LINEAR EXTENDERS AND THE AXIOM OF CHOICE

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Abstract. In set theory without the axiom of Choice **ZF**, we prove that for every commutative field $\mathbb{K}$, the following statement $\mathbf{D}_{\mathbb{K}}$: "On every non null $\mathbb{K}$-vector space, there exists a non null linear form" implies the existence of a "$\mathbb{K}$-linear extender" on every vector subspace of a $\mathbb{K}$-vector space. This solves a question raised in [9]. In the second part of the paper, we generalize our results in the case of spherically complete ultrametric valued fields, and show that Ingleton's statement is equivalent to the existence of "isometric linear extenders".

1. Introduction

We work in **ZF**, set theory without the Axiom of Choice (**AC**). Given a commutative field $\mathbb{K}$ and two $\mathbb{K}$-vector spaces E, F, we denote by $L_{\mathbb{K}}(E,F)$ (or $L(E,F)$) the set of $\mathbb{K}$-linear mappings $T : E \to F$. Thus, $L(E,F)$ is a vector subspace of the product vector space F^E. A *linear form* on the $\mathbb{K}$-vector space E is a $\mathbb{K}$-linear mapping $f : E \to \mathbb{K}$; we denote by E^* the *algebraic dual* of E, *i.e.* the vector space $L(E,\mathbb{K})$. Given two $\mathbb{K}$-vector spaces E_1 and E_2, and a linear mapping $T : E_1 \to E_2$, we denote by $T^t : E_2^* \to E_1^*$ the mapping associating to every $g \in E_2^*$ the linear mapping $g \circ T : E_1 \to \mathbb{K}$: the mapping T^t is $\mathbb{K}$-linear and is called the *transposed* mapping of T. Given a vector subspace F of the $\mathbb{K}$-vector space E, a *linear extender* for F in E is a linear mapping $T : F^* \to E^*$ associating to each linear form $f \in F^*$ a linear form $\tilde{f} : E \to \mathbb{K}$ extending f. If G is a *complementary subspace* of F in E (*i.e.* $F + G = E$ and $F \cap G = \{0\}$), which we denote by $F \oplus G = E$, if $p : E \to F$ is the linear mapping fixing every element of F and which is null on G, then the transposed mapping $p^t : F^* \to E^*$ associating to every $f \in F^*$ the linear form $f \circ p : E \to \mathbb{K}$ is a linear extender for F in E. However, given a commutative field $\mathbb{K}$, the existence of a complementary subspace of every subspace of a $\mathbb{K}$-vector space implies **AC** in **ZF**. More precisely, denoting by **ZFA** (see [7, p. 44]) the set theory **ZF** with the axiom of extensionality weakened to allow the existence of atoms, the existence of a complementary subspace for every subspace of a $\mathbb{K}$-vector space implies, in **ZFA** (see [2, Lemma 2]), the following *Multiple Choice* axiom **MC** (see [7, p. 133] and form 37 of [4, p. 35]): *"For every infinite family $(X_i)_{i\in I}$ of nonempty sets, there exists a family $(F_i)_{i\in I}$ of nonempty finite sets such that for each $i \in I$, $F_i \subseteq X_i$."* It is known that **MC** is equivalent to **AC** in **ZF**, but **MC** does not imply **AC** in **ZFA**.

Given a commutative field $\mathbb{K}$, we consider the following consequences of the Axiom of Choice:

- $\mathbf{BE}_{\mathbb{K}}$: *"Every linearly independent subset of a vector space E over $\mathbb{K}$ is included in a basis of E."*
- $\mathbf{B}_{\mathbb{K}}$: *"Every vector space over $\mathbb{K}$ has a basis."*

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- $\mathbf{LE}_{\mathbb{K}}$: (Linear Extender) *"For every subspace F of a $\mathbb{K}$-vector space E, there exists a linear mapping $T : F^* \to E^*$ associating to every $f \in F^*$ a linear mapping $T(f) : E \to \mathbb{K}$ extending f.*
- $\mathbf{D}_{\mathbb{K}}$: *"For every non null $\mathbb{K}$-vector space E there exists a non null linear form $f : E \to \mathbb{K}$.*

In **ZF**, $\mathbf{BE}_{\mathbb{K}} \Rightarrow \mathbf{B}_{\mathbb{K}} \Rightarrow \mathbf{LE}_{\mathbb{K}} \Rightarrow \mathbf{D}_{\mathbb{K}}$ (see [9, Proposition 4]). In this paper, we show (see Theorem 2.5 in Section 2) that for each commutative field $\mathbb{K}$, $\mathbf{D}_{\mathbb{K}}$ implies $\mathbf{LE}_{\mathbb{K}}$, and this solves Question 2 of [9]. In Section 3 we provide several other statements which are equivalent to $\mathbf{D}_{\mathbb{K}}$ and we introduce the following consequence $w\mathbf{D}_{\mathbb{K}}$ of $\mathbf{D}_{\mathbb{K}}$: *"For every $\mathbb{K}$-vector space E and every $a \in E \backslash \{0\}$, there exists an additive mapping $f : E \to \mathbb{K}$ such that $f(a) = 1$."*

Question 1.1. Given a commutative field $\mathbb{K}$, does the statement $w\mathbf{D}_{\mathbb{K}}$ imply $\mathbf{D}_{\mathbb{K}}$?

Blass ([1]) has shown that the statement $\forall \mathbb{K} \mathbf{B}_{\mathbb{K}}$ (form 66 of [4]: "For every commutative field, every $\mathbb{K}$-vector space has a basis") implies **MC** in **ZFA** (and thus implies **AC** in **ZF**), but it is an open question to know whether there exists a commutative field $\mathbb{K}$ such that $\mathbf{B}_{\mathbb{K}}$ implies **AC**. In **ZFA**, the statement **MC** implies $\mathbf{D}_{\mathbb{K}}$ for every commutative field $\mathbb{K}$ with null characteristic (see [9, Proposition 1]). Thus in **ZFA**, the statement "For every commutative field $\mathbb{K}$ with null characteristic, $\mathbf{D}_{\mathbb{K}}$" does not imply **AC**. Denoting by **BPI** the *Boolean prime ideal*: "Every non null boolean algebra has an ultrafilter" (see form 14 in [4]), Howard and Tachtsis (see [5, Theorem 3.14]) have shown that for every finite field $\mathbb{K}$, **BPI** implies $\mathbf{D}_{\mathbb{K}}$. Since $\mathbf{BPI} \not\Rightarrow \mathbf{AC}$, the statement "For every finite field $\mathbb{K}$, $\mathbf{D}_{\mathbb{K}}$" does not imply **AC**. They also have shown (see [5, Corollary 4.9]) that in **ZFA**, $\forall \mathbb{K} \mathbf{D}_{\mathbb{K}}$ ("For every commutative field, for every non null $\mathbb{K}$-vector space E, there exists a $\mathbb{K}$-linear form $f : E \to \mathbb{K}$") does not imply $\forall \mathbb{K} \mathbf{B}_{\mathbb{K}}$, however, the following questions seem to be open in **ZF**:

Question 1.2. Does the statement $\forall \mathbb{K} \mathbf{D}_{\mathbb{K}}$ imply **AC** in **ZF**? Is there a (necessarily infinite) commutative field $\mathbb{K}$ such that $\mathbf{D}_{\mathbb{K}}$ implies **AC** in **ZF**?

In Section 4, we extend Proposition 1 of Section 2 to the case of spherically complete ultrametric valued fields (see Lemma 4.9) and prove that Ingleton's statement, which is a "Hahn-Banach type" result for ultrametric semi-normed spaces over spherically complete ultrametric valued fields $\mathbb{K}$, follows from **MC** when $\mathbb{K}$ has a null characteristic. In Section 5, we prove that Ingleton's statement is equivalent to the existence of "isometric linear extenders".

2. $\mathbf{LE}_{\mathbb{K}}$ and $\mathbf{D}_{\mathbb{K}}$ are equivalent

2.1. **Reduced powers of a commutative field $\mathbb{K}$.** Given a set E and a filter $\mathcal{F}$ on a set I, we denote by $E_{\mathcal{F}}$ the quotient of the set E^I by the equivalence relation $=_{\mathcal{F}}$ on E^I satisfying for every $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I} \in E^I$, $x =_{\mathcal{F}} y$ if and only if $\{i \in I : x_i = y_i\} \in \mathcal{F}$. If $\mathbb{L}$ is a first order language and if E carries a $\mathbb{L}$-structure, then the quotient set $E_{\mathcal{F}}$ also carries a quotient $\mathbb{L}$-structure: this $\mathbb{L}$-structure is a *reduced power* of the $\mathbb{L}$-structure E (see [3, Section 9.4]). Denoting by $\delta : E \to E^I$ the *diagonal mapping* associating to each $x \in E$ the constant family $i \mapsto x$, and denoting by $\mathrm{can}_{\mathcal{F}} : E^I \to E_{\mathcal{F}}$ the canonical quotient mapping, then we denote by $j_{\mathcal{F}} : E \to E_{\mathcal{F}}$ the one-to-one mapping $\mathrm{can}_{\mathcal{F}} \circ \delta$. Notice that $j_{\mathcal{F}}$ is a morphism of $\mathbb{L}$-structures.

Example 2.1 (The reduced power $\mathbb{K}_{\mathcal{F}}$ of a field $\mathbb{K}$). Given a commutative field $\mathbb{K}$ and a filter $\mathcal{F}$ on a set I, we consider the unitary $\mathbb{K}$-algebra $\mathbb{K}^I$, then the quotient $\mathbb{K}$-algebra $\mathbb{K}_{\mathcal{F}}$ is the quotient of the $\mathbb{K}$-algebra $\mathbb{K}^I$ by the following ideal $nul_{\mathcal{F}}$ of *$\mathcal{F}$-almost everywhere null* elements of $\mathbb{K}^I$: $\{x = (x_i)_{i\in I} \in \mathbb{K}^I : \{i \in I : x_i = 0\} \in \mathcal{F}\}$. The mapping $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$ is a one-to-one unitary morphism of $\mathbb{K}$-algebras, thus $\mathbb{K}$ can be viewed as the one-dimensional unitary $\mathbb{K}$-subalgebra of the $\mathbb{K}$-algebra $\mathbb{K}_{\mathcal{F}}$. Notice that the $\mathbb{K}$-algebra $\mathbb{K}_{\mathcal{F}}$ is a field if and only if $\mathcal{F}$ is an ultrafilter.

Notation 2.2. For every set E, we denote by $fin(E)$ the set of finite subsets of E, and we denote by $fin^*(E)$ the set of nonempty finite subsets of E.

Given two sets E and I, a binary relation $R \subseteq E \times I$ is said to be *concurrent* (see [8]) if for every $G \in fin^*(E)$, the set $R[G] := \cap_{x\in G} R(x)$ is nonempty; in this case, $\{R(x) : x \in I\}$ satisfies the finite intersection property, and we denote by $\mathcal{F}_R$ the filter on I generated by the sets $R(x)$, $x \in E$.

2.2. **$\mathbf{D}_{\mathbb{K}}$ implies linear extenders.**

Remark 2.3. *It is known (see* [9, Theorem 2]*), that $\mathbf{D}_{\mathbb{K}}$ is equivalent to the following statement: "For every vector subspace of a $\mathbb{K}$-vector space E and every linear mapping $f : F \to \mathbb{K}$, there exists a $\mathbb{K}$-linear mapping $f : E \to \mathbb{K}$ extending f."*

Proposition 2.4. *Given a commutative field $\mathbb{K}$, the following statements are equivalent:*

i) $\mathbf{D}_{\mathbb{K}}$

ii) For every filter $\mathcal{F}$ on a set I, the linear embedding $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$ has a $\mathbb{K}$-linear retraction $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$.

Proof. $\Rightarrow$ Let $f : j_{\mathcal{F}}[\mathbb{K}] \to \mathbb{K}$ be the mapping $x \mapsto j_{\mathcal{F}}^{-1}(x)$. Then f is $\mathbb{K}$-linear and $j_{\mathcal{F}}[\mathbb{K}]$ is a vector subspace of the $\mathbb{K}$-vector space $\mathbb{K}_{\mathcal{F}}$. Using Remark 2.3, let $\tilde{f} : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be a $\mathbb{K}$-linear mapping extending f; then $\tilde{f}$ is a $\mathbb{K}$-linear retraction of $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$.
$\Leftarrow$ Let E be a non null $\mathbb{K}$-vector space. Let a be a non-null element of E. Using Lemma 1 in [9], there exists a filter $\mathcal{F}$ on the set $I = \mathbb{K}^E$, and a $\mathbb{K}$-linear mapping $g : E \to \mathbb{K}_{\mathcal{F}}$ such that $g(a) = j_{\mathcal{F}}(1)$. Using (ii), let $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be a $\mathbb{K}$-linear retraction of the linear embedding $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$. It follows that $f = r \circ g : E \to \mathbb{K}$ is a $\mathbb{K}$-linear mapping such that $f(a) = 1$. □

Theorem 2.5. $\mathbf{D}_{\mathbb{K}} \Leftrightarrow \mathbf{LE}_{\mathbb{K}}$.

Proof. The implication $\mathbf{LE}_{\mathbb{K}} \Rightarrow \mathbf{D}_{\mathbb{K}}$ is trivial. We shall prove $\mathbf{D}_{\mathbb{K}} \Rightarrow \mathbf{LE}_{\mathbb{K}}$. Given some vector subspace F of a vector space E, let I be the set of mappings $\Phi : F^* \to E^*$ and let R be the binary relation on $fin(F^*) \times I$ such that for every $Z \in fin(F^*)$ and every $\Phi \in I$, $R(Z, \Phi)$ if and only if for every $f \in Z$, the linear form $\Phi(f)$ extends f and Φ is $\mathbb{K}$-linear on $\text{span}_{F^*}(Z)$. Then the binary relation R is concurrent: given m finite subsets $Z_1, \dots, Z_m \in fin(F^*)$, let $B = \{f_1, \dots, f_p\}$ be a (finite) basis of the $\mathbb{K}$-vector subspace of F^* generated by the finite set $\cup_{1\le i\le m} Z_i$; then using $\mathbf{D}_{\mathbb{K}}$ (see Remark 2.3), let $\tilde{f}_1, \dots, \tilde{f}_p$ be linear forms on E extending $f_1, \dots, f_p$; let $L : \text{span}(\{f_1, \dots, f_p\}) \to E^*$ be the linear mapping such that for each $i \in \{1, \dots, p\}$, $L(f_i) = \tilde{f}_i$, and let $\Phi : F^* \to E^*$ be some mapping extending L (for example, define $\Phi(f) = 0$ for every $f \in F^* \setminus \text{span}(\{f_1, \dots, f_p\})$. Then for every $i \in \{1, \dots, m\}$, $R(Z_i, \Phi)$. Consider the filter $\mathcal{F}$ on I generated by $\{R[Z]; Z \in fin(F^*)\}$. Then the mapping $\Phi : F^* \to L(E, \mathbb{K}_{\mathcal{F}})$ associating to each $f \in F^*$ the $\mathbb{K}$-linear mapping $\Phi(f) : E \to \mathbb{K}_{\mathcal{F}}$ associating to each $x \in E$ the class of $(f(x))_{f\in I}$ in $\mathbb{K}_{\mathcal{F}}$ is linear, and for every

$f \in F^*$, $\Phi(f) : E \to \mathbb{K}_{\mathcal{F}}$ extends f. Using $\mathbf{D}_{\mathbb{K}}$, there exists (see Proposition 2.4) a $\mathbb{K}$-linear retraction $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ of $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$. Let $T : F^* \to E^*$ be the mapping $f \mapsto r \circ \Phi(f)$. Then the mapping $T : F^* \to E^*$ is a linear extender for F in E. □

Remark 2.6. *Notice that the axiom $\mathbf{LE}_{\mathbb{K}}$ is "multiple": given a family $(E_i)_{i\in I}$ of $\mathbb{K}$-vector spaces and a family $(F_i)_{i\in I}$ such that for each $i \in I$, F_i is a vector subspace of E_i, then there exists a family $(T_i)_{i\in I}$ such that for each $i \in I$, $T_i : F_i^* \to E_i^*$ is a linear extender for F_i in E_i: apply $\mathbf{LE}_{\mathbb{K}}$ to the vector subspace $\oplus_{i\in I} F_i$ of $\oplus_{i\in I} E_i$.*

Corollary 2.7. *Given a commutative field $\mathbb{K}$, then $\mathbf{D}_{\mathbb{K}}$ implies (and is equivalent to) the following statement: for every $\mathbb{K}$-vector space E, for every vector subspace F of E, denoting by* can $: F \to E$ *the canonical mapping, then the double transposed mapping* $\mathrm{can}^{tt} : F^{**} \to E^{**}$ *is one-to-one and has a $\mathbb{K}$-linear retraction $r : E^{**} \to F^{**}$.*

Proof. The mapping $\mathrm{can}^{tt} : F^{**} \to E^{**}$ associates to every $\Phi \in F^{**}$ the mapping $\overline{\Phi} : E^* \to \mathbb{K}$ such that for every $g \in E^*$, $\overline{\Phi}(g) = \Phi(g_{\restriction F})$. Given some $\Phi \in \ker(\mathrm{can}^{tt})$, then, for every $g \in E^*$, $(\mathrm{can}^{tt}(\Phi))(g) = 0$ *i.e.* $\Phi(g_{\restriction F}) = 0$; using $\mathbf{D}_{\mathbb{K}}$, for every $f \in F^*$ there exists some $g \in E^*$ such that $f = g_{\restriction F}$, thus for every $f \in F^*$, $\Phi(f) = 0$, so $\Phi = 0$. It follows that $can^{tt} : F^{**} \to E^{**}$ is one-to-one. Using the equivalent form $\mathbf{LE}_{\mathbb{K}}$ of $\mathbf{D}_{\mathbb{K}}$, let $T : F^* \to E^*$ be a $\mathbb{K}$-linear extender *i.e.* a $\mathbb{K}$-linear mapping such that for each $f \in F^*$, $T(f) : E \to \mathbb{K}$ extends f. Then the transposed mapping $T^t : E^{**} \to F^{**}$ is $\mathbb{K}$-linear and for every $\Phi \in F^{**}$, $T^t(\mathrm{can}^{tt}(\Phi)) = \mathrm{can}^{tt}(\Phi) \circ T = \Phi$. □

It follows that $\mathbf{D}_{\mathbb{K}}$ implies (and is equivalent to) the following statement: "For every vector space E, for every vector subspace F of of E, the canonical linear mapping $F^{**} \to E^{**}$ is one-to-one and there exists a $\mathbb{K}$-vector space G such that G is a complement of F^{**} in E^{**}.

Remark 2.8. *Corollary 2.7 is equivalent to its "multiple form": given a family $(E_i)_{i\in I}$ of $\mathbb{K}$-vector spaces and a family $(F_i)_{i\in I}$ such that for each $i \in I$, F_i is a vector subspace of E_i, then for each $i \in I$, the canonical mapping* $\mathrm{can}_i : F_i^{**} \to E_i^{**}$ *is one-to-one and there exists a family $(r_i)_{i\in I}$ such that for each $i \in I$, $r_i : E_i^{**} \to F_i^{**}$ is a $\mathbb{K}$-linear retraction of* $\mathrm{can}_i : F_i^{**} \to E_i^{**}$.

3. Other statements equivalent to $\mathbf{D}_{\mathbb{K}}$

3.1. $\mathbb{K}$-linearity and additive retractions.

Proposition 3.1. *Let $\mathbb{K}$ be a commutative field and let a be a non-null element of a $\mathbb{K}$-vector space E. Let $j_a : \mathbb{K} \hookrightarrow E$ be the mapping $\lambda \mapsto \lambda.a$. Given any additive mapping $r : E \to \mathbb{K}$ such that $r(a) = 1$, then r is $\mathbb{K}$-linear if and only if r is a retraction of the mapping $j_a : \mathbb{K} \hookrightarrow E$.*

Proof. The direct implication is easy to prove. We show the converse statement. Assuming that $r : E \to \mathbb{K}$ is an additive retraction of j_a, let us check that r is $\mathbb{K}$-linear. Since r is a retraction of j_a, $\ker(r) \cap \mathbb{K}.a = \{0\}$. Thus $\ker(r) \oplus \mathbb{K}.a = E$ is the direct sum of groups with the unique decomposition $x = (x - r(x).a) + r(x).a$ for every $x \in E$. Therefore, $\ker(r)$ is a maximal subgroup H of E such that $H \cap \mathbb{K}.a = \{0\}$. Also notice that, from $\ker(r) \cap \mathbb{K}.a = \{0\}$, it follows that $\mathbb{K}.\ker(r) \cap \mathbb{K}.a = \{0\}$, thus $\mathbb{K}.\ker(r) = \ker(r)$. We now check that the additive mapping r is $\mathbb{K}$-linear: given $z \in E$, then $z = x \oplus t.a$ where $x \in \ker(r)$ and $t \in \mathbb{K}$; thus for every $\lambda \in \mathbb{K}$, $r(\lambda.z) = r(\lambda.x \oplus \lambda t.a) = r(\lambda.x) + r(\lambda t.a) = r(\lambda t.a) = \lambda t = \lambda.r(z)$. □

3.2. **Additivity statements equivalent to $\mathbf{D}_{\mathbb{K}}$.** Given a commutative field $(\mathbb{K}, +, \times, 0, 1)$, we consider the following statements:

$\mathbf{A}_{\mathbb{K}}$: *"For every $\mathbb{K}$-vector space E and every subgroup F of $(E, +)$, for every additive mapping $f : F \to (\mathbb{K}, +)$, there exists an additive mapping $\tilde{f} : E \to \mathbb{K}$ extending f.*

$\mathbf{A}'_{\mathbb{K}}$: *"For every $\mathbb{K}$-vector space E and every vector subspace F of E, for every $\mathbb{K}$-linear mapping $f : F \to \mathbb{K}$, there exists an additive mapping $\tilde{f} : E \to \mathbb{K}$ extending f.*

$\mathbf{A}''_{\mathbb{K}}$: *"For every $\mathbb{K}$-vector space E and every $a \in E\backslash\{0\}$, there exists an additive mapping $f : E \to \mathbb{K}$ such that for every $\lambda \in \mathbb{K}$, $f(\lambda.a) = \lambda$."*

Proposition 3.2. *For every commutative field $\mathbb{K}$, $\mathbf{A}_{\mathbb{K}} \Leftrightarrow \mathbf{A}'_{\mathbb{K}} \Leftrightarrow \mathbf{A}''_{\mathbb{K}} \Leftrightarrow \mathbf{D}_{\mathbb{K}}$.*

Proof. The implications $\mathbf{A}_{\mathbb{K}} \Rightarrow \mathbf{A}'_{\mathbb{K}}$ and $\mathbf{A}'_{\mathbb{K}} \Rightarrow \mathbf{A}''_{\mathbb{K}}$ are trivial. We prove $\mathbf{A}''_{\mathbb{K}} \Rightarrow \mathbf{D}_{\mathbb{K}}$. In view of Proposition 2.4, we prove that for every filter $\mathcal{F}$ on a set I, the canonical mapping $j_{\mathcal{F}} : \mathbb{K} \hookrightarrow \mathbb{K}_{\mathcal{F}} := \mathbb{K}^I/\mathcal{F}$ has a $\mathbb{K}$-linear retraction. Let $f : \mathbb{K} \to \mathbb{K}$ be the identity mapping. Using $\mathbf{A}''_{\mathbb{K}}$, let $\tilde{f} : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be an additive mapping extending f. Using Proposition 3.1, $\tilde{f}$ is $\mathbb{K}$-linear. It follows that $\tilde{f}$ is a $\mathbb{K}$-linear retraction of $j_{\mathcal{F}} : \mathbb{K} \hookrightarrow \mathbb{K}_{\mathcal{F}}$.
$\mathbf{D}_{\mathbb{K}} \Rightarrow \mathbf{A}_{\mathbb{K}}$. Let E be a $\mathbb{K}$-vector space, let F be a subgroup of the additive group $(E, +)$, and let $f : F \to (\mathbb{K}, +)$ be an additive mapping. Using a concurrent relation (the proof is similar to the proof of Lemma 1 in [9]), let $\mathcal{F}$ be a filter on the set $I = \mathbb{K}^E$ and let $\iota : E \to \mathbb{K}_{\mathcal{F}}$ be an additive mapping extending f. Using $\mathbf{D}_{\mathbb{K}}$, let $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be an additive retraction of $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$. Then $\tilde{f} := r \circ \iota : E \to \mathbb{K}$ is additive and extends f. □

3.3. **A consequence of $\mathbf{D}_{\mathbb{K}}$.**

Proposition 3.3. *Given a commutative field $\mathbb{K}$, the following statements are equivalent:*

i) $w\mathbf{D}_{\mathbb{K}}$*:* "For every $\mathbb{K}$-vector space E and every $a \in E\backslash\{0\}$, there exists an additive mapping $f : E \to \mathbb{K}$ such that $f(a) = 1$."

ii) "For every non null $\mathbb{K}$-vector space E, there exists a non null additive mapping $f : E \to \mathbb{K}$."

iii) "For every filter $\mathcal{F}$ on a set I, there exists a non null additive mapping $f : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$."

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are easy. We prove (iii) $\Rightarrow$ (i) Given a non null element a of a $\mathbb{K}$-vector space E, let $\mathcal{F}$ be a filter on a set I and $g : E \to \mathbb{K}_{\mathcal{F}}$ be a $\mathbb{K}$-linear mapping such that $g(a) = 1$. Using (iii), let $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be a non null additive mapping. Let $\alpha \in \mathbb{K}_{\mathcal{F}}$ such that $r(\alpha) \neq 0$. Let $m_\alpha : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}_{\mathcal{F}}$ be the additive mapping $x \mapsto \alpha x$. Then $g_1 := r \circ m_\alpha \circ g : E \to \mathbb{K}$ is additive; let $f := \frac{1}{r(\alpha)}.g_1 : E \to \mathbb{K}$; then f is additive and $f(a) = 1$. □

Remark 3.4. *Given a commutative field $\mathbb{K}$ with prime field k, then $\mathbf{D}_k$ implies $w\mathbf{D}_{\mathbb{K}}$.*

Proof. Given a $\mathbb{K}$-vector space E, a mapping $f : E \to \mathbb{K}$ is additive if and only if f is k-linear. □

Question 3.5. Given a commutative field $\mathbb{K}$ with null characteristic, does $\mathbf{D}_{\mathbb{Q}}$ imply $\mathbf{D}_{\mathbb{K}}$?

Question 3.6. Given a prime number p and a commutative field $\mathbb{K}$ with characteristic p, and denoting by $\mathbb{F}_p$ the finite field with p elements, does $\mathbf{D}_{\mathbb{F}_p}$ imply $\mathbf{D}_{\mathbb{K}}$? Does **BPI** (which implies $\mathbf{D}_{\mathbb{F}_p}$) imply $\mathbf{D}_{\mathbb{K}}$?

4. Ingleton's statement for ultrametric valued fields

4.1. **Semi-norms on vector spaces over a valued field.**

4.1.1. *Pseudo metric spaces.* Given a set X, a *pseudo-metric* on X is a mapping $d : X \times X \to \mathbb{R}_+$ such that for every $x, y, z \in X$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$. If d satisfies the extra property $(d(x, y) = 0 \Rightarrow x = y)$, then d is a *metric* on X. A pseudo-metric d on X is said to be *ultrametric* if for every $x, y, z \in X$, $d(x, z) \leq \max(d(x, y), d(y, z))$.

Given a pseudo-metric space (X, d), for every $a \in X$ and every $r \in \mathbb{R}_+$, we denote by $B_s(a, r)$ the "strict" ball $\{x \in X : d(x, a) < r\}$ and we denote by $B(a, r)$ the "large" ball $\{x \in X : d(x, a) \leq r\}$. Notice that large balls of a pseudo-metric space are nonempty. A pseudo-metric space (X, d) is *spherically complete* if every chain (*i.e.* set which is linearly ordered for the inclusion) of large balls of X has a nonempty intersection.

Example 4.1. Given a nonempty set X, the discrete metric d_{disc} on X, associating to each $(x, y) \in X \times X$ the real number 1 if $x \neq y$ and 0 else is ultrametric and the associated metric space (X, d_{disc}) is spherically complete since large balls for this metric are singletons and the whole space X.

4.1.2. *Group semi-norms.* Given a commutative group $(G, +, 0)$, a *group semi-norm* on G is a mapping $N : G \to \mathbb{R}_+$ which is sub-additive (for every $x, y \in G$, $N(x+y) \leq N(x)+N(y)$) and symmetric (for every $x \in G$, $N(-x) = N(x)$) and such that $N(0) = 0$. If for every $x \in G$, $(N(x) = 0 \Rightarrow x = 0)$, then N is a *norm*. Given a group semi-norm N on an abelian group $(G, +, 0)$, the mapping $d : G \times G \to \mathbb{R}_+$ associating to each $(x, y) \in G \times G$ the real number $N(x - y)$ is a pseudo-metric on G. Moreover, if N is a norm, then d is a metric on G. The semi-norm N is said to be ultrametric if the pseudo-metric d_N is ultrametric, which is equivalent to say that for every $x, y \in G$, $N(x + y) \leq \max(N(x), N(y))$. The topology on G associated to the pseudo-metric d_N is a group topology on G, which is Hausdorff if and only if N is a norm.

4.1.3. *Absolute values on a commutative field.* Given a commutative unitary ring $(R, +, \times, 0, 1)$, a *ring semi-norm* (see [14, p. 137]) on R is a group semi-norm on $(R, +, 0)$ which is sub-multiplicative: for every $x, y \in R$, $N(x \times y) \leq N(x)N(y)$; if a ring semi-norm $N : R \to \mathbb{R}_+$ is non null, then $N(1) \geq 1$. If R is a commutative field and if N is a ring semi-norm on R which is multiplicative (for every $x, y \in R$, $N(x \times y) = N(x)N(y)$) and non null, then $N(1) = 1$, N is a norm and N is called an *absolute value* on the commutative field R, and (R, N) is a *valued field.*

Example 4.2. Given a commutative field $\mathbb{K}$, the mapping $|.|_{disc} : \mathbb{K} \to \mathbb{R}_+$ associating to each $x \in \mathbb{K}$ the real number 1 if $x \neq 0$ and 0 else is an absolute value on $\mathbb{K}$, which is called the *trivial* absolute value on $\mathbb{K}$. The metric associated to this absolute value is the discrete metric d_{disc} on $\mathbb{K}$, thus, the discrete field $(\mathbb{K}, |.|_{disc})$ is spherically complete.

4.1.4. *Vector semi-norms on vector spaces over a valued field.* Given a valued field $(\mathbb{K}, |.|)$ and a $\mathbb{K}$-vector space E, a *vector semi-norm* on E (see [14, p. 210]) is a group semi-norm $p : E \to \mathbb{R}_+$ on the abelian group $(E, +)$ such that for every $\lambda \in \mathbb{K}$ and every $x \in E$, $p(\lambda.x) = |\lambda|p(x)$; if $(\mathbb{K}, |.|)$ is ultrametric, then the semi-norm $p : E \to \mathbb{R}_+$ is said to be *ultrametric* if and only if for every $x, y \in E$, $p(x + y) \leq \max(p(x), p(y))$. Given a $\mathbb{K}$-algebra $(A, +, \times, 0, 1, \lambda.)$, an *algebra semi-norm* on A is a ring semi-norm on the ring A which is also a vector semi-norm on the $\mathbb{K}$-vector space A.

4.2. **Ingleton's statement for spherically complete ultrametric valued fields.** Given a valued field $(\mathbb{K},|.|)$, a set E and a mapping $f : E \to \mathbb{K}$, we denote by $|f| : E \to \mathbb{R}_+$ the mapping $x \mapsto |f(x)|$. We endow $\mathbb{R}^E$ with the product order, thus given mappings $p : E \to \mathbb{R}$ and $q : E \to \mathbb{R}$, $p \leq q$ means that for every $x \in E$, $p(x) \leq q(x)$. In particular, $|f| \leq p$ means that for every $x \in E$, $|f(x)| \leq p(x)$.

Lemma (Ingleton, [6])**.** *Let $(\mathbb{K},|.|)$ be a ultrametric spherically complete valued field, let E be a $\mathbb{K}$-vector space and let $p : E \to \mathbb{R}$ be an ultrametric vector semi-norm. Let F be a vector subspace of E and let $f : F \to \mathbb{K}$ be a linear mapping such that $|f| \leq p$. Then, for every $a \in E \backslash F$, there exists a linear mapping $\tilde{f} : F \oplus \mathbb{K}a \to \mathbb{K}$ extending f such that $\tilde{f} \leq p_{\restriction F \oplus \mathbb{K}a}$.*

Proof. Ingleton's proof of this Lemma holds in **ZF**. For sake of completeness, we give a proof of this Lemma. We search for some $\alpha \in \mathbb{K}$ such that:

$$\forall z \in F\ \forall \lambda \in \mathbb{K}\ |f(z) + \lambda.\alpha| \leq p(z + \lambda.a) \tag{1}$$

Given some $\alpha \in \mathbb{K}$, and denoting by $\mathbb{K}^*$ the set $\mathbb{K}\backslash\{0\}$, Condition (1) is equivalent to:

$$\forall z \in F\ \forall \lambda \in \mathbb{K}^*\ |\lambda.(f(\frac{z}{\lambda}) + \alpha)| \leq p(\lambda(\frac{z}{\lambda} + a))$$

$$\forall z \in F\ \forall \lambda \in \mathbb{K}^*\ |\lambda|.|f(\frac{z}{\lambda}) + \alpha)| \leq |\lambda|.p(\frac{z}{\lambda} + a)$$

$$\forall z \in F\ \forall \lambda \in \mathbb{K}^*\ |(f(\frac{z}{\lambda}) + \alpha)| \leq p(\frac{z}{\lambda} + a)$$

$$\forall z' \in F\ |f(z') + \alpha)| \leq p(z' + a)$$

$$\forall z \in F\ |f(z) - \alpha)| \leq p(z - a)$$

Let $\mathcal{B}_a$ the set of large balls $B(f(z), p(z - a))$ for $z \in F$. The element $\alpha \in \mathbb{K}$ satisfies (1) if and only if $\alpha \in \cap\mathcal{B}_a$. Given two elements $z_1, z_2 \in F$, then $|f(z_1) - f(z_2)| = |f(z_1 - z_2)| \leq p(z_1 - z_2) \leq \max(p(z_1 - a), p(z_2 - a))$ since p is ultrametric. If $p(z_1 - a) \leq p(z_2 - a)$, it follows that $f(z_1) \in B(f(z_2), p(z_2 - a))$ thus, since $(\mathbb{K},|.|)$ is ultrametric, $B(f(z_1), p(z_2 - a)) \subseteq B(f(z_2), p(z_2 - a))$, thus $B(f(z_1), p(z_1 - a)) \subseteq B(f(z_2), p(z_2 - a))$, so the set of large balls $\mathcal{B}_a$ is a chain. Since $(\mathbb{K},|.|)$ is spherically complete, it follows that $\cap\mathcal{B}_a$ is nonempty. ☐

Given a spherically complete ultrametric valued field $(\mathbb{K},|.|)$, we now consider Ingleton's statement:

$\mathbf{I}_{(\mathbb{K},|.|)}$: *"For every $\mathbb{K}$-vector space E endowed with an ultrametric vector semi-norm $p : E \to \mathbb{R}_+$, for every vector subspace F of E and every linear mapping $f : F \to \mathbb{K}$ such that $|f| \leq p_{\restriction F}$, there exists a linear mapping $\tilde{f} : E \to \mathbb{K}$ extending f such that $|\tilde{f}| \leq p$."*

Corollary (Ingleton, [6])**.** *For every spherically complete ultrametric valued field $(\mathbb{K},|.|)$,* $\mathbf{AC} \Rightarrow \mathbf{I}_{(\mathbb{K},|.|)}$.

Proof. Use Zorn's lemma and the previous Lemma. ☐

Remark 4.3. *Notice that in* **ZF***, given a spherically complete ultrametric valued field $(\mathbb{K},|.|)$ and a $\mathbb{K}$-vector space E which has a well-orderable basis (for example a finitely generated $\mathbb{K}$-vector space), then every ultrametric vector semi-norm $p : E \to \mathbb{R}_+$ satisfies the Ingleton statement.*

4.3. **MC implies Ingleton's statement for some null characteristic fields.**

Proposition 4.4. *Given a spherically complete ultrametric valued field* $(\mathbb{K}, |.|)$ *with null characteristic such that the restricted absolute value* $|.|_{\restriction \mathbb{Q}}$ *is trivial,* **MC** *implies* $\mathbf{I}_{(\mathbb{K},|.|)}$. *It follows that in* **ZFA**, *the Ingleton statement restricted to null characteristic ultrametric valued fields* $(\mathbb{K}, |.|)$ *such that the restricted absolute value* $|.|_{\restriction \mathbb{Q}}$ *is trivial does not imply* **AC**.

Proof. Assume that E is a $\mathbb{K}$-vector space endowed with an ultrametric semi-norm $p : E \to \mathbb{R}_+$, and assume that F is a vector subspace of E and that $f : F \to \mathbb{K}$ is a linear mapping such that $|f| \leq p$. Using **MC**, there exists an ordinal α and some partition $(F_i)_{i\in\alpha}$ in finite sets of $E\backslash F$. This implies that there is an ordinal β and a strictly increasing family $(V_i)_{i\in\beta}$ of vector subspaces of E such that $V_0 = F$, for every $i \in \beta$ such that $i + 1 \in \beta$, V_{i+1}/V_i is finite-dimensional, and for every non null limit ordinal $i \in \beta$, $V_i = \cup_{j<i} V_j$. Let Z be the set of 3-uples (V, W, l) such that V, W are subspaces of E satisfying $V \subseteq W$, W/V is finite-dimensional and $l : V \to \mathbb{K}$ is a linear mapping satisfying $|l| \leq p$. For each $(V, W, l) \in Z$, using the previous Lemma (which holds in **ZFA**), the set $A_{(V,W,l)}$ of linear mappings $\tilde{l} : W \to \mathbb{K}$ extending l and satisfying $|\tilde{l}| \leq p$ is non-empty; using **MC**, there is a mapping associating to every $i = (V, W, l) \in Z$ a nonempty finite subset B_i of $A_{(V,W,l)}$. Then, for every $i \in Z$, define $\Phi(i) := \frac{1}{\#B_i} \sum_{u\in B_i} u$ where $\#B_i$ is the cardinal of the nonempty finite set B_i (here we use the fact that the characteristic of $\mathbb{K}$ is null); notice that $\Phi(i)$ is linear and that that for every $x \in W$, $|\Phi(i)(x)| = |\sum_{u\in B_i} u(x)| \leq \max_{u\in B_i} |u(x)| \leq p(x)$. Using the function Φ, we define by transfinite recursion a family $(f_i)_{i\in\beta}$ such that for each $i \in \beta$, $f_i : V_i \to \mathbb{K}$ is linear, f_i extends f, $|f_i| \leq p$, and for every $i < j \in \beta$, f_j extends f_i. Let $\tilde{f} := \cup_{i\in\beta} f_i$. Then $\tilde{f} : E \to \mathbb{K}$ is linear, $\tilde{f}$ extends f and $|\tilde{f}| \leq p$. □

Question 4.5. Given a spherically complete ultrametric valued field $(\mathbb{K}, |.|)$, does $\mathbf{B}_{\mathbb{K}}$ imply $\mathbf{I}_{(\mathbb{K},|.|)}$? Does $\mathbf{BE}_{\mathbb{K}}$ imply $\mathbf{I}_{(\mathbb{K},|.|)}$?

Question 4.6 (van Rooij, see [13])**.** Does the full Ingleton statement (*i.e.* $\mathbf{I}_{(\mathbb{K},|.|)}$ for every spherically complete ultrametric valued field $(\mathbb{K}, |.|)$) imply **AC**?

Remark 4.7. *If a commutative field* $\mathbb{K}$ *is endowed with the trivial absolute value* $|.|_{disc}$, *then* $\mathbf{I}_{(\mathbb{K},|.|_{disc})}$ *is the statement* $\mathbf{D}_{\mathbb{K}}$, *thus the "full Ingleton statement" implies* $\mathbf{D}_{\mathbb{K}}$ *for every commutative field* $\mathbb{K}$.

Proof. $\mathbf{I}_{(\mathbb{K},|.|_{disc})} \Rightarrow \mathbf{D}_{\mathbb{K}}$. Given a $\mathbb{K}$-vector space E and a non null vector $a \in E$, consider the trivial ultrametric vector semi-norm $p : E \to \mathbb{R}_+$ defined by $p(0_E) = 0$ and for every $x \in E\backslash\{0\}$, $p(x) = 1$. Let $f : \mathbb{K}.a \to \mathbb{K}$ be the linear mapping such that $f(a) = 1$; then $|f(0_E)|_{disc} = 0 = p(0_E)$ and, for every $\lambda \in \mathbb{K}\backslash\{0\}$, $|f(\lambda.a)|_{disc} = |\lambda|_{disc} = 1 = p(\lambda.a)$. Using $\mathbf{I}_{(\mathbb{K},|.|_{disc})}$, there exists a linear mapping $\tilde{f} : E \to \mathbb{K}$ extending f such that for every $x \in E$, $|\tilde{f}(x)|_{disc} \leq p(x)$. Thus $\tilde{f}$ is a linear form on E such that $\tilde{f}(a) = 1$.
$\mathbf{D}_{\mathbb{K}} \Rightarrow \mathbf{I}_{(\mathbb{K},|.|_{disc})}$. Let E be a $\mathbb{K}$-vector space and let $p : E \to \mathbb{R}_+$ be an ultrametric vector semi-norm with respect to the valued field $(\mathbb{K}, |.|_{disc})$. Assume that F is a vector subspace of E and that $f : F \to \mathbb{K}$ is a linear form such that for every $x \in F$, $|f(x)|_{disc} \leq p(x)$. If f is null, then the null mapping $\tilde{f} : E \to \mathbb{K}$ extends f and satisfies $|\tilde{f}(x)|_{disc} \leq p(x)$ for every $x \in E$. If f is not null, let $a \in F$ such that $f(a) = 1$. Denoting by V the vector subspace $\{x \in E : p(x) < 1\}$, then $a \notin V + \ker(f)$ because if $a = a_1 + a_2$ with $a_1 \in V$ and $a_2 \in \ker(f)$, then $a_1 = a - a_2 \in F$ thus $1 = f(a) = f(a_1) \leq p(a_1)$ so $a_1 \notin V$, which is contradictory!

Let $\mathrm{can} : E \to E/(V + \ker(f))$ be the quotient mapping. Since $a \notin V + \ker(f)$, it follows that $\mathrm{can}(a) \neq 0$. Using $\mathbf{D}_{\mathbb{K}}$, let $g : E/(V + \ker(f)) \to \mathbb{K}$ be a linear mapping such that $g(\mathrm{can}(a)) = 1$. Then $\tilde{f} := g \circ \mathrm{can} : E \to \mathbb{K}$ is a linear mapping which is null on $V + \ker(f)$ and such that $\tilde{f}(a) = 1$. Since $F = \ker(f) \oplus \mathbb{K}.a$, it follows that $\tilde{f}$ extends f. We now show that $|\tilde{f}| \leq p$. Given some $x \in E$, if $x \in V$ then $\tilde{f}(x) = 0$ so $|\tilde{f}(x)|_{disc} \leq p(x)$; else $p(x) \geq 1$, thus $|\tilde{f}(x)|_{disc} \leq 1 \leq p(x)$. □

4.4. **Ingleton's statement for ultrametric fields with compact large balls.** Given a valued field $(\mathbb{K}, |.|)$ and a filter $\mathcal{F}$ on a set I, we denote by $|.|_{\mathcal{F}} : \mathbb{K}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ the quotient mapping associating to each $x \in \mathbb{K}_{\mathcal{F}}$ which is the equivalence class of some $(x_i)_{i\in I} \in \mathbb{K}^I$, the class of $(|x_i|)_{i\in I}$ in $\mathbb{R}_{\mathcal{F}}$. We denote by $(\mathbb{K}_{\mathcal{F}})_b$ the following unitary subalgebra of "bounded elements" of the unitary $\mathbb{K}$-algebra $\mathbb{K}_{\mathcal{F}}$: $\{x \in \mathbb{K}_{\mathcal{F}} : \exists t \in \mathbb{R}_+ : |x|_{\mathcal{F}} \leq_{\mathcal{F}} t\}$. We also denote by $N_{|.|,\mathcal{F}} : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{R}_+$ the mapping associating to each $x \in (\mathbb{K}_{\mathcal{F}})_b$ the real number $\inf\{t \in \mathbb{R}_+ : |x|_{\mathcal{F}} \leq_{\mathcal{F}} t\}$. The mapping $N_{|.|,\mathcal{F}}$ is a unitary algebra semi-norm on $(\mathbb{K}_{\mathcal{F}})_b$; moreover, if the valued field $(\mathbb{K}, |.|)$ is ultrametric, the vector semi-norm $N_{|.|,\mathcal{F}}$ is also ultrametric.

Lemma 4.8. *Let $(\mathbb{K}, |.|)$ be a spherically complete ultrametric valued field, let E be a $\mathbb{K}$-vector space and let $p : E \to \mathbb{K}$ be an ultrametric vector semi-norm. Assume that F is a vector subspace of E and that $f : F \to \mathbb{K}$ is a linear form such that $|f| \leq p$. Let $I := \mathbb{K}^E$. There exists a filter $\mathcal{F}$ on I and a $\mathbb{K}$-linear mapping $\iota : E \to (\mathbb{K}_{\mathcal{F}})_b$ definable from E, p and f such that ι extends f and such that $N_{|.|,\mathcal{F}} \circ \iota \leq p$.*

Proof. Let $R \subseteq (fin(E) \times I)$ be the following binary relation: given $Z \in fin(E)$ and given some mapping $u : E \to \mathbb{K}$, then $R(Z, u)$ iff u extends f, $|u| \leq p$ and $u_{\restriction Z}$ is linear *i.e.* for every $x, y \in Z$ and $\lambda \in \mathbb{K}$, $(x + y \in Z \Rightarrow u(x + y) = u(x) + u(y))$ and $(\lambda x \in Z \Rightarrow u(\lambda x) = \lambda u(x))$. Using Ingleton's Lemma in Section 4.2, the binary relation R is concurrent, thus it generates a filter $\mathcal{F}$ on I. Let $\iota : E \to \mathbb{K}_{\mathcal{F}}$ be the mapping associating to each $x \in E$, the equivalence class of $(i(x))_{i\in I}$ in $\mathbb{K}_{\mathcal{F}}$. Then, for every $x \in F$, $\iota(x) = f(x)$ and for every $x \in E$, $|\iota(x)|_{\mathcal{F}} \leq p(x)$ whence $N_{|.|,\mathcal{F}}(\iota(x)) \leq p(x)$. Moreover, ι is $\mathbb{K}$-linear: given $x, y \in E$ and $\lambda \in \mathbb{K}$, let $Z := \{x, y, \lambda y, x + \lambda y\}$; by definition of ι, the set $J := \{i \in I : R(Z, i)\}$ belongs to $\mathcal{F}$, and $J \subseteq \{i \in I : i(x + \lambda y) = i(x) + \lambda i(y)\}$; thus $\iota(x + \lambda y) = \iota(x) + \lambda\iota(y)$ so ι is $\mathbb{K}$-linear. □

Lemma 4.9. *Given a spherically complete ultrametric valued field $(\mathbb{K}, |.|)$, the following statements are equivalent:*

(i) $\mathbf{I}_{(\mathbb{K},|.|)}$
(ii) For every filter $\mathcal{F}$ on a set I, the $\mathbb{K}$-linear mapping $j_{\mathcal{F}} : \mathbb{K} \to (\mathbb{K}_{\mathcal{F}})_b$ has an additive retraction $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$ such that for every $x \in (\mathbb{K}_{\mathcal{F}})_b$, $|r(x)| \leq N_{|.|,\mathcal{F}}(x)$.

Proof. (i) ⇒ (ii) Given a filter $\mathcal{F}$ on a set I, consider the $\mathbb{K}$-linear mapping $f : \mathbb{K}.1_{\mathcal{F}} \to \mathbb{K}$ associating 1 to $1_{\mathcal{F}}$. Then $|f| \leq p_{\mathcal{F}}$. Since the vector semi-norm $p_{\mathcal{F}} : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{R}_+$ is ultrametric, (ii) implies a $\mathbb{K}$-linear mapping $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$ extending f such that $r \leq N_{|.|,\mathcal{F}}$. Since r is $\mathbb{K}$-linear and fixes $1_{\mathcal{F}}$, r fixes every element of $\mathbb{K}$ in $\mathbb{K}_{\mathcal{F}}$ thus r is a retraction of $j_{\mathcal{F}} : \mathbb{K} \to (\mathbb{K}_{\mathcal{F}})_b$.
(ii) ⇒ (i) Given an ultrametric vector semi-norm p on a $\mathbb{K}$-vector space E, and a linear mapping f defined on a vector subspace F of E such that $|f| \leq p$, using Lemma 4.8, consider a linear mapping $\iota : E \to (\mathbb{K}_{\mathcal{F}})_b$ extending f such that $N_{|.|,\mathcal{F}} \circ \iota \leq p$. Using (ii),

let $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be an additive retraction such that $|r| \leq N_{|.|,\mathcal{F}}$; using Proposition 3.1, r is $\mathbb{K}$-linear. Then the $\mathbb{K}$-linear mapping $\tilde{f} := r \circ \iota : E \to \mathbb{K}$ extends f and $|\tilde{f}| \leq p$. $\square$

Remark 4.10 (Hahn-Banach)**.** *Consider the Hahn-Banach statement* **HB***: "Given a vector space E over $\mathbb{R}$, given a subadditive mapping $p : E \to \mathbb{R}$ such that for every $\lambda \in \mathbb{R}_+$ and every $x \in E$, $p(\lambda.x) = \lambda p(x)$, and given a linear form f defined on a vector subspace F of E satisfying $|f| \leq p$, there exists a linear mapping $\tilde{f} : E \to \mathbb{R}$ extending f such that $|\tilde{f}| \leq p$". It is known (see* [4]*) that* **BPI** $\Rightarrow$ **HB** *and that* **HB** $\not\Rightarrow$ **BPI***. It is also known (see* [8]*) that* **HB** *is equivalent to the following statement: "For every filter $\mathcal{F}$ on a set I, there exists a $\mathbb{R}$-linear mapping $r : (\mathbb{R}_{\mathcal{F}})_b \to \mathbb{R}$ such that $r(1) = 1$ and r is positive.*

Question 4.11. Is there an ultrametric spherically complete valued field $(\mathbb{K}, |.|)$ such that **HB** is equivalent to $\mathbf{I}_{(\mathbb{K},|.|)}$? Given two distinct prime numbers p and q, are the statements $\mathbf{I}_{\mathbb{Q}_p}$ and $\mathbf{I}_{\mathbb{Q}_q}$ equivalent? Are they equivalent to **HB**? Here we denote by $\mathbb{Q}_p$ (see [14, p. 186]) the valued field which is the Cauchy-completion of $\mathbb{Q}$ endowed with the p-adic absolute value.

Remark 4.12. *Every ultrametric valued field in which every large ball is compact is spherically complete.*

Corollary (van Rooij, [13])**.** *For every ultrametric valued field $(\mathbb{K}, |.|)$ such that every large ball of $\mathbb{K}$ is compact, then* **BPI** *implies* $\mathbf{I}_{(\mathbb{K},|.|)}$.

Proof. For sake of completeness, we give the proof sketched by van Rooij. Using Lemma 4.9, it is sufficient to show that given a filter $\mathcal{F}$ on a set I, there is a $\mathbb{K}$-linear mapping $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$ such that $|r| \leq N_{|.|,,\mathcal{F}}$. Using **BPI**, let $\mathcal{U}$ be an ultrafilter on I such that $\mathcal{F} \subseteq \mathcal{U}$. For every $x \in (\mathbb{K}_{\mathcal{F}})_b$, the large ball $B(0, N_{|.|,\mathcal{F}}(x) + 1)$ of $\mathbb{K}$ is compact and Hausdorff, whence for every $(u_i)_{i\in I} \in \mathbb{K}^I$ and $(v_i)_{i\in I} \in \mathbb{K}^I$ such that x is the class of $(u_i)_{i\in I}$ and the class of $(v_i)_{i\in I}$ in $\mathbb{K}_{\mathcal{F}}$, then $(u_i)_{i\in I}$ and $(v_i)_{i\in I}$ both converge through the ultrafilter $\mathcal{U}$ to the same element of the ball $B(0, N_{|.|,\mathcal{F}}(x) + 1)$: we denote by $r(x)$ this element of $\mathbb{K}$. Since the class of $(u_i)_{i\in I}$ in $\mathbb{K}_{\mathcal{F}}$ is x, for every real number $\varepsilon > 0$, $|r(x)| \leq N_{|.|,,\mathcal{F}}(x) + \varepsilon$, thus $|r(x)| \leq N_{|.|,,\mathcal{F}}(x)$. We have defined a mapping $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$. Then $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$ is additive and fixes every element of $\mathbb{K}$, thus, using Proposition 3.1, r is $\mathbb{K}$-linear. And $|r| \leq N_{|.|,,\mathcal{F}}$ by construction. $\square$

Remark 4.13. *In particular, given a finite field $\mathbb{K}$ endowed with the trivial absolute value, then $\mathbb{K}$ is compact thus* **BPI** *implies* $\mathbf{D}_{\mathbb{K}}$*: this is Howard and Tachtsis's result (see* [5, Theorem 3.14]*).*

Question 4.14 (van Rooij, see [13])**.** Does **BPI** imply the full Ingleton statement?

5. Isometric linear extenders

5.1. **Bounded dual of a semi-normed vector space over a valued field.** Given a valued field $(\mathbb{K}, |.|)$ and two semi-normed $\mathbb{K}$-vector spaces (E, p) and (F, q), a linear mapping $T : E \to F$ is said to be *bounded* with respect to the semi-norms p and q if and only if there exists a real number $M \in \mathbb{R}_+$ satisfying $q(T(x)) \leq Mp(x)$ for every $x \in E$.

Proposition ([11, Proposition 3.1. p. 13])**.** *Let $(\mathbb{K}, |.|)$ be a valued field and let (E, p) and (F, q) be two semi-normed $\mathbb{K}$-vector spaces. Let $T : E \to F$ be a $\mathbb{K}$-linear mapping.*

(1) If T is bounded with respect to the semi-norms p and q, then T is continuous with respect to the topologies associated to the semi-norms p and q.

(2) *If T is continuous with respect to the topologies associated to the semi-norms p and q and if the absolute value $|.|$ on $\mathbb{K}$ is not trivial, then T is bounded with respect to the semi-norms p and q.*

Proof. (1) If T is bounded, then T is continuous at the point 0_E of E. Since translations of E are continuous with respect to p, it follows that T is continuous at every point of E.
(2) We assume that the absolute value $|.|$ on $\mathbb{K}$ is not trivial. Let G be the subgroup $\{|x| : x \in \mathbb{K}\backslash\{0\}\}$ of $(\mathbb{R}_+^*, \times)$. Since $|.|$ is not trivial, there exists $a \in \mathbb{K}^*$ such that $|a| \neq 1$. Using $\frac{1}{a}$ instead of a, we may assume that $0 < |a| < 1$. It follows that the sequence $(|a^n|)_{n\in\mathbb{N}}$ of $\mathbb{R}_+^*$ converges to 0. Since T is continuous at point 0_E, let $\eta \in \mathbb{R}_+^*$ such that for every $x \in E$, $(p(x) < \eta \Rightarrow q(T(x)) < 1)$. Let $n_0 \in \mathbb{N}$ such that $|a^{n_0}| < \eta$ and let $M := \frac{1}{|a^{n_0+1}|}$. Then for every $x \in E$, let us check that $q(T(x)) \leq Mp(x)$. If $p(x) = 0$, then for every $\lambda \in \mathbb{K}^*$, $p(\lambda.x) = 0$ hence $q(T(\lambda x)) < 1$, thus for every $\lambda \in \mathbb{K}^*$, $q(T(x)) < \frac{1}{|\lambda|}$ so $q(T(x)) = 0$. If $p(x) > 0$, let $n \in \mathbb{N}$ such that $|a^{n+1}| \leq p(x) < |a^n|$; then $p(\frac{x}{a^n}) < 1$ thus $p(\frac{a^{n_0}x}{a^n}) < |a^{n_0}|$ so $q(T(\frac{a^{n_0}x}{a^n})) < 1$ *i.e.* $q(T(x)) < |a^{n-n_0}| = \frac{|a^{n+1}|}{|a^{n_0+1}|} \leq \frac{p(x)}{|a^{n_0+1}|} = Mp(x)$. □

Remark 5.1 ([10, Example 3 p.77-78])**.** *Given a commutative valued field $\mathbb{K}$ endowed with the trivial absolute value $|.|_{disc}$, and two semi-normed $\mathbb{K}$-vector spaces (E,p) and (F,q), a continuous linear mapping $T : E \to F$ is not necessarily bounded with respect to the semi-norms p and q. For sake of completeness, we sketch the argument. Let E be the ring $\mathbb{K}[X]$ of polynomials with coefficients in $\mathbb{K}$, let p be the trivial norm on $\mathbb{K}[X]$ and let $q : \mathbb{K}[X] \to \mathbb{R}_+$ be the mapping associating to each polynomial P the number $deg(P) + 1$ if P is not null, and 0 else. Then q is an ultrametric semi-norm on the vector space $\mathbb{K}[X]$ over the valued field $(\mathbb{K}, |.|_{disc})$. Now the "identity transformation"* Id $: (\mathbb{K}[X], p) \to (\mathbb{K}[X], q)$ *is continuous (because the topology of the semi-normed space $(\mathbb{K}[X], p)$ is discrete), but* Id *is not bounded with respect to the semi-norms p and q, since for every $n \in \mathbb{N}$, $p(X^n) = 1$ and $q(X^n) = n+1$.*

Given a valued field $(\mathbb{K}, |.|)$, and two semi-normed $\mathbb{K}$-vector spaces (E,p) and (F,q), we denote by $\mathrm{BL}(E,F)$ the vector space of bounded linear mappings from E to F. Given some bounded linear mapping $T : E \to F$, the real number $\inf\{M \in \mathbb{R}_+ : \forall x \in E\ q(T(x)) \leq Mp(x)\}$ is called the *semi-norm* of the operator T, and is denoted by $\|T\|_{\mathrm{BL}(E,F)}$ (or $\|T\|$). The mapping $\|.\| : \mathrm{BL}(E,F) \to \mathbb{R}_+$ associating to each bounded operator $T \in \mathrm{BL}(E,F)$ its semi-norm $\|T\|$ is a vector semi-norm, which is ultrametric if the semi-norm q of F is ultrametric.

Remark 5.2. *Given a spherically complete ultrametric valued field $(\mathbb{K}, |.|)$, the Ingleton statement $\mathbf{I}_{(\mathbb{K},|.|)}$ can be reformulated as follows: for every ultrametric semi-normed space (E,p) over the valued field $(\mathbb{K}, |.|)$, for every vector subspace F of E and for every bounded linear mapping $f : (F,p) \to (\mathbb{K}, |.|)$, there exists a bounded linear mapping $\tilde{f} : (E,p) \to (\mathbb{K}, |.|)$ extending f such that $\|\tilde{f}\| = \|f\|$.*

Given a valued field $(\mathbb{K}, |.|)$, a semi-normed $\mathbb{K}$-vector space (E,p) and a vector subspace F of E, a *continuous linear extender* from $\mathrm{BL}(F,\mathbb{K})$ to $\mathrm{BL}(E,\mathbb{K})$ is a continuous linear mapping $T : \mathrm{BL}(F,\mathbb{K}) \to \mathrm{BL}(E,\mathbb{K})$ such that for every $f \in \mathrm{BL}(F,\mathbb{K})$, $T(f)$ extends f; moreover, if for every $f \in \mathrm{BL}(F,\mathbb{K})$, $T(f)$ has the same semi-norm as f, then the continuous linear extender T is said to be *isometric.*

5.2. **Orthogonal basis of a finite dimensional ultrametric semi-normed space.**

Lemma ([12, Ex. 3.R p. 63])**.** *Let* $(\mathbb{K}, |.|)$ *be a spherically complete ultrametric valued field. Let* E *be a* $\mathbb{K}$*-vector space endowed with a semi-norm* $p : E \to \mathbb{R}$*. Given two vector subspaces* F *and* G *of* E *such that* $F \oplus G = E$*, and denoting by* $P_F : E \to F$ *and* $P_G : E \to G$ *the associated projections, the following statements are equivalent:*

(1) For every $x \in E$ $p(P_F(x)) \leq p(x)$ *(*i.e. P_F *is bounded and* $\|P_F\| \leq 1$*)*
(2) For every $x \in E$ $p(P_G(x)) \leq p(x)$ *(*i.e. P_G *is bounded and* $\|P_G\| \leq 1$*)*
(3) For every $x \in F$ *and every* $x \in G$*,* $p(x_F \oplus x_G) = \max(p(x_F), p(x_G))$*.*

Definition 5.3. Given an ultrametric valued field $(\mathbb{K}, |.|)$, a $\mathbb{K}$-vector space E and an ultrametric semi-norm $p : E \to \mathbb{R}$, two vector subspaces F and G of E satisfying the conditions of Lemma 5.2 are said to be *orthocomplemented*.

Lemma. *Let* $(\mathbb{K}, |.|)$ *be a spherically complete ultrametric valued field. Let* E *be a finite dimensional* $\mathbb{K}$*-vector space endowed with an ultrametric semi-norm* $p : E \to \mathbb{R}$*. Every one-dimensional vector subspace* D *of* E *has an orthocomplemented subspace in* E*.*

Proof. Let D be a one-dimensional vector subspace of E. If $p_{\restriction D}$ is null, then every vector subspace H of E such that $H \oplus D = E$ is an orthocomplement of D in E. Assume that $p_{\restriction D}$ is not null. Let $a \in D \backslash \{0\}$. Let $f : D \to \mathbb{K}$ be the linear mapping such that $f(a) = 1$: then for every $x \in D$, $|f(x)| \leq \frac{1}{p(a)} p(x)$. Since the vector space E is finite dimensional, using Remark 4.3, there exists in **ZF** a linear mapping $\tilde{f} : E \to \mathbb{K}$ extending f such that $|\tilde{f}| \leq \frac{1}{p(a)} p$. Then $H := \ker(\tilde{f})$ is an orthocomplement of the subspace D in E: for every $x \in E$, $x = (x - \tilde{f}(x).a) \oplus \tilde{f}(x).a$ where $x - \tilde{f}(x).a \in H$; moreover, $p(\tilde{f}(x).a) = |\tilde{f}(x)| p(a) \leq p(x)$, so, denoting by P_D the projection onto the subspace D with kernel H, $\|P_D\| \leq 1$. □

Remark 5.4. *Given a spherically complete ultrametric valued field* $(\mathbb{K}, |.|)$*, and an ultrametric semi-normed* $\mathbb{K}$*-vector space* (E, p)*, then the Ingleton statement* $\mathbf{I}_{(\mathbb{K}, |.|)}$ *implies that every one-dimensional subspace of* E *is orthocomplemented in* E*.*

Definition 5.5. Given an ultrametric valued field $(\mathbb{K}, |.|)$, a $\mathbb{K}$-vector space E and an ultrametric semi-norm $p : E \to \mathbb{R}$, a sequence $(e_i)_{0 \leq i \leq p}$ of E is said to be p-orthogonal if for every sequence $(s_i)_{0 \leq i \leq p}$ of $\mathbb{K}$, $p(\sum_{0 \leq i \leq p} s_i.e_i) = \sup_{0 \leq i \leq p} p(s_i.e_i)$.

Lemma ([12, Lemma 5.3 p.169])**.** *Let* $(\mathbb{K}, |.|)$ *be a spherically complete ultrametric valued field. For every finite dimensional* $\mathbb{K}$*-vector space* E *and every ultrametric semi-norm* $p : E \to \mathbb{R}$*, there exists a* p*-orthogonal basis in the vector space* E*.*

Proof. We prove the Lemma by recursion over the dimension of E. If $dim(E) = 1$, then every basis of E is p-orthogonal. Assume that the result holds for some natural number $n \geq 1$ and assume that E is a $\mathbb{K}$-vector space with dimension $n+1$, and that $p : E \to \mathbb{R}$ is a ultrametric semi-norm. Let $a \in E \backslash \{0\}$, and let D be the line $\mathbb{K}.a$. Using the previous Lemma, let H be a p-orthocomplemented subspace of D in E. Using the recursion hypothesis, let $(e_i)_{1 \leq i \leq n}$ be a p-orthogonal basis of the subspace H. Let $e_{n+1} := a$. Since D and H are orthocomplemented, $(e_i)_{0 \leq i \leq n+1}$ is a p-orthogonal basis of E. □

5.3. **Ultrametric isometric linear extenders.** In this Section, we shall show that given a spherically complete valued field $(\mathbb{K}, |.|)$, the statement $\mathbf{I}_{(\mathbb{K}, |.|)}$ is equivalent to the following

"isometric linear extender" statement:
$\mathbf{LE}_{(\mathbb{K},|.|)}$: *"For every vector subspace F of an ultrametric semi-normed $\mathbb{K}$-vector space (E,p), there exists an isometric linear extender $T : \mathrm{BL}(F,\mathbb{K}) \to \mathrm{BL}(E,\mathbb{K})$.*

Theorem 5.6. *Given a spherically complete ultrametric valued field $(\mathbb{K},|.|)$, the statements $\mathbf{I}_{(\mathbb{K},|.|)}$ and $\mathbf{LE}_{(\mathbb{K},|.|)}$ are equivalent.*

Proof. $\mathbf{I}_{(\mathbb{K},|.|)} \Rightarrow \mathbf{LE}_{(\mathbb{K},|.|)}$ Let (E,p) be an ultrametric semi-normed vector space (E,p) over the valued field $(\mathbb{K},|.|)$. We endow the $\mathbb{K}$-vector space $\mathrm{BL}(E,\mathbb{K})$ with its ultrametric semi-norm $\|.\|$. Given some vector subspace F of E, let I be the set of mappings $\Phi : \mathrm{BL}(F,\mathbb{K}) \to \mathrm{BL}(E,\mathbb{K})$ and let R be the binary relation on $fin(\mathrm{BL}(F,\mathbb{K})) \times I$ such that for every $Z \in fin(\mathrm{BL}(F,\mathbb{K}))$ and every $\Phi \in I$, $R(Z,\Phi)$ if and only if for every $f \in Z$, the bounded linear form $\Phi(f)$ extends f, $\|\Phi(f)\| = \|f\|$, and Φ is $\mathbb{K}$-linear on $\mathrm{span}_{\mathrm{BL}(F,\mathbb{K})}(Z)$. Then the binary relation R is concurrent: given m finite subsets $Z_1,\dots,Z_m \in fin(\mathrm{BL}(F,\mathbb{K}))$, let $B = \{f_1,\dots,f_n\}$ be a (finite) $\|.\|$-orthogonal basis of the $\mathbb{K}$-vector subspace of $\mathrm{BL}(F,\mathbb{K})$ generated by the finite set $\cup_{1\leq i\leq m} Z_i$; then using $\mathbf{I}_{(\mathbb{K},|.|)}$, let $\tilde{f}_1, \dots, \tilde{f}_n$ be bounded linear forms on E extending $f_1, \dots, f_n$ such that for each i, $\|\tilde{f}_i\| = \|f_i\|$, and let $L : \mathrm{span}(\{f_1,\dots,f_n\}) \to \mathrm{BL}(E,\mathbb{K})$ be the linear mapping such that for each $i \in \{1,\dots,n\}$, $L(f_i) = \tilde{f}_i$. For every $f \in \mathrm{span}(\{f_1,\dots,f_n\})$, let us check that $\|L(f)\| = \|f\|$. Given $f \in \mathrm{span}(\{f_1,\dots,f_n\})$, f is of the form $\sum_{1\leq i\leq n} s_i f_i$ where $s_1, \dots, s_n \in \mathbb{K}$, thus $L(f) = \sum_{1\leq i\leq n} s_i \tilde{f}_i$; since $\|.\|$ is ultrametric, it follows that $\|L(f)\| = \|\sum_{1\leq i\leq p} s_i \tilde{f}_i\| \leq \max_{1\leq i\leq n} \|s_i \tilde{f}_i\| = \max_{1\leq i\leq n} |s_i| \|\tilde{f}_i\| = \max_{1\leq i\leq n} |s_i| \|f_i\| = \max_{1\leq i\leq n} \|s_i f_i\| = \|\sum_{1\leq i\leq n} s_i f_i\|$ (because the sequence $(f_i)_{1\leq i\leq n}$ is $\|.\|$-orthogonal). Let $\Phi : \mathrm{BL}(F,\mathbb{K}) \to \mathrm{BL}(E,\mathbb{K})$ be some mapping extending L (for example, define $\Phi(f) = 0$ for every $f \in \mathrm{BL}(F,\mathbb{K}) \setminus \mathrm{span}(\{f_1,\dots,f_n\})$. Then for every $i \in \{1,\dots,m\}$, $R(Z_i,\Phi)$. Consider the filter $\mathcal{F}$ on I generated by the set $\{R(Z); Z \in fin(\mathrm{BL}(F,\mathbb{K}))\}$. Then the mapping $\Phi : \mathrm{BL}(F,\mathbb{K}) \to L(E,\mathbb{K}_{\mathcal{F}})$ associating to each $f \in \mathrm{BL}(F,\mathbb{K})$ the $\mathbb{K}$-linear mapping $\Phi(f) : E \to \mathbb{K}_{\mathcal{F}}$ associating to each $x \in E$ the class of $(f(x))_{f\in I}$ in $\mathbb{K}_{\mathcal{F}}$ is linear, and for every $f \in \mathrm{BL}(F,\mathbb{K})$, $\Phi(f) : E \to \mathbb{K}_{\mathcal{F}}$ extends f and for every $x \in E$, $N_{|.|,\mathcal{F}}(\Phi(f)(x)) \leq \|f\|\|x\|$. Using $\mathbf{I}_{(\mathbb{K},|.|)}$ and Lemma 4.9, consider a $\mathbb{K}$-linear retraction $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$ of $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$ such that for every $x \in (\mathbb{K}_{\mathcal{F}})_b$, $|r(x)| \leq N_{|.|,\mathcal{F}}(x)$. Thus, for every $f \in \mathrm{BL}(F,\mathbb{K})$, for every $x \in F$, $|r \circ \Phi(f)(x)| \leq \|f\|\|x\|$, so $\|r \circ \Phi(f)\| \leq \|f\|$; since $r \circ \Phi(f) : E \to \mathbb{K}$ extends $f : F \to \mathbb{K}$, it follows that $\|r \circ \Phi(f)\| = \|f\|$. Let $T : \mathrm{BL}(F,\mathbb{K}) \to \mathrm{BL}(E,\mathbb{K})$ be the mapping $f \mapsto r \circ \Phi(f)$. Then the mapping $T : \mathrm{BL}(F,\mathbb{K}) \to \mathrm{BL}(E,\mathbb{K})$ is an isometric linear extender. The implication $\mathbf{LE}_{(\mathbb{K},|.|)} \Rightarrow \mathbf{I}_{(\mathbb{K},|.|)}$ is trivial. □

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