf{u}_f|^2)\mathbf{n} & = 0 & \text{in } H^{-1/2}(\Gamma_2 \cap \partial\Omega_f, \mathbb{R}^3), \\ -p_\mathrm{p} & = 0 & \text{in } H^{1/2}(\Gamma_2 \cap \partial\Omega_\mathrm{p}), \\ \left(\nu \nabla \mathbf{u}_f - p_f \mathbb{I}_3 - \frac{1}{2}|\mathbf{u}_f|^2 \mathbb{I}_3\right) \mathbf{n}_{\mathrm{p}f} & = -p_\mathrm{p} \mathbf{n}_{\mathrm{p}f} & \text{in } \left(W^{1-1/q,q}(\Gamma_{\mathrm{p},f}, \mathbb{R}^3)\right)', \end{array}\right. \tag{3.11}$$

where $q = 2p/(2-p)$ *for any* $p \in [1, 3/2]$.

Proof. The four first equations of (3.11) follow from (3.10) by passing to the limit in the variational formulation (3.3). To pass to the limit in the non-linear term N_f, we use the continuity of N_f together with strong L^2 and weak H^1 convergence of $(u_{\tau,f})_\tau$. For the non-linear term N_p, it goes to zero thanks to (3.9). The Dirichlet boundary condition on Γ_1 is obtained thanks to the compactness of the trace operator. The boundary condition on $\Gamma_2 \cap \partial\Omega_f$ and on $\Gamma_2 \cap \partial\Omega_\mathrm{p}$ are easily recovered by integration by part in the variational formulation.

To recover the transmissions condition on $\Gamma_{p,f}$ and the boundary conditions on $\Gamma_1 \cap \partial\Omega_\mathrm{p}$, we first introduce the reflexive Banach space $L^p_{\mathrm{div}}(\Omega, \mathrm{Hom}(\mathbb{R}^3))$ which is defined by

$$L^p_{\mathrm{div}}(\Omega, \mathrm{Hom}(\mathbb{R}^3)) = \left\{ \mathbb{G} \in L^p(\Omega, \mathrm{Hom}(\mathbb{R}^3)) \mid \mathrm{div}\, \mathbb{G} \in L^p(\Omega)^3 \right\}.$$

Note that any $\mathbf{v} \in W^{1,q}(\Omega, \mathbb{R}^3)$, for $q = 1/(1-p^{-1})$, has a trace $\mathbf{v}|_{\partial\Omega}$ that belong to $W^{1-1/q,q}(\Omega)$. Therefore, for any $(\mathbb{G}, \mathbf{v}) \in L^p_{\mathrm{div}}(\Omega, \mathrm{Hom}(\mathbb{R}^3)) \times W^{1,q}(\Omega, \mathbb{R}^3)$, the following Green's identity holds

$$\int_\Omega \mathbf{v} \cdot \mathrm{div}\, \mathbb{G} dx + \sum_{j=1}^3 \int_\Omega \mathbb{G}_j \cdot \nabla \mathbf{v}_j dx = \sum_{j=1}^3 \langle \mathbb{G}_j \cdot \mathbf{n}, \mathbf{v} \rangle, \tag{3.12}$$

where $\langle ., . \rangle$ is the duality product between $\left(W^{1-1/q,q}(\Omega)\right)'$ and $W^{1-1/q,q}(\Omega)$. To get the matching of the normal trace on $\Gamma_{p,f}$, one only has to remark that $\mathbf{u}_\tau \in L^2_{\mathrm{div}}(\Omega, \mathbb{R}^3)$ with uniform bound with respect to τ since $\mathrm{div}\, \mathbf{u}_\tau = 0$. Therefore, one has $\mathbf{u}_{\tau,f} \cdot \mathbf{n}_{\mathrm{p}f} = \mathbf{u}_{\tau,s} \cdot \mathbf{n}_{\mathrm{p}f}$ and, using the Green formula, we then get that $\mathbf{u}_\mathrm{p} \cdot \mathbf{n}_{\mathrm{p}f} = \mathbf{u}_f \cdot \mathbf{n}_{\mathrm{p}f}$ in $H^{-1/2}(\Gamma_2)$. Similarly, since $\mathbf{u}_{\mathrm{p},\tau} = 0$ in $H^{1/2}\left(\Gamma_1 \cap \partial\Omega_\mathrm{p}, \mathbb{R}^3\right)$, we end up with $\mathbf{u}_\mathrm{p} \cdot \mathbf{n} = 0$ in $H^{-1/2}\left(\Gamma_1 \cap \partial\Omega_\mathrm{p}\right)$.

We now consider the distribution $\mathbb{F}_\tau$ defined below

$$\mathbb{F}_\tau = \left(\nu \mathrm{e}^{-\tau\alpha} \nabla \mathbf{u}_\tau - p_\tau \mathbb{I}_3 - \frac{1}{2} \mathrm{e}^{-2\tau\alpha} |\mathbf{u}_\tau|^2 \mathbb{I}_3\right).$$

Theorem 3.1 and the Sobolev embedding (2.7) ensure that $\mathbb{F}_\tau \in L^2(\Omega, \mathrm{Hom}(\mathbb{R}^3))$ and that there exists a positive constant $C > 0$ that does not depend on τ such that

$$\|\mathbb{F}_\tau\|_{L^2(\Omega, \mathrm{Hom}(\mathbb{R}^3))} \leq C.$$

We now turn our attention on proving that $\mathrm{div}(\mathbb{F}_\tau)$ is also bounded uniformly with respect to τ. Starting from Equation (2.3), Theorem 3.1 and Holder inequality, we infer that

$$\begin{aligned}\|\mathrm{div}\mathbb{F}_\tau\|_{L^p(\Omega)^3} &= \left\|-\mathbf{f}+\alpha\mathbf{u}_\tau - \mathrm{e}^{-2\tau\alpha}\mathbf{u}_\tau\times(\nabla\times\mathbf{u}_\tau)\right\|_{L^p(\Omega)^3}\\ &\le \|\mathbf{f}\|_{L^2(\Omega)^3}+\|\alpha\|_{L^\infty(\Omega)}\|\mathbf{u}_\tau\|_{L^2(\Omega)^3}\\ &+C\left\|\mathrm{e}^{-\tau\alpha}\mathbf{u}_\tau\right\|_{L^q(\Omega)^3}\left\|\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_\tau\right\|_{L^2(\Omega)^3},\end{aligned}$$

where $C>0$ does not depend on τ, $q=2p/(2-p)$ and $1\le p\le p_{\max}$ for some $p_{\max}$ that is going to be specified below. Theorem 3.1 shows that $\|\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_\tau\|_{L^2(\Omega)^3}$ is bounded uniformly with respect to τ. It only remains to show that $\|\mathrm{e}^{-\tau\alpha}\mathbf{u}_\tau\|_{L^q(\Omega)^3}$ is also bounded independently of τ. To prove this, we recall the continuous embedding $H^1(\mathcal{O})\subset L^6(\mathcal{O})$ (see e.g. [44] Chap. II p.159) for any locally Lipchitz set $\mathcal{O}\subset\mathbb{R}^d$ for $d=2,3$. Using this together with Theorem 3.1 for the H^1 bound on Ω_f and (3.8) for the H^1 bound on Ω_{p}, we obtain

$$\begin{aligned}\left\|\mathrm{e}^{-\tau\alpha}\mathbf{u}_\tau\right\|_{L^q(\Omega)} &\le \|\mathbf{u}_{f,\tau}\|_{L^q(\Omega_f)^3}+\left\|\mathrm{e}^{-\tau\alpha}\mathbf{u}_{\mathrm{p},\tau}\right\|_{L^q(\Omega_{\mathrm{p}})^3}\\ &\le C\left(\|\nabla\mathbf{u}_{f,\tau}\|_{H^1(\Omega_f)^3}+\left\|\nabla\left(\mathrm{e}^{-\tau\alpha}\mathbf{u}_{\mathrm{p},\tau}\right)\right\|_{H^1(\Omega_{\mathrm{p}})^3}\right)\\ &\le M,\end{aligned}$$

where $C,M>0$ does not depend on τ. Note that, thanks to the Sobolev embedding, we require $q\le 6$ which yield $p_{\max}=3/2$. This finally prove that $\mathbb{F}_\tau$ is uniformly bounded in $L^p_{\mathrm{div}}(\Omega,\mathrm{Hom}(\mathbb{R}^3))$ and thus there exists a subsequence (still denoted $\mathbb{F}_\tau$) which converges weakly toward some $\mathbb{F}\in L^p_{\mathrm{div}}(\Omega,\mathrm{Hom}(\mathbb{R}^3))$. From (3.10), we get that $\mathbb{F}_\tau$ converge weakly in $L^2(\Omega,\mathbb{R}^3)$ toward

$$\mathbb{F}=\begin{cases}\left(\nu\nabla\mathbf{u}_f-p_f\mathbb{I}_3-\frac{1}{2}|\mathbf{u}_f|^2\mathbb{I}_3\right) & \text{in } \Omega_f\\ \nabla p_{\mathrm{p}} & \text{in } \Omega_{\mathrm{p}}.\end{cases}$$

The uniqueness of the weak limit and the Green's formula (3.12) then shows that $\sum_{j=1}^3\mathbb{F}_j\mathbf{n}$ weakly converge to $\sum_{j=1}^3\mathbb{F}_j\mathbf{n}$ in $\left(W^{1-1/q,q}(\Gamma_{p,f},\mathbb{R}^3)\right)'$ which complete the proof. □

Remark 3.3 (Two dimensional case). *The penalized model in a two-dimensional setting is given in Remark 2.1. One can see that all the proofs of the Theorem 2.3, 3.1, 3.2 remain the same using that*

$$\begin{pmatrix}\mathbf{u}_2\partial_2\mathbf{u}_1-\mathbf{u}_2\partial_1\mathbf{u}_2\\ \mathbf{u}_1\partial_1\mathbf{u}_2-\mathbf{u}_1\partial_2\mathbf{u}_1\end{pmatrix}\cdot\mathbf{u}=0.$$

Also, the limiting Problem (3.11) is exactly the same.

Note that the first equation of Problem (3.11) gives $p_{\mathrm{p}}\in H^1(\Omega_{\mathrm{p}})$ since $\mathbf{u}_{\mathrm{p}}\in L^2(\Omega_{\mathrm{p}})$. Also, since $\mathbf{u}_f\in\mathbb{X}_f$, the second equation is valid in $L^2(\Omega_f)$. Now, using that $\mathbf{u}_{\mathrm{p}}=-\alpha^{-1}\nabla p$, we obtain the same interface condition on $\Gamma_{p,f}$ that have been used in [29]. Keeping both the velocity and the pressure in the porous medium, they read:

$$\begin{aligned}&\mathbf{u}_f\cdot\mathbf{n}_{\mathrm{p}f}=\mathbf{u}_{\mathrm{p}}\cdot\mathbf{n}_{\mathrm{p}f},\\ &\left(\nu\nabla\mathbf{u}_f-p_f\mathbb{I}_3-\frac{1}{2}|\mathbf{u}_f|^2\mathbb{I}_3\right)\mathbf{n}_{\mathrm{p}f}=-p_{\mathrm{p}}\mathbf{n}_{\mathrm{p}f}.\end{aligned}$$

Let $(\mathbf{t}_1,\mathbf{t}_2)$ be any orthonormal basis of the tangent plane to $\Gamma_{p,f}$ so that one can write $\mathbf{v}=\mathbf{v}_{\mathbf{n}_{\mathrm{p}f}}\mathbf{n}_{\mathrm{p}f}+\mathbf{v}_1\mathbf{t}_1+\mathbf{v}_2\mathbf{t}_2$ for any vector field. Projecting the interface condition according to this decomposition, one gets:

$$\begin{aligned}&\left(\nu\nabla\mathbf{u}_f\mathbf{n}_{\mathrm{p}f}\cdot\mathbf{n}_{\mathrm{p}f}-p_f\mathbb{I}_3-\tfrac{1}{2}|\mathbf{u}_f|^2\mathbb{I}_3\right)=-p_{\mathrm{p}} \text{ on } \Gamma_{\mathrm{p},f}\\ &\left(\nu\nabla\mathbf{u}_f\mathbf{n}_{\mathrm{p}f}\cdot\mathbf{t}_j\right)\mathbf{t}_j=0,\ j=1,2,\quad \text{on } \Gamma_{\mathrm{p},f}.\end{aligned}\tag{3.13}$$

The variational formulation to the limit problem can be directly obtained from (3.3) by letting $\tau \to +\infty$ and reads

$$\begin{aligned}
&\textit{Find } (\mathbf{u}_f, p_f, \mathbf{u}_\mathrm{p}, p_\mathrm{p}) \in \mathbb{X}_f \times L^2(\Omega_f) \times \mathbb{X}_\mathrm{p} \times L^2(\Omega_\mathrm{p}) \textit{ such that}\\
&a_f(\mathbf{u}_f, \mathbf{v}_f) + b_f(\mathbf{v}_f, p_f) + \int_{\Omega_\mathrm{p}} \alpha \mathbf{u}_\mathrm{p} \cdot \mathbf{v}_\mathrm{p} + N_f(\mathbf{u}_f, \mathbf{u}_f, \mathbf{v}_f) + b_\mathrm{p}(\mathbf{v}_\mathrm{p}, p_\mathrm{p})\\
&- \int_{\Gamma_{\mathrm{p},f}} (p_\mathrm{p} - p_f)\, \mathbf{n}_{\mathrm{p}f} \cdot (\mathbf{v}_\mathrm{p} - \mathbf{v}_f) d\sigma\\
&= \int_{\Omega_f} \mathbf{f}(x) \cdot \mathbf{v}_f(x) dx + \int_{\Omega_\mathrm{p}} \mathbf{f}(x) \cdot \mathbf{v}_\mathrm{p}(x) dx\\
&b_f(\mathbf{u}_f, q_f) + b_\mathrm{p}(\mathbf{u}_\mathrm{p}, q_\mathrm{p}) = 0,\\
&\forall (\mathbf{v}, \mathbf{v}_\mathrm{p}) \in \mathbb{X}_f \times \mathbb{X}_\mathrm{p}\ \forall (q_f, q_\mathrm{p}) \in L^2(\Omega_f) \times L^2(\Omega_\mathrm{p}).
\end{aligned} \tag{3.14}$$

The latter can be used to get some *a priori* estimate as well as the existence and uniqueness of $(\mathbf{u}_f, p_f, \mathbf{u}_\mathrm{p}, p_\mathrm{p})$. Note also that the well-posedness of Problem (3.11)-(3.13) have been studied in [19].

Remark 3.4 (Link with Beaver-Joseph-Saffman condition). *The transmission condition on the tangential part of $\nu \nabla \mathbf{u}_f \mathbf{n}_{\mathrm{p}f}$ is mentioned in [19] as a simplified form of the so-called Beavers-Joseph-Saffman interface condition which is used when dealing with fluid-porous media interface problem (see e.g. [13, 29]). The latter reads*

$$\partial_{\mathbf{n}_{\mathrm{p}f}} \mathbf{u}_f \cdot \mathbf{t}_j = \frac{C_{BJ}}{\sqrt{K}} (\mathbf{u}_f - \mathbf{u}_\mathrm{p}) \cdot \mathbf{t}_j \text{ on } \Gamma_{\mathrm{p},f}, \tag{3.15}$$

where C_{BJ} is a dimensionless constant depending on the structure of the porous media and K is the permeability of the porous media. Condition (3.15) has been mathematically justified in [35] using two-scale convergence.

In this paper, we derive the interface fluid-porous media problem through a singular perturbation approach where the porous media is modeled by adding the term $\alpha \mathbf{u}$ in the Navier-Stokes equation, also called a penalization model [14]. This approach allow to recover the interface fluid-porous media problem when $\alpha = K/\eta$, as η goes to zero, which explain why we only obtained the first order approximation of the Beavers-Joseph-Saffman condition (3.15) in the limiting problem (3.11).

Remark 3.5 (On the computation of the order of convergence). *The strongly penalized Darcy-Navier-Stokes problem (2.3) is a singular perturbation problem. Therefore, we cannot expect to have a regular expansion of $(\mathbf{u}_\tau, p_\tau)$ as $\tau \to +\infty$ that is valid everywhere on Ω since boundary layers occurs at the interface between the fluid and the porous medium. In the context of matched asymptotic expansion (see e.g. [17]) the solution to the limit problem (3.11) can be view as the zeroth order exterior expansion and is used to approximate the solution outside the boundary layer. Similar singular perturbation problem involving Stokes-Brinkman interface model has been studied in [5] where the author compute boundary layer corrector as well as the complete asymptotic expansion. Nevertheless, this paper aims at using the model (2.3) for topology optimization to be able to look for optimal design that are porous medium. The complete asymptotic expansion of $(\mathbf{u}_\tau, p_\tau)$, as $\tau \to +\infty$, is thus not needed in the present paper.*

3.3. Direct extensions. We present below several direct extension that can be drawn, without additional proof, from the above analysis.

Tensorial porosity. One can easily consider $\alpha \in \mathrm{Lip}(\Omega_\mathrm{p}, \mathrm{Hom}(\mathbb{R}^d))$ for $d = 2, 3$. The coercivity assumption over the porous part of the domain now reads

$$\text{a.e. } x \in \Omega_\mathrm{p},\ \alpha(x)^t = \alpha(x),\ \alpha_{\min} \mathbb{I}_d \lesssim \alpha(x),$$

for $\alpha_{\min} > 0$. The latter means that the permeability tensor is symmetric and comparable to the identity over the solid part of the domain. Note that the limiting problem (3.11) is exactly the same.

Penalization model using Cauchy-Stress tensor. Since div$\mathbf{u} = 0$, one has $\Delta\mathbf{u} = \mathrm{div}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) = \mathrm{div}(2D(\mathbf{u}))$ where $D(\mathbf{u})$ is the symmetric part of the Jacobian matrix of vector field $x \in \Omega \mapsto \mathbf{u}(x) \in \mathbb{R}^d$. As a result, considering the trace of $2D(\mathbf{u})\mathbf{n}$ is equivalent to considering $\partial_{\mathbf{n}}\mathbf{u}$ either in the boundary conditions (see e.g. [16] p.4 Remark 1) or in the interface condition and we could have use div $(2\nu e^{-\tau\alpha}D(\mathbf{u}))$ instead of div $(\nu e^{-\tau\alpha}\nabla\mathbf{u})$. It is worth noting that the above analysis remains the same thanks to the following formula

$$\int_\Omega \mathrm{div}\left(2\nu e^{-\tau\alpha}D(\mathbf{u})\right)\cdot\mathbf{v}dx = \int_{\partial\Omega}\left(2\nu e^{-\tau\alpha}D(\mathbf{u})\mathbf{n}\right)\cdot\mathbf{v}d\sigma - 2\nu\int_\Omega D(\mathbf{u}) : D(\mathbf{v})dx,$$

where $A : B = \sum_{i,j=1}^d A_{ij}B_{ij}$.

Extension to Stokes-Darcy model. The above analysis extends without any change to the case of Stokes-Darcy model. The resulting penalized model is given by

$$\left\{\begin{array}{rll} -\mathrm{div}\left(\nu e^{-\tau\alpha}\nabla\mathbf{u} - p\mathbb{I}_3\right) + \alpha\mathbf{u} & = \mathbf{f} & \text{in } \Omega, \\ \mathrm{div}(\mathbf{u}) & = 0 & \text{in } \Omega, \\ \mathbf{u} & = 0 & \text{on } \Gamma_1, \\ \nu e^{-\tau\alpha}\partial_{\mathbf{n}}\mathbf{u} - p\mathbf{n} & = 0 & \text{on } \Gamma_2. \end{array}\right. \tag{3.16}$$

Since we do not have to pass to the limit in the non-linear term, we obtain the following fluid-porous media interface limit problem

$$\left\{\begin{array}{rll} \nabla p_{\mathrm{p}} + \alpha\mathbf{u}_{\mathrm{p}} & = \mathbf{f} & \text{in } \mathcal{D}'(\Omega_{\mathrm{p}}, \mathbb{R}^3), \\ \mathrm{div}(\mathbf{u}_{\mathrm{p}}) & = 0 & \text{in } \mathcal{D}'(\Omega_{\mathrm{p}}), \\ -\mathrm{div}\left(\nu\nabla\mathbf{u}_f - p_f\mathbb{I}_3\right) & = \mathbf{f} & \text{in } \mathcal{D}'(\Omega_f, \mathbb{R}^3), \\ \mathrm{div}(\mathbf{u}_f) & = 0 & \text{in } \mathcal{D}'(\Omega_f), \\ \mathbf{u}_f & = 0 & \text{in } H^{1/2}(\Gamma_1 \cap \partial\Omega_f, \mathbb{R}^3), \\ \mathbf{n}\cdot\mathbf{u}_{\mathrm{p}} & = 0 & \text{in } H^{-1/2}(\Gamma_1 \cap \partial\Omega_{\mathrm{p}}), \\ \mathbf{u}_f\cdot\mathbf{n}_{\mathrm{p}f} & = \mathbf{u}_{\mathrm{p}}\cdot\mathbf{n}_{\mathrm{p}f} & \text{in } H^{-1/2}(\Gamma_{\mathrm{p},f}), \\ \nu\partial_{\mathbf{n}}\mathbf{u}_f - p_f\mathbf{n} & = 0 & \text{in } H^{-1/2}(\Gamma_2 \cap \partial\Omega_f, \mathbb{R}^3), \\ -p_{\mathrm{p}} & = 0 & \text{in } H^{1/2}(\Gamma_2 \cap \partial\Omega_{\mathrm{p}}), \\ \left(\nu\nabla\mathbf{u}_f - p_f\mathbb{I}_3\right)\mathbf{n}_{\mathrm{p}f} & = -p_{\mathrm{p}}\mathbf{n}_{\mathrm{p}f} & \text{in } H^{-1/2}(\Gamma_{\mathrm{p},f}, \mathbb{R}^3), \end{array}\right. \tag{3.17}$$

where, as seen before, the above system can be written in a stronger sense than in the sense of distribution.

4. Application to topology optimization. We now work with fixed $\tau > 0$ and we consider the following general PDE-constrained optimization problem

$$(\mathcal{P}) \quad \inf_{(\mathbf{u},p,\alpha)\in\mathbb{X}\times L^2(\Omega)\times U} \mathcal{J}(\mathbf{u}, p, \alpha), \text{ such that } (\mathbf{u}, p) \text{ satisfy (2.5)}, \ \alpha \in U_{\mathrm{ad}}, \tag{4.1}$$

where $\mathcal{J}$ is a real-valued cost functional. The set $U_{\mathrm{ad}} \subset U$ is a set of admissible design defined thank to the following sets

$$\Omega_f(\alpha) = \{x \in \Omega \mid \alpha(x) = 0\}, \ \Omega_{\mathrm{p}}(\alpha) = \{x \in \Omega \mid \alpha(x) \geq \alpha_{\min}\},$$

where $\alpha_{\min} > 0$. Note that both $\Omega_f(\alpha)$ and $\Omega_{\mathrm{p}}(\alpha)$ are Lebesgue mesurable set if $\alpha \in L^\infty(\Omega)$ since such functions are defined almost everywhere. The set of control is then

$$U := \left\{\alpha \in L^\infty(\Omega) \mid \overline{\Omega} = \Omega_f(\alpha) \cup \Omega_{\mathrm{p}}(\alpha) \text{ and } \alpha \in \mathrm{Lip}(\Omega_{\mathrm{p}})\right\}.$$

Theorem 2.3 ensures the existence of solution to Problem (2.5) for any $\alpha \in U$ and thus the set of constraints is not empty. Nevertheless, Theorem 3.2 which gives the limiting problem demands, for any $\alpha \in U$, that $\Omega_f(\alpha)$ and $\Omega_{\mathrm{p}}(\alpha)$ are bounded open set with Lipschitz boundary because the derivation of the limiting problem involves the study of the interface problem (3.1). We emphasize that such a regularity on the level set of α is hard to impose and that U as defined above is clearly not convex. To bypass these difficulties, we propose to work with a fixed decomposition of Ω hence assuming that the location of the fluid/porous media is given. Therefore, the set U we work with is now

$$U = \{\alpha \in L^\infty(\Omega) \mid \alpha(x) = 0 \ a.e \ x \in \Omega_f, \ \alpha(x) \geq \alpha_{\min} \ a.e \ x \in \Omega_{\mathrm{p}}, \ \alpha \in \mathrm{Lip}(\Omega_{\mathrm{p}})\},$$

where $\Omega = \Omega_{\mathrm{p}} \cup \Omega_f$. Note that the above set is a convex subset of $L^\infty(\Omega)$. It is also closed with respect to the strong topology of L^∞ and thus weakly closed.

This section is now devoted to prove some existence results for the optimization problem (4.1) and to give next first order optimality conditions.

4.1. Existence of optimal control. We show here the existence of optimal design for Problem (4.1). Recall that for any non-linear function h on a reflexive Banach space and α_n a weak-convergent sequence, one has $h(\alpha_n) \rightharpoonup h(\alpha_0)$ if and only if the function is affine. Also, it is classical in PDE-constrained optimization problem to require the state equation to be weakly continuous with respect to the each of its variable [31, 34]. We then need to consider U_{ad} to be a strong compact of $L^\infty(\Omega)$ since we have to pass to the limit in terms of the form $\mathrm{e}^{-\tau\alpha}$. We give after the proof some example of set U satisfying this assumption.

THEOREM 4.1. *Assume that*

i) $\mathcal{J}$ *is lower-semicontinous with respect to the* (weak, weak, weak*) *topology of* $H^1(\Omega)^d \times L^2(\Omega) \times L^\infty(\Omega)$.
ii) $\forall(\mathbf{u}, p, \alpha) \in \mathbb{X} \times L^2(\Omega) \times L^\infty(\Omega) \times U$, *one has* $\mathcal{J}(\mathbf{u}, p, \alpha) > -\infty$.
iii) The set U_{ad} *is a closed convex bounded subset of* U. *It is also a strong compact of* U *for the* L^∞ *norm.*

Then the optimization Problem (4.1) has at least one optimal solution.

Proof. The proof is based on standard weak compactness argument for minimizing sequence and on the continuity of the mapping $\alpha \in U_{\mathrm{ad}} \mapsto (\mathbf{u}, p) \in \mathbb{X} \times L^2(\Omega)$ where $(\mathbf{u}, p)$ is a solution to Problem (2.5). Let $(\mathbf{u}_n, p_n, \alpha_n)$ be a minimizing sequence for Problem (4.1). Since $\alpha_n \in U_{\mathrm{ad}}$, Theorem 2.3 show that there exists some $(\mathbf{u}_n, p_n) \in \mathbb{X} \times L^2(\Omega)$ satisfying Equation (2.5) such that $\|\mathbf{u}_n\|_{H^1(\Omega)^d}$ and $\|p_n\|_{L^2(\Omega)}$ are both uniformly bounded with respect to n. One then has a subsequence of $(\mathbf{u}_n, p_n)$ which converges weakly in $H^1(\Omega)^d \times L^2(\Omega)$ toward some $(\mathbf{u}_0, p_0)$ and, thanks to (2.7), $\mathbf{u}_n \to \mathbf{u}_0$ strongly in $L^p(\Omega)^d$ for $1 \leq p \leq 4$. From *iii*), we infer that α_n converges strongly toward $\alpha_0 \in U_{\mathrm{ad}}$ in $L^\infty(\Omega)$. Owning to this strong convergence in $L^\infty(\Omega)$, we get that $\mathrm{e}^{-\tau\alpha_n} \to \mathrm{e}^{-\tau\alpha_0}$ strongly in $L^\infty(\Omega)$ and, passing to the limit in the variational formulation (2.5), we obtain that $(\mathbf{u}_0, p_0)$ is a solution to Problem (2.5) with $\alpha = \alpha_0$. Assumptions *i*), *ii*) then give

$$\mathcal{J}(\mathbf{u}_0, p_0, \alpha_0) \leq \liminf_{n \to +\infty} \mathcal{J}(\mathbf{u}_n, p_n, \alpha_n) = \inf(\mathcal{P}),$$

which shows that $(\mathbf{u}_0, p_0, \alpha_0)$ is a solution to Problem (4.1). ◻

We now give some examples where the control set U_{ad} is a strong compact of L^∞. We recall that we assumed a fixed location of fluid/porous media and now prove that *iii*) holds. Indeed, if $\alpha_n \in U_{\mathrm{ad}}$ is bounded, it is also bounded in $\mathrm{Lip}(\Omega_{\mathrm{p}})$ and vanishes on Ω_f, Ascoli's theorem then ensures that there exists a subsequence (still denoted α_n) and $\alpha_0 \in \mathrm{Lip}(\Omega_{\mathrm{p}})$ such that

$$\alpha_n \to \alpha_0 \text{ strongly in } L^\infty(\Omega_{\mathrm{p}}), \ \alpha_n \to 0 \text{ strongly in } L^\infty(\Omega_f).$$

This gives the strong convergence of α_n toward some $\bar{\alpha}$ in $L^\infty(\Omega)$ and the proof of Theorem 4.1 remains the same.

Another way to get assumption *iii*) satisfied is to consider α that belong to some higher order Sobolev spaces that are compactly embedded into $\mathcal{C}(\overline{\Omega})$. Such set U are given (see e.g. [24]) by the two following Banach spaces

$$U = \begin{cases} W_0^{1,q}(\mathcal{O}) & \hookrightarrow\hookrightarrow \mathcal{C}(\overline{\mathcal{O}}) \text{ if } q > n \\ W^{k,q}(\mathcal{O}) & \hookrightarrow \mathcal{C}^{r,\gamma}(\mathcal{O}) \hookrightarrow\hookrightarrow \mathcal{C}(\overline{\mathcal{O}}) \text{ for } \gamma \in (0,1), k - r - \gamma = d/q, \end{cases} \tag{4.2}$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a bounded open set with Lipschitz boundary and the compactness of the last embedding follows from Ascoli's Theorem. As a result, one can get the required regularity by adding suitable Sobolev norms of α in the objective functional. These additional terms can be seen as regularization terms and read as follow

$$\mathcal{J}(\mathbf{u}, p, \alpha) = \mathcal{J}_0(\mathbf{u}, p) + \begin{cases} \dfrac{c_1}{q} \|\alpha\|^q_{L^q(\Omega_{\rm p})} + \dfrac{c_2}{q} \|\nabla\alpha\|^q_{L^q(\Omega_{\rm p})} + \dfrac{c_3}{2} \|\nabla\alpha\|^2_{L^2(\Omega_{\rm p})}, \\ \dfrac{c_1}{2} \|\alpha\|^2_{L^2(\Omega_{\rm p})} + \dfrac{c_2}{2} \|\nabla\alpha\|^2_{L^2(\Omega_{\rm p})} + \dfrac{c_3}{2} \displaystyle\sum_{i,j=1}^{d} \|\partial_i\partial_j\alpha\|^2_{L^2(\Omega_{\rm p})}, \end{cases} \tag{4.3}$$

where $c_1, c_2, c_3 > 0$ and we used $r = 0, q = 2, \gamma = 2 - d/2$ for the parameters of the second embedding to have a Sobolev space involving integer derivatives. The main steps of the existence proof then remains the same since, assuming that $J_0(\mathbf{u}, p) > -\infty$, ensure that α is bounded in U and thus has a subsequence that vanishes in Ω_f and converges strongly in $\mathcal{C}(\overline{\Omega_{\rm p}})$.

4.2. First order optimality condition. The next result gives some general assumptions yielding first order optimality condition for a PDE-constrained optimization problem and can be found in [34] p. 64, Section 5.2.

THEOREM 4.2. *Let U, Y, Z be Banach spaces and consider the following general nonlinear problem*

$$\min_{(y,u)\in Y\times U} f(y, u) \text{ subject to } E(y, u) = 0,\ u \in U_{\rm ad}.$$

Assume that the following hold

1. *$U_{\rm ad} \subset U$ is non-empty and convex.*
2. *$f : Y \times U \to \mathbb{R}$ and $E : Y \times U \to Z$ are continuously Fréchet differentiable.*
3. *For all $u \in V$ in a neighborhood $V \subset U$ of $U_{\rm ad}$, the state equation $E(y, u) = 0$ has a unique solution $y = y(u) \in Y$.*
4. *$\partial_y E(y(u), u) \in \mathcal{L}(Y, Z)$ has a bounded inverse for all $u \in U_{\rm ad}$.*

Then, denoting $\widetilde{f}(u) = f(y(u), u)$, any local solution $\bar{u}$ of the reduced problem

$$\min_{u\in U} \widetilde{f}(u) \text{ such that } u \in U_{\rm ad},$$

satisfies the variational inequality

$$\left\langle \partial_u \widetilde{f}(\bar{u}), u - \bar{u} \right\rangle_{U^*,U} \geq 0,\ \forall u \in U_{\rm ad}.$$

The reduced problem associated to (4.1) reduces to find the minimum of $\mathcal{J}_1(\alpha) = \mathcal{J}(\mathbf{u}(\alpha), p(\alpha), \alpha)$ over the closed convex set $U_{\rm ad}$. We show below that the mapping $\alpha \mapsto S(\alpha) = (\mathbf{u}(\alpha), p(\alpha))$ is $\mathcal{C}^1$ with respect to the topology of the strong convergence and thus, assuming that $\mathcal{J}$ is differentiable, one get that the optimum $(\bar{\mathbf{u}}, \bar{\mathbf{p}}, \bar{\alpha})$ satisfies the following first order optimality condition

$$\mathcal{J}_1'(\bar{\alpha})[\delta\alpha - \bar{\alpha}] \geq 0,\ \forall \delta\alpha \in U_{\rm ad}.$$

Since $\mathcal{J}_1(\alpha) = \mathcal{J}(S(\alpha), \alpha)$, the chain rules gives

$$\mathcal{J}_1'(\alpha)[\delta\alpha] = \langle \nabla_{\mathbf{u},p}\mathcal{J}(S(\alpha), \alpha), S'(\alpha)[\delta\alpha]\rangle + \langle \nabla_\alpha \mathcal{J}(S(\alpha), \alpha), \delta\alpha\rangle, \tag{4.4}$$

where the brackets stand for the duality product. We emphasize that, among the assumptions of Theorem 4.2, only the inversibility of the linearized problem has to be proved.

Derivability of the design to flow mapping. We study first the linearization of the penalized Navier-Stokes-Darcy system. Computing the Fréchet derivative of the operator involved in Problem (2.5) at $(\mathbf{u}, p) \in \mathbb{X} \times L^2(\Omega)$, we get that $(\mathbf{w}, q)$ is a solution to the linearization of (2.5) if it satisfies

$$\begin{aligned} &\textit{Find } (\mathbf{w}, q) \in \mathbb{X} \times L^2(\Omega) \textit{ such that} \\ &\forall \mathbf{v} \in \mathbb{X},\ a(\mathbf{w}, \mathbf{v}) + N'(\mathbf{u}, \mathbf{w}, \mathbf{v}) + b(\mathbf{v}, q) = \int_\Omega \mathbf{f}(x) \cdot \mathbf{v}(x) dx + \int_{\Gamma_2} \varphi \cdot \mathbf{v} d\sigma, \\ &\forall \pi \in L^2(\Omega),\ b(\mathbf{w}, \pi) = 0, \end{aligned} \tag{4.5}$$

where

$$N'(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -\int_\Omega \mathrm{e}^{-2\tau\alpha} \left(\mathbf{u} \cdot \mathbf{w} \mathrm{div}(\mathbf{v}) + \mathbf{u} \times (\nabla \times \mathbf{w}) \cdot \mathbf{v} + \mathbf{w} \times (\nabla \times \mathbf{u}) \cdot \mathbf{v} \right) dx.$$

Integrating by parts then gives the following strong formulation of Problem (4.5)

$$\begin{aligned} -\mathrm{div} \left(\nu \mathrm{e}^{-\tau\alpha} \nabla \mathbf{w} - q \mathbb{I}_3 - \mathrm{e}^{-2\tau\alpha} \mathbf{u} \cdot \mathbf{w} \mathbb{I}_3 \right) & \\ -\mathrm{e}^{-2\tau\alpha} \left(\mathbf{u} \times (\nabla \times \mathbf{w}) + \mathbf{w} \times (\nabla \times \mathbf{u}) \right) + \alpha \mathbf{w} &= f \text{ in } \Omega, \\ \mathrm{div}(\mathbf{w}) &= 0 \text{ in } \Omega, \\ \mathbf{w} &= 0 \text{ on } \Gamma_1, \\ \nu \mathrm{e}^{-\tau\alpha} \partial_{\mathbf{n}} \mathbf{w} - (q + \mathrm{e}^{-2\tau\alpha} \mathbf{u} \cdot \mathbf{w}) \mathbf{n} &= \varphi \text{ on } \Gamma_2. \end{aligned} \tag{4.6}$$

The well-posedness of the linear Problem (4.5) can be obtained with standard method and its precise statement depends on the regularity of $(\mathbf{u}, p)$ as one can see in the next result.

Theorem 4.3. *Let $(\mathbf{u}, p) \in \mathbb{X} \times L^2(\Omega)$ and $(\mathbf{f}, \varphi) \in \mathbb{X}' \times H^{-1/2}(\Gamma_2, \mathbb{R}^3)$ be given. We have the following*

i) Assume that $(\mathbf{u}, p) \in (\mathbb{X} \cap L^\infty(\Omega)) \times L^2(\Omega)$ and consider the unbounded operator of $\mathbb{X}'$, $(\mathcal{L}, \mathbb{V})$ defined by

$$\langle \mathcal{L} \mathbf{w}, \mathbf{v} \rangle = \langle \mathcal{A} \mathbf{w} + \mathcal{N}' \mathbf{w}, \mathbf{v} \rangle = a(\mathbf{w}, \mathbf{v}) + N'(\mathbf{u}, \mathbf{w}, \mathbf{v}),\ \forall \mathbf{v} \in \mathbb{V}.$$

Then the spectrum of $(\mathcal{L}, \mathbb{V})$ is discrete with no accumulation point and if $0 \notin \sigma(\mathcal{L})$, then Problem (4.6) is well-posed for any $\nu > 0$.

ii) If $\nu > 0$ is only large enough, then Problem (4.6) is well-posed.

Proof. From the *inf-sup* condition (2.9), the pressure is determined as soon as the velocity is. Also, the unicity can be obtained using standard technique since the underlying problem is linear. Note that the velocity is a zero of the linear mapping $\Phi : \mathbb{V} \to \mathbb{V}'$ defined by

$$\langle \Phi(\mathbf{w}), \mathbf{v} \rangle := a(\mathbf{w}, \mathbf{v}) + N'(\mathbf{u}, \mathbf{w}, \mathbf{v}) - \int_\Omega \mathbf{f}(x) \cdot \mathbf{v}(x) dx - \int_{\Gamma_2} \varphi \cdot \mathbf{v} d\sigma.$$

We detail below both cases.

i) Assume that $(\mathbf{u}, p) \in (\mathbb{X} \cap L^\infty(\Omega)) \times L^2(\Omega)$. We first prove that the spectrum of the unbounded operator $(\mathcal{L}, \mathbb{V})$ is discrete. Using $2ab \leq a^2 t + b^2 t^{-1}$ for any $t > 0$, we obtain

$$\begin{aligned} \langle \mathcal{L} \mathbf{w}, \mathbf{w} \rangle &= a(\mathbf{w}, \mathbf{w}) + N'(\mathbf{u}, \mathbf{w}, \mathbf{w}) \\ &\geq \nu C_1(\alpha_{\min}, \Omega) \|\mathbf{u}\|^2_{H^1(\Omega)} - \int_\Omega \mathrm{e}^{-2\tau\alpha} \mathbf{u} \times (\nabla \times \mathbf{w}) \cdot \mathbf{w} dx \\ &\geq (\nu C_1(\alpha_{\min}, \Omega) - \|\mathbf{u}\|_{L^\infty(\Omega)} \frac{t}{2}) \|\mathbf{w}\|^2_{H^1(\Omega)} - \|\mathbf{u}\|_{L^\infty(\Omega)} \frac{1}{2t} \|\mathbf{w}\|^2_{L^2(\Omega)^3}, \end{aligned}$$

where $t > 0$ is small enough to ensure that

$$(\nu C_1(\alpha_{\min}, \Omega) - \|\mathbf{u}\|_{L^\infty(\Omega)} t/2) > \nu C_1(\alpha_{\min}, \Omega)/2.$$

Therefore, using Lax-Milgram lemma, the operator $(\mathcal{L}+\lambda\mathbb{I})$ is invertible with a continuous inverse if $\lambda>0$ is large enough. This shows that the resolvent set of $(\mathcal{L},\mathbb{V})$ is non-empty. Since the embedding $\mathbb{V}\hookrightarrow\mathbb{X}'$ is compact, one has that the resolvent of $(\mathcal{L},\mathbb{V})$ is compact and thus its spectrum is discrete with no accumulation point. Remark now that if $0\notin\sigma(\mathcal{L})$ then, for all $\nu>0$, there exists a unique $\mathbf{w}\in\mathbb{V}$ such that $\mathcal{L}\mathbf{w}=\mathbf{f}$ for any $\mathbf{f}\in\mathbb{X}'$. This gives $\langle\Phi(\mathbf{w}),\mathbf{v}\rangle=0$ for all $\mathbf{v}\in\mathbb{V}$ and thus the existence and uniqueness of $\mathbf{w}$.

ii) Due to the lack of regularity of $(\mathbf{u},p)$, we cannot get a Garding inequality for the operator $\mathcal{L}$ as above. Instead, to get a bound on $N'(\mathbf{u},\mathbf{w},\mathbf{w})$, we split $\int_\Omega$ using that $\Omega=\Omega_f\cup\Omega_{\mathrm{p}}$ and apply Holder inequality

$$\begin{aligned}\left|\int_\Omega \mathrm{e}^{-2\tau\alpha}\mathbf{u}\times(\nabla\times\mathbf{w})\cdot\mathbf{w}dx\right| &\leq C(\Omega)\left(\mathrm{e}^{-\alpha_{\min}}\left\|\nabla\mathbf{w}_{\mathrm{p}}\right\|^2_{L^2(\Omega_{\mathrm{p}})}\left\|\mathrm{e}^{-\tau\alpha}\nabla\mathbf{u}_{\mathrm{p}}\right\|_{L^2(\Omega_{\mathrm{p}})}\right)\\ &\quad+C(\Omega)\left(\left\|\nabla\mathbf{w}_f\right\|^2_{L^2(\Omega_f)}\left\|\nabla\mathbf{u}_f\right\|_{L^2(\Omega_f)}\right).\end{aligned}$$

Using Young inequality, we then infer that Φ satisfies a Garding inequality if $\nu>0$ is large enough owning to Theorem 3.1. The existence is then proved as in *ii*) and the unicity holds for large enough $\nu>0$. ☐

Remark 4.4. *Theorem 4.3 giving the well-posedness of the linearized Problem (4.5) can be extended for non vanishing divergence as well (see e.g. [7, 42] for Stokes and Navier-Stokes equations). To handle such case, consider the following orthogonal decomposition*

$$\mathbb{X}=\mathbb{V}\oplus\mathbb{V}^\perp, \tag{4.7}$$

and the associated projectors $P_0:\mathbb{X}\to\mathbb{V}$, $P_\perp:\mathbb{X}\to\mathbb{V}^\perp$. *Projecting Problem (4.5) according to the decomposition (4.7) then gives the following equivalent system*

$$\begin{aligned} b(P_\perp\mathbf{w},q) &= (g,q)_{L^2(\Omega)}, && \forall q\in L^2(\Omega),\\ a(P_0\mathbf{w},\mathbf{v}_0)+N'(\mathbf{u},P_0\mathbf{w},\mathbf{v}_0) &= \langle\mathbf{f},\mathbf{v}_0\rangle, && \forall\mathbf{v}_0\in\mathbb{V},\\ b(\mathbf{v},p)+a(P_0\mathbf{w}+P_\perp\mathbf{w},\mathbf{v})+N'(\mathbf{u},P_0\mathbf{w}+P_\perp\mathbf{w},\mathbf{v}) &= \langle\mathbf{f},\mathbf{v}\rangle, && \forall\mathbf{v}\in\mathbb{X},\end{aligned} \tag{4.8}$$

Note that the second equation of (4.8) is well-posed as soon as any assumption of Theorem 4.3 hold. This gives the existence and uniqueness of $P_0\mathbf{w}$. *Now, let* $B:\mathbb{X}\to L^2(\Omega)$ *be the operator defined by* $\langle B\mathbf{u},q\rangle=b(\mathbf{u},q)$. *The inf-sup condition (2.9) with Lemma 4.1 p.58 [28] then ensures that the first and last equations of (4.8) are well-posed and thus the linearized problem (4.5) with non-homogeneous divergence is well-posed under the same assumptions as those of Theorem 4.3.*

We can now apply the implicit function theorem to get some smoothness of the design-to-flow mapping.

Theorem 4.5. *Let* $\beta\in U$ *and* $(\mathbf{u}(\beta),p(\beta))\in\mathbb{X}\times L^2(\Omega)$ *be a solution to Problem (2.5) with* $\alpha=\beta$. *Assume that Problem (4.6) is well-posed (see Theorem 4.3). Then there exists an open neighborhood* $\mathcal{O}(\mathbf{u})\times\mathcal{O}(p)\subset\mathbb{X}\times L^2(\Omega)$ *, an open neighborhood* $\mathcal{O}(\beta)\subset U$ *and a* $\mathcal{C}^1$ *mapping*

$$S:\alpha\in\mathcal{O}(\beta)\mapsto(\mathbf{u}(\alpha),p(\alpha))\in\mathcal{O}(\mathbf{u})\times\mathcal{O}(p),$$

where $(\mathbf{u}(\alpha),p(\alpha))$ *satisfy Problem (2.5).*

First order optimality condition and adjoint model. We now use Theorems 4.2,4.3,4.5 to get the first order optimality condition involving the adjoint model. This technique is standard and can be found for instance in [34] Corollary 5.2.4 p. 66.

Recall first that $S(\alpha)=(\mathbf{u}(\alpha),p(\alpha))$ satisfying Problem (2.5) are also solution to $F(\mathbf{u},p,\alpha)=0$ where $F:\mathbb{X}\times L^2(\Omega)\times U_{\mathrm{ad}}\mapsto\mathbb{X}'\times L^2(\Omega)$ is defined by

$$\langle F(\mathbf{u},p,\alpha),(\mathbf{v},q)\rangle=a(\mathbf{u},\mathbf{v})+N(\mathbf{u},\mathbf{u},\mathbf{v})+b(\mathbf{v},p)+b(\mathbf{u},q)-\int_\Omega\mathbf{f}(x)\cdot\mathbf{v}(x)dx-\int_{\Gamma_2}\varphi\cdot\mathbf{v}d\sigma.$$

The implicit function theorem then gives that

$$S'(\alpha)[\delta\alpha]=-\left(\partial_{(\mathbf{u},p)}F(u(\alpha),p(\alpha),\alpha)\right)^{-1}\circ\partial_\alpha F(\mathbf{u}(\alpha),p(\alpha),\alpha)[\delta\alpha], \tag{4.9}$$

where $\partial_{.}$ is the Fréchet derivative with respect to the subscripted variable. Using now (4.4), we obtain the derivative of $\mathcal{J}_1(\alpha) = \mathcal{J}(S(\alpha), \alpha)$ at any $\alpha \in U_{\rm ad}$

$$\begin{aligned}\mathcal{J}_1'(\alpha)[\delta\alpha] &= \left\langle \nabla_{(\mathbf{u},p)}\mathcal{J}(S(\alpha),\alpha), S'(\alpha)[\delta\alpha]\right\rangle + \langle \nabla_\alpha \mathcal{J}(S(\alpha),\alpha), \delta\alpha\rangle \\ &= \langle \nabla_\alpha \mathcal{J}(S(\alpha),\alpha), \delta\alpha\rangle - \\ &\quad \left\langle \left(\partial_{(\mathbf{u},p)}F(S(\alpha),\alpha)\right)^{-*} \nabla_{(\mathbf{u},p)}\mathcal{J}(S(\alpha),\alpha), \partial_\alpha F(S(\alpha),\alpha)[\delta\alpha]\right\rangle,\end{aligned} \tag{4.10}$$

where the $.^{-*}$ denotes the adjoint of the inverse of the linear operator $\partial_{(\mathbf{u},p)}F(S(\alpha),\alpha)$.

Let $(\bar{\mathbf{u}}, \bar{p}, \bar{\alpha}) \in \mathbb{X} \times L^2(\Omega) \times U_{\rm ad}$ be a solution to the optimal control problem (4.1). Since $U_{\rm ad}$ is convex, one has that optimal solution satisfies

$$\mathcal{J}_1'(\bar{\alpha})[\delta\alpha - \bar{\alpha}] \geq 0, \ \forall \delta\alpha \in U_{\rm ad}, \tag{4.11}$$

where the derivative of $\mathcal{J}_1$ is given in (4.10) and involves the resolution of the so-called adjoint problem

$$\begin{aligned}&Find\ (\mathbf{w}, q) \in \mathbb{X} \times L^2(\Omega)\ such\ that \\ &\forall \mathbf{v} \in \mathbb{X},\ a(\mathbf{v},\mathbf{w}) + N'(\mathbf{u},\mathbf{v},\mathbf{w}) + b(\mathbf{v},q) = \int_\Omega \mathbf{f}(x)\cdot\mathbf{v}(x)dx + \int_{\Gamma_2} \varphi\cdot\mathbf{v}d\sigma, \\ &\forall \pi \in L^2(\Omega),\ b(\mathbf{w},\pi) = -(g_a,\pi)_{L^2(\Omega)},\end{aligned} \tag{4.12}$$

which is the variational formulation to the Problem

$$\begin{aligned}-\mathrm{div}\left(\nu e^{-\tau\alpha}\nabla\mathbf{w} - q\mathbb{I}_3\right) + \nabla\times\left(e^{-2\tau\alpha}\mathbf{u}\times\mathbf{w}\right)& \\ -e^{-2\tau\alpha}\left(\mathbf{u}\,\mathrm{div}\,\mathbf{w} + \nabla\times\mathbf{u}\times\mathbf{w}\right) + \alpha\mathbf{w} &= \mathbf{f}_a \text{ in } \Omega, \\ \mathrm{div}(\mathbf{w}) &= g_a \text{ in } \Omega, \\ \mathbf{w} &= 0 \text{ on } \Gamma_1, \\ \nu e^{-\tau\alpha}\partial_{\mathbf{n}}\mathbf{w} - (q\mathbf{n} + e^{-2\tau\alpha}\mathbf{n}\times(\mathbf{u}\times\mathbf{w})) &= \varphi_a \text{ on } \Gamma_2.\end{aligned} \tag{4.13}$$

The well-posedness of the adjoint equation (4.12) can be proved similarly as Theorem 4.3. For non-homogeneous divergence, this can be done as in Remark 4.4.

Now assume that the derivative of the cost functional can be written as

$$\begin{aligned}\langle \partial_{\mathbf{u}}\mathcal{J}(\mathbf{u},p,\alpha), \delta\mathbf{u}\rangle &= \langle \partial_{\mathbf{u}}\mathcal{J}_\Omega(\mathbf{u},p,\alpha), \delta\mathbf{u}\rangle_{\mathbb{X}^*\times\mathbb{X}} + \langle \partial_{\mathbf{u}}\mathcal{J}_\Gamma(\mathbf{u},p,\alpha), \delta\mathbf{u}\rangle_{H^{-1/2}\times H^{1/2}}, \\ \langle \partial_{\rm p}\mathcal{J}(\mathbf{u},p,\alpha), \delta p\rangle &= \left(\partial_{\rm p}\mathcal{J}(\mathbf{u},p,\alpha), \delta p\right)_{L^2(\Omega)},\end{aligned}$$

where $(\partial_{\mathbf{u}}\mathcal{J}_\Omega(\bar{\mathbf{u}},\bar{p},\bar{\alpha}), \partial_{\mathbf{u}}\mathcal{J}_\Gamma(\bar{\mathbf{u}},\bar{p},\bar{\alpha})) \in \mathbb{X}^* \times H^{-1/2}(\Gamma_2,\mathbb{R}^3)$. The first order optimality condition (4.11) can finally be recast as follow

$$\begin{aligned}&(\bar{\mathbf{u}},\bar{p}) \in \mathbb{X}\times L^2(\Omega) \text{ satisfy (2.5)}, \\ &(\bar{\mathbf{w}},\bar{q}) \in \mathbb{X}\times L^2(\Omega) \text{ satisfy (4.13) with } \mathrm{div}\,\bar{\mathbf{w}} = g_a, \\ &\text{where } g_a = \partial_{\rm p}\mathcal{J}(\bar{\mathbf{u}},\bar{p},\bar{\alpha}) \text{ and } (\mathbf{f}_a,\varphi_a) = \left(\partial_{\mathbf{u}}\mathcal{J}_\Omega(\bar{\mathbf{u}},\bar{p},\bar{\alpha}), \partial_{\mathbf{u}}\mathcal{J}_\Gamma(\bar{\mathbf{u}},\bar{p},\bar{\alpha})\right), \\ &\left(\partial_\alpha\mathcal{J}(\bar{\mathbf{u}},\bar{p},\bar{\alpha}) - \partial_\alpha a(\bar{\mathbf{u}},\bar{\mathbf{w}}) - \partial_\alpha N(\mathbf{u},\mathbf{u},\mathbf{w})\right)[\bar{\alpha} - \delta\alpha] \geq 0,\ \forall \delta\alpha \in U_{\rm ad},\end{aligned} \tag{4.14}$$

with

$$\begin{aligned}\partial_\alpha a(\mathbf{u},\mathbf{w})[\delta\alpha] &= \nu\sum_{j=1}^d \int_\Omega (-\tau\delta\alpha)e^{-\tau\alpha}\nabla\mathbf{u}_j\cdot\nabla\mathbf{v}_j dx + \int_\Omega \delta\alpha\mathbf{u}\cdot\mathbf{v}dx \\ \partial_\alpha N(\mathbf{u},\mathbf{u},\mathbf{w})[\delta\alpha] &= \tau\int_\Omega \left(e^{-2\tau\alpha}|\mathbf{u}|^2\mathrm{div}(\mathbf{w}) + 2e^{-2\tau\alpha}\mathbf{u}\times(\nabla\times\mathbf{u})\cdot\mathbf{w}\right)\delta\alpha dx.\end{aligned}$$

We finally emphasize that the first order optimality conditions (4.14) correspond to the standard adjoint method (see e.g. [34] Corollary 5.2.4).

Remark 4.6 (Derivatives of $\mathcal{J}$ with regularization terms). *We give here the derivative of the cost functional (4.3) involving regularization terms which ensures that α belong to a strong compact of $L^\infty(\Omega)$. In the case where $\mathcal{J}_1(\alpha) = \frac{c_1}{q}\left\|\alpha\right\|^q_{L^q(\Omega_{\rm p})} + \frac{c_2}{q}\left\|\nabla\alpha\right\|^q_{L^q(\Omega_{\rm p})} + \frac{c_3}{2}\left\|\nabla\alpha\right\|^2_{L^2(\Omega_{\rm p})}$ for $\alpha \in W^{1,q}_0(\Omega_{\rm p})$ with $q > d$, one has $\forall \delta\alpha \in W^{1,q}_0(\Omega_{\rm p})$,*

$$\begin{aligned}\partial_\alpha \mathcal{J}_1(\alpha)[\delta\alpha] &= \int_{\Omega_{\rm p}} c_1|\alpha|^{q-2}\alpha\delta\alpha + c_2|\nabla\alpha|^{q-2}\nabla\alpha\cdot\nabla\delta\alpha + c_3\nabla\alpha\cdot\nabla\delta\alpha dx\\ &= -\int_{\Omega_{\rm p}} \left(\mathrm{div}((c_3 + c_2|\nabla\alpha|^{q-2})\nabla\alpha) - c_1|\alpha|^{q-2}\alpha\right)\delta\alpha\, dx.\end{aligned}$$

Therefore we end up with solving a nonlinear coercive elliptic problem involving the q-Laplacian.

In the case $\mathcal{J}_1(\alpha) = \frac{c_1}{2}\left\|\alpha\right\|^2_{L^2(\Omega_{\rm p})} + \frac{c_2}{2}\left\|\nabla\alpha\right\|^2_{L^2(\Omega_{\rm p})} + \frac{c_3}{2}\sum_{i,j=1}^d \left\|\partial_i\partial_j\alpha\right\|^2_{L^2(\Omega_{\rm p})}$ with $\alpha \in H^2_0(\Omega_{\rm p})$ we get

$$\forall\delta\alpha \in H^2_0(\Omega_{\rm p}),\ \partial_\alpha\mathcal{J}_1(\alpha)[\delta\alpha] = \int_{\Omega_{\rm p}} \left(c_1\alpha - c_2\Delta\alpha + c_3\Delta^2\alpha\right)\delta\alpha.$$

We then need to solve a fourth order problem.

5. Numerical illustration: Velocity tracking. In this section, we wish to give an illustration of the penalized Navier-Stokes-Darcy model applied to topology optimization. We would like to emphasize that the goal of this section is to show some numerical illustration and thus we do not intend to prove that all the algorithms and discretization used converges. Nevertheless, the convergence of finite element method applied to the penalized Stokes-Darcy model can be deduced from [28] since both problems have a coercive principal part. Having this in mind, we are going to use FreeFEM ++[32].

Let $\Omega = [0,1]\times[0,2]$ be an enclosure where air flows comes from the left of the room with a given velocity $\mathbf{u}_0 = (1,0)$. The velocity and pressure of air in the room then satisfy a steady state Stokes equation with inhomogeneous Dirichlet boundary condition. We are interested in finding the location of a Darcy porous medium with porosity α such that the following velocity-tracking cost functional is minimized

$$\mathcal{J}(\alpha) = \frac{1}{2}\int_\Omega |\mathbf{u}-\mathbf{u}_d|^2 dx$$

where $\mathbf{u}_d = (1,0)$, $\mathbf{u}_b = (1,0)$. We chose to take into account the porous media with the following penalized Stokes-Darcy model

$$\left\{\begin{array}{rll} -\mathrm{div}\left(\nu \mathrm{e}^{-\tau\alpha}\nabla\mathbf{u} - p\mathbb{I}_3\right) + \alpha\mathbf{u} & = 0 & \text{in } \Omega,\\ \mathrm{div}(\mathbf{u}) & = 0 & \text{in } \Omega,\\ \mathbf{u} & = \mathbf{u}_0 & \text{on } \{0\}\times[0,1],\\ \mathbf{u} & = 0 & \text{on } [0,2]\times(\{0\}\cup\{1\}),\\ \nu\mathrm{e}^{-\tau\alpha}\partial_{\mathbf{n}}\mathbf{u} - p\mathbf{n} & = 0 & \text{on } \{2\}\times[0,1]. \end{array}\right. \tag{5.1}$$

The velocity-tracking optimisation problem is now given as

$$\min_{(\mathbf{u},p,\alpha)\in\mathbb{X}\times L^2(\Omega)\times U} \mathcal{J}(\alpha),\ (\mathbf{u},p) \text{ satisfies (5.1)},\ \alpha\in U_{\rm ad}, \tag{5.2}$$

where the set $U_{\rm ad}$ is defined below

$$U_{\rm ad} = \left\{\alpha \in L^\infty(\Omega) \mid \alpha(x) = 0 \text{ or } \alpha(x) \geq \alpha_{\min}\right\}.$$

We are going to need the adjoint equation associated to problem (5.2). The latter can be obtained from (4.13) and reads here

$$\left\{\begin{aligned} -\mathrm{div}\,(\nu \mathrm{e}^{-\tau\alpha}\nabla \mathbf{v} - q\mathbb{I}_3) + \alpha\mathbf{v} &= \mathbf{f} && \text{in } \Omega,\\ \mathrm{div}(\mathbf{v}) &= 0 && \text{in } \Omega,\\ \mathbf{v} &= 0 && \text{on } \{0\}\times[0,1],\\ \mathbf{v} &= 0 && \text{on } [0,2]\times(\{0\}\cup\{1\}),\\ \nu \mathrm{e}^{-\tau\alpha}\partial_{\mathbf{n}}\mathbf{v} - q\mathbf{n} &= 0 && \text{on } \{2\}\times[0,1], \end{aligned}\right. \tag{5.3}$$

where $\mathbf{f} = -(\mathbf{u} - \mathbf{u}_d)$.

The variational formulation of the direct and adjoint problems can be respectively found in Equations (2.5) and (4.12) setting $N = N' = 0$. The finite element discretisation is done with a regular mesh composed of triangles build from a uniform $n \times n$ grid. Both direct and adjoint states are computed with $\mathbb{P}^2$ finite element for the velocity fields and $\mathbb{P}^1$ for pressure. The porosity fields α is sought in $\mathbb{P}^0$ and is thus a piecewise constant function.

The optimization problem (5.2) is now solved with a steepest descent algorithm. The derivative of $\mathcal{J}$ with respect to α can be computed thanks to (4.10) (see also (4.14)) which is actually the standard adjoint algorithm for PDE-constrained optimization problem [33, 39].

Algorithm 1:

1. $\alpha = 0 \in \mathbb{P}^0$ is given.
2. Compute $(\mathbf{u}_h, p_h) \in \mathbb{P}^2 \times \mathbb{P}^1$ satisfying Problem (5.1).
3. Compute $(\mathbf{v}_h, q_h) \in \mathbb{P}^2 \times \mathbb{P}^1$ satisfying the adjoint problem (5.3).
4. Update the porosity thanks to the optimality condition (4.14)

$$\begin{aligned} \beta &= \max\left\{0, \alpha - \lambda\Pi_h\left(-\nu\tau \mathrm{e}^{-\tau\alpha}\sum_{j=1}^{d}(\nabla \mathbf{u}_{j,h}\cdot\nabla \mathbf{v}_{j,h}) + \mathbf{u}_h\cdot\mathbf{v}_h\right)\right\},\\ \alpha &= \max\{\alpha_{\min}, \beta\}\,\chi_{(\beta>0)} \end{aligned}$$

   where $\Pi_h : L^2(\Omega) \to \mathbb{P}^0$ is the finite element projector and the max has to be understood on each cell of the mesh. Also, $\chi_{(\beta>0)}$ is defined on each cell of the mesh and return zero if $\beta = 0$ and the value of β if $\beta > 0$.
5. Return to Step 2.

REMARK 5.1. *We use the Crout solver to solve both direct and adjoint problem. The latter needs every sub-matrices to be invertible and we thus add the term $\varepsilon\pi$, $\pi = p, q$, in the incompressibility condition of both direct and adjoint problems. The convergence, as $\varepsilon \to 0$, of the solution to the discrete perturbed problems to the solution of the discrete original one is of the order of ε aas proved in [43]. The analysis can be extended for our penalized Navier-Stokes-Darcy model since both problem share the same mathematical structure.*

The following parameters are used in the numerical simulations

$$\alpha_{\min} = 10,\ \nu = 1,\ \tau = 10,\ n = 100,\ \lambda = 10,\ n_{\text{iter}} = 1500, \varepsilon = 0.000001.$$

We chose $\nu = 1$ since we work with Stokes equation that can be used as a model for fluid flow at low Reynolds number. We emphasize that we run Algorithm 1 with 1500 iterations since, as we show below in Figure 5.1 the values of the cost functional are stationary, when rounding up to four digits, before reaching the end of iterates. Values of the cost function without porous medium embedded in Ω is $\mathcal{J} = 0.1801$ and after 1500 iterations is $\mathcal{J} = 0.0043$ hence reducing the initial value of the cost function by a factor 42.

We represent in Figure 5.2 the velocity field when no porosity is embedded in Ω and in Figure 5.3 after 1500 iteration of the steepest descent algorithm. The optimal porosity design can be found in Figure 5.4. Note that most of the porous media is added at the outlet of the room. This can be explained by the fact

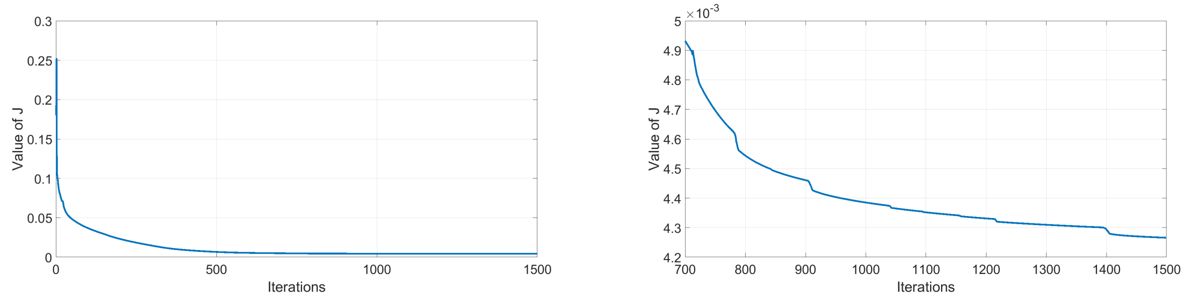

FIG. 5.1. *Value of cost functions over iteration.*

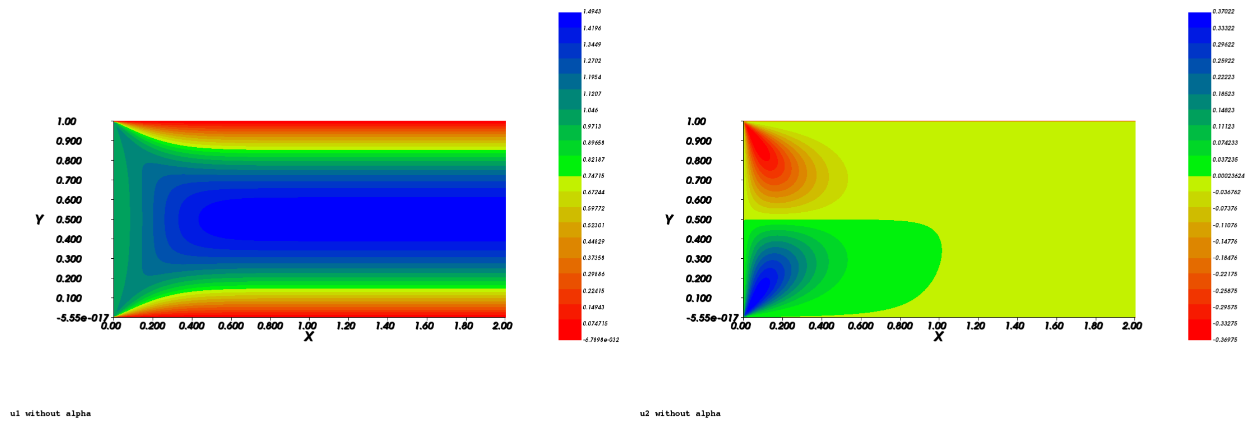

FIG. 5.2. *Velocity fields without porosity. Left: u_1, Right: u_2.*

that we impose $\mathbf{u}_0 = (1,0) = \mathbf{u}_d$ at the entrance. Finally, we represent the term $\exp(-\tau\alpha)$ in Figure 5.4. This clearly shows where the the porous media is located. Note that, when $\alpha > 0$, $\exp(-\tau\alpha)$ numerically vanished and so the penalized model (5.1) is actually the Darcy model for porous media. As a result, we have the Stokes model inside the blue zone and the Darcy model inside the red zones.

6. Conclusions and Perspectives. This paper proposed a penalization model for the Navier-Stokes-Darcy system and its application to topology optimization problems. The main advantage of this approach is that one can use the porosity as optimization parameter and thus find optimal design that are porous medium. We first prove that we recover the standard Navier-Stokes-Darcy model as the weak-limit of our penalization model. We then gave necessary condition on both the control set and the cost functional to have at least one optimal solution. We also derive the first order optimality condition and finally end this paper with some numerical simulations, on Stokes flow, to show the interest of the approach.

A first perspective is to perform more intensive numerical simulation of the penalized Navier-Stokes-Darcy model applied to topology optimization. Indeed, we used in this paper one of the simplest optimization algorithm (steepest descent) and this could be worth using, for instance, Newton method. Also, we emphasize that, at the continuous level, one needs the porosity to belong to a strong compact of L^∞ to get at least on optimal design and we thus propose to add some regularization terms (see Remark 4.6) that were not considered in the numerical simulation. The impact of those terms on the optimal design could be studied as well. Another perspective could be to consider other methods for solving both primal and adjoint problem. For instance, domain decomposition methods should be more suitable to handle the heterogeneous nature of the fluid-porous media transmission problem.

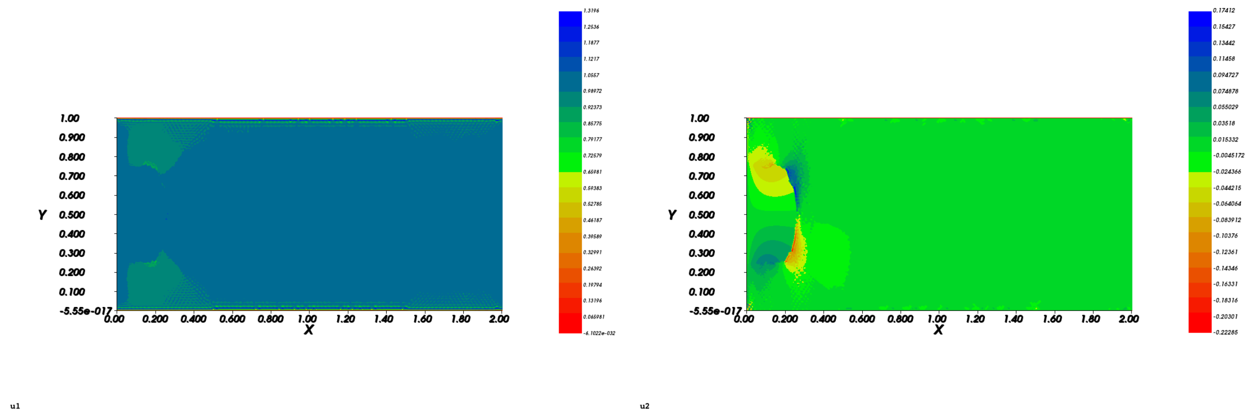

FIG. 5.3. *Velocity fields after optimization. Left:* u_1*, Right:* u_2*.*

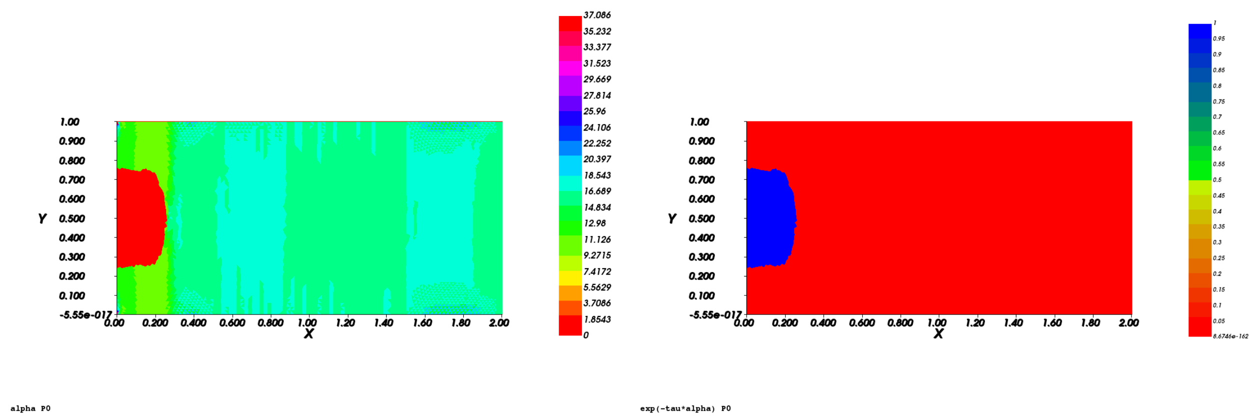

FIG. 5.4. α *after* 1500 *iteration. Left:* α*, Right:* $\exp(-\tau\alpha)$*.*

Error estimates for the numerical approximation of the optimality condition could also be investigated for finite element or finite volume method. Such analysis has already been carried out for finite element approximation of distributed control problem for Navier-Stokes equation [15, 30]. Convergence of finite volume method has been proved in [21] for a control problem in linear conduction coefficient. All of these previous works can serve as a basis to handle the case of topology optimization with our penalized model.

Another possible future work is linked to the topological asymptotic expansion [2, 3] which compute the so-called topological derivative as the first order term in the asymptotic expansion of the solution to PDE in domain with hole with vanishing size. An idea could be to replace the hole by a porous medium with a given permeability and compute the asymptotic expansion of the solution to the fluid-porous medium transmission problem as the characteristic size of the porous medium goes to zero.

A last direction for future work is related to the derivation of similar penalization model for other modeling of porous media (see e.g. [20]). For instance the Brinkman model [5] involves a second order term, with an effective viscosity, in the porous medium and can be used to account high porosity of the porous media. Another example is the Forchheimer model [14] whose model involves an additional non-linear term inside the porous media to deal with microscopic inertial effects. The case of time-dependent model is also an interesting continuation of the present work.

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