LINEAR EXTENDERS AND THE AXIOM OF CHOICE

Marianne Morillon

LINEAR EXTENDERS AND THE AXIOM OF CHOICE

MARIANNE MORILLON

Abstract. In set theory without the axiom of Choice **ZF**, we prove that for every commutative field $\mathbb{K}$, the following statement $\mathbf{D}_{\mathbb{K}}$: "On every non nul $\mathbb{K}$-vector space, there exists a non null linear form" implies the existence of a "$\mathbb{K}$-linear extender" on every vector subspace of a $\mathbb{K}$-vector space. This solves a question raised in [9]. In the second part of the paper, we generalize our results in the case of spherically complete ultrametric valued fields, and show that Ingleton's statement is equivalent to the existence of "continuous linear extenders".

1. Introduction

We work in **ZF**, set theory without the Axiom of Choice (**AC**). Given a commutative field $\mathbb{K}$ and two $\mathbb{K}$-vector spaces E, F, we denote by $L_{\mathbb{K}}(E, F)$ (or $L(E, F)$) the set of $\mathbb{K}$-linear mappings $T : E \to F$: then $L(E, F)$ is a vector subspace of the product vector space F^E. A *linear form* on the $\mathbb{K}$-vector space E is a $\mathbb{K}$-linear mapping $f : E \to \mathbb{K}$; we denote by E^* the *algebraic dual* of E *i.e.* the vector space $L(E, \mathbb{K})$. Given two $\mathbb{K}$-vector spaces E_1 and E_2, and a linear mapping $T : E_1 \to E_2$, we denote by $T^t : E_2^* \to E_1^*$ the mapping associating to every $g \in E_2^*$ the linear mapping $g \circ T : E_1 \to \mathbb{K}$: the mapping T^t is $\mathbb{K}$-linear and is called the *transposed* mapping of T. Given a vector subspace F of the $\mathbb{K}$-vector space E, a *linear extender* for F in E is a linear mapping $T : F^* \to E^*$ associating to each linear form $f \in F^*$ a linear form $\tilde{f} : E \to \mathbb{K}$ extending f. If G is a *complementary subspace* of F in E (*i.e.* $F + G = E$ and $F \cap G = \{0\}$), which we denote by $F \oplus G = E$, if $p : E \to F$ is the linear mapping fixing every element of F and which is null on G, then the transposed mapping $p^t : F^* \to E^*$ associating to every $f \in F^*$ the linear form $f \circ p : E \to \mathbb{K}$ is a linear extender for F in E. However, given a commutative field $\mathbb{K}$, the existence of a complementary subspace of every subspace of a $\mathbb{K}$-vector space implies **AC** in **ZF**; more precisely, denoting by **ZFA** (see [7, p. 44]) the set theory **ZF** with the axiom of extensionality weakened to allow the existence of atoms, the existence of a complementary subspace of every subspace of a $\mathbb{K}$-vector space implies in **ZFA** (see [2, Lemma 2]) the following *Multiple Choice* axiom **MC** (see [7, p. 133] and form 37 of [4, p. 35]): *"For every infinite family $(X_i)_{i \in I}$ of nonempty sets, there exists a family $(F_i)_{i \in I}$ of nonempty finite sets such that for each $i \in I$, $F_i \subseteq X_i$."* It is known that **MC** is equivalent to **AC** in **ZF**, but **MC** does not imply **AC** in **ZFA**.

Given a commutative field $\mathbb{K}$, we consider the following consequences of the Axiom of Choice:

- $\mathbf{BE}_{\mathbb{K}}$: *"Every linearly independent subset of a vector space E over $\mathbb{K}$ is included in a basis of E."*
- $\mathbf{B}_{\mathbb{K}}$: *"Every vector space over $\mathbb{K}$ has a basis."*

2000 *Mathematics Subject Classification.* Primary 03E25 ; Secondary 46S10.

Key words and phrases. Axiom of Choice, extension of linear forms, non-Archimedean fields, Ingleton's theorem.

- $\mathbf{LE}_{\mathbb{K}}$: (Linear Extender) *"For every subspace F of a $\mathbb{K}$-vector space E, there exists a linear mapping $T : F^* \to E^*$ associating to every $f \in F^*$ a linear mapping $T(f) : E \to \mathbb{K}$ extending f.*
- $\mathbf{D}_{\mathbb{K}}$: *"For every non null $\mathbb{K}$-vector space E there exists a non null linear form $f : E \to \mathbb{K}$.*

In **ZF**, $\mathbf{BE}_{\mathbb{K}} \Rightarrow \mathbf{B}_{\mathbb{K}} \Rightarrow \mathbf{LE}_{\mathbb{K}} \Rightarrow \mathbf{D}_{\mathbb{K}}$ (see [9, Proposition 4]). In this paper, we show (see Theorem 1 in Section 2) that for every commutative field $\mathbb{K}$, $\mathbf{D}_{\mathbb{K}}$ implies $\mathbf{LE}_{\mathbb{K}}$, and this solves Question 2 of [9]. In Section 3 we provide several other statements which are equivalent to $\mathbf{D}_{\mathbb{K}}$ and we introduce the following consequence $w\mathbf{D}_{\mathbb{K}}$ of $\mathbf{D}_{\mathbb{K}}$: *"For every $\mathbb{K}$-vector space E and every $a \in E \backslash \{0\}$, there exists an additive mapping $f : E \to \mathbb{K}$ such that $f(a) = 1$."*

Question 1. Given a commutative field $\mathbb{K}$, does the statement $w\mathbf{D}_{\mathbb{K}}$ imply $\mathbf{D}_{\mathbb{K}}$?

Blass ([1]) has shown that the statement $\forall \mathbb{K} \mathbf{B}_{\mathbb{K}}$ (form 66 of [4]: "For every commutative field, every non $\mathbb{K}$-vector space has a basis") implies **MC** in **ZFA** (and thus implies **AC** in **ZF**), but it is an open question to know if there exists a commutative field $\mathbb{K}$ such that $\mathbf{B}_{\mathbb{K}}$ implies **AC**. In **ZFA**, the statement **MC** implies $\mathbf{D}_{\mathbb{K}}$ for every commutative field $\mathbb{K}$ with null characteristic (see [9, Proposition 1]) thus in **ZFA**, the axioms $\mathbf{D}_{\mathbb{K}}$ for commutative fields $\mathbb{K}$ with null characteristic do not imply **AC**. Denoting by **BPI** the *Boolean prime ideal*: "Every boolean non null algebra has a ultrafilter" (see form 14 in [4]), Howard and Tachtsis (see [5, Theorem 3.14]) have shown that for every finite field $\mathbb{K}$, **BPI** implies $\mathbf{D}_{\mathbb{K}}$; since $\mathbf{BPI} \not\Rightarrow \mathbf{AC}$, the statements $\mathbf{D}_{\mathbb{K}}$ for finite fields $\mathbb{K}$ do not imply **AC**. They also have shown (see [5, Corollary 4.9]) that in **ZFA**, $\forall \mathbb{K} \mathbf{D}_{\mathbb{K}}$ ("For every commutative field, for every non null $\mathbb{K}$-vector space E, there exists a $\mathbb{K}$-linear form $f : E \to \mathbb{K}$") does not imply $\forall \mathbb{K} \mathbf{B}_{\mathbb{K}}$, however, the following questions seem to be open in **ZF**:

Question 2. Does the statement $\forall \mathbb{K} \mathbf{D}_{\mathbb{K}}$ imply **AC** in **ZF**? Is there a (necessarily infinite) commutative field $\mathbb{K}$ such that $\mathbf{D}_{\mathbb{K}}$ implies **AC** in **ZF**?

In Section 4, we extend Proposition 1 of Section 2 to the case of spherically complete ultrametric valued fields (see Lemma 2) and prove that Ingleton's statement, which is a "Hahn-Banach type" result for ultrametric semi-normed spaces over spherically complete ultrametric valued fields $\mathbb{K}$, follows from **MC** when $\mathbb{K}$ has a null characteristic. In Section 5, we prove that Ingleton's statement is equivalent to the existence of "isometric linear extenders".

2. $\mathbf{LE}_{\mathbb{K}}$ and $\mathbf{D}_{\mathbb{K}}$ are equivalent

2.1. **Reduced powers of a commutative field $\mathbb{K}$.** Given a set E and a filter $\mathcal{F}$ on a set I, we denote by $E_{\mathcal{F}}$ the quotient of the set E^I by the equivalence relation $=_{\mathcal{F}}$ on E^I satisfying for every $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I} \in E^I$, $x =_{\mathcal{F}} y$ if and only if $\{i \in I : x_i = y_i\} \in \mathcal{F}$. If $\mathbb{L}$ is a first order language and if E carries a $\mathbb{L}$-structure, then the quotient set $E_{\mathcal{F}}$ also carries a quotient $\mathbb{L}$-structure: this $\mathbb{L}$-structure is a *reduced power* of the $\mathbb{L}$-structure E (see [3, Section 9.4]). Denoting by $\delta : E \to E^I$ the *diagonal mapping* associating to each $x \in E$ the constant family $i \mapsto x$, and denoting by $can_{\mathcal{F}} : E^I \to E_{\mathcal{F}}$ the canonical quotient mapping, then we denote by $j_{\mathcal{F}} : E \to E_{\mathcal{F}}$ the one-to-one mapping $can_{\mathcal{F}} \circ \delta$. Notice that $j_{\mathcal{F}}$ is a morphism of $\mathbb{L}$-structures.

Example 1 (The reduced power $\mathbb{K}_{\mathcal{F}}$ of a field $\mathbb{K}$). Given a commutative field $\mathbb{K}$ and a filter $\mathcal{F}$ on a set I, we consider the unitary $\mathbb{K}$-algebra $\mathbb{K}^I$, then the quotient $\mathbb{K}$-algebra $\mathbb{K}_{\mathcal{F}}$ is the quotient of the $\mathbb{K}$-algebra $\mathbb{K}^I$ by the following ideal $nul_{\mathcal{F}}$ of *$\mathcal{F}$-almost everywhere null* elements of $\mathbb{K}^I$: $\{x = (x_i)_{i\in I} \in \mathbb{K}^I : \{i \in I : x_i = 0\} \in \mathcal{F}\}$. The mapping $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$ is a one-to-one unitary morphism of $\mathbb{K}$-algebras, thus $\mathbb{K}$ can be viewed as the one-dimensional unitary $\mathbb{K}$-subalgebra of the $\mathbb{K}$-algebra $\mathbb{K}_{\mathcal{F}}$. Notice that the $\mathbb{K}$-algebra $\mathbb{K}_{\mathcal{F}}$ is a field if and only if $\mathcal{F}$ is a ultrafilter.

Notation 1. For every set E, we denote by $fin(E)$ the set of finite subsets of E, and we denote by $fin^*(E)$ the set of nonempty finite subsets of E.

Given two sets E and I, a binary relation $R \subseteq E \times I$ is said to be *concurrent* (see [8]) if for every $G \in fin^*(E)$, the set $R[G] := \cap_{x\in G} R(x)$ is nonempty; in this case, $\{R(x) : x \in I\}$ satisfies the finite intersection property, and we denote by $\mathcal{F}_R$ the filter on I generated by the sets $R(x)$, $x \in E$.

2.2. $\mathbf{D}_{\mathbb{K}}$ implies linear extenders.

Remark 1. It is known (see [9, Theorem 2]), that $\mathbf{D}_{\mathbb{K}}$ is equivalent to the following statement: "For every vector subspace of a $\mathbb{K}$-vector space E and every linear mapping $f : F \to \mathbb{K}$, there exists a $\mathbb{K}$-linear mapping $f : E \to \mathbb{K}$ extending f."

Proposition 1. *Given a commutative field $\mathbb{K}$, the following statements are equivalent:*

i) $\mathbf{D}_{\mathbb{K}}$

ii) For every filter $\mathcal{F}$ on a set I, the linear embedding $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$ has a $\mathbb{K}$-linear retraction $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$.

Proof. $\Rightarrow$ Let $f : \mathbb{K} \to \mathbb{K}$ be the identity mapping. Then f is $\mathbb{K}$-linear and $\mathbb{K}$ is a vector subspace of the $\mathbb{K}$-vector space $\mathbb{K}_{\mathcal{F}}$. Using Remark 1, let $\tilde{f} : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be a $\mathbb{K}$-linear mapping extending f; then $\tilde{f}$ is a $\mathbb{K}$-linear retraction of $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$.
$\Leftarrow$ Let E be a non null $\mathbb{K}$-vector space. Let a be a non-null element of E. Using Lemma 1 in [9], there exists a filter $\mathcal{F}$ on a set I, and a $\mathbb{K}$-linear mapping $g : E \to \mathbb{K}_{\mathcal{F}}$ such that $g(a) = j_{\mathcal{F}}(1)$. Using (ii), let $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be a $\mathbb{K}$-linear retraction of the linear embedding $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$. It follows that $f = r \circ g : E \to \mathbb{K}$ is a $\mathbb{K}$-linear mapping such that $f(a) = 1$. □

Theorem 1. $\mathbf{D}_{\mathbb{K}}$ *implies* $\mathbf{LE}_{\mathbb{K}}$.

Proof. Given some vector subspace F of a vector space E, let I be the set of mappings $\Phi : F^* \to E^*$ and let R be the binary relation on $fin(F^*) \times I$ such that for every $Z \in fin(F^*)$ and every $\Phi \in I$, $R(Z, \Phi)$ if and only if for every $f \in Z$, the linear form $\Phi(f)$ extends f and Φ is $\mathbb{K}$-linear on $\text{span}_{F^*}(Z)$. Then the binary relation R is concurrent: given m finite subsets $Z_1, \dots, Z_m \in fin(F^*)$, let $B = \{f_1, \dots, f_p\}$ be a (finite) basis of the $\mathbb{K}$-vector subspace of F^* generated by the finite set $\cup_{1\le i\le m} Z_i$; then using $\mathbf{D}_{\mathbb{K}}$ (see Remark 1), let $\tilde{f}_1, \dots, \tilde{f}_p$ be linear forms on E extending $f_1, \dots, f_p$; let $L : \text{span}(\{f_1, \dots, f_p\}) \to E^*$ be the linear mapping such that for each $i \in \{1, \dots, p\}$, $L(f_i) = \tilde{f}_i$, and let $\Phi : F^* \to E^*$ be some mapping extending L (for example, define $\Phi(f) = 0$ for every $f \in F^* \setminus \text{span}(\{f_1, \dots, f_p\})$. Then for every $i \in \{1, \dots, m\}$, $R(Z_i, \Phi)$. Consider the filter $\mathcal{F}$ on I generated by $\{R[Z]; Z \in fin(F^*)\}$. Then the mapping $\Phi : F^* \to L(E, \mathbb{K}_{\mathcal{F}})$ associating to each $f \in F^*$ the $\mathbb{K}$-linear mapping $\Phi(f) : E \to \mathbb{K}_{\mathcal{F}}$ associating to each $x \in E$ the class of $(f(x))_{f\in I}$ in $\mathbb{K}_{\mathcal{F}}$ is linear, and for every $f \in F^*$, $\Phi(f) : E \to \mathbb{K}_{\mathcal{F}}$ extends f. Using $\mathbf{D}_{\mathbb{K}}$, there exists (see Proposition 1) a $\mathbb{K}$-linear retraction $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ of $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$. Let $T : F^* \to E^*$ be the mapping $f \mapsto r \circ \Phi(f)$. Then the mapping $T : F^* \to E^*$ is a linear extender for F in E. □

Remark 2. Notice that the axiom $\mathbf{LE}_{\mathbb{K}}$ is "multiple": given a family $(E_i)_{i\in I}$ of $\mathbb{K}$-vector spaces and a family $(F_i)_{i\in I}$ such that for each $i \in I$, F_i is a vector subspace of E_i, then there exists a family $(T_i)_{i\in I}$ such that for each $i \in I$, $T_i : F_i^* \to E_i^*$ is a linear extender for F_i in E_i: apply $\mathbf{LE}_{\mathbb{K}}$ to the vector subspace $\oplus_{i\in I} F_i$ of $\oplus_{i\in I} E_i$.

Corollary 1. *Given a commutative field $\mathbb{K}$, then $\mathbf{D}_{\mathbb{K}}$ implies (and is equivalent to) the following statement: for every $\mathbb{K}$-vector space E, for every vector subspace F of E, denoting by $can : F \to E$ the canonical mapping, then the double transposed mapping $can^{tt} : F^{**} \to E^{**}$ is one-to-one and has a $\mathbb{K}$-linear retraction $r : E^{**} \to F^{**}$.*

Proof. The mapping $can^{tt} : F^{**} \to E^{**}$ associates to every $\Phi \in F^{**}$ the mapping $\overline{\Phi} : E^* \to \mathbb{K}$ such that for every $g \in E^*$, $\overline{\Phi}(g) = \Phi(g_{\restriction F})$. Given some $\Phi \in \ker(can^{tt})$, then, for every $g \in E^*$, $(can^{tt}(\Phi))(g) = 0$ *i.e.* $\Phi(g_{\restriction F}) = 0$; using $\mathbf{D}_{\mathbb{K}}$, for every $f \in F^*$ there exists some $g \in E^*$ such that $f = g_{\restriction F}$, thus for every $f \in F^*$, $\Phi(f) = 0$, so $\Phi = 0$. It follows that $can^{tt} : F^{**} \to E^{**}$ is one-to-one. Using the equivalent form $\mathbf{LE}_{\mathbb{K}}$ of $\mathbf{D}_{\mathbb{K}}$, let $T : F^* \to E^*$ be a $\mathbb{K}$-linear extender *i.e.* a $\mathbb{K}$-linear mapping such that for each $f \in F^*$, $T(f) : E \to \mathbb{K}$ extends f. Then the transposed mapping $T^t : E^{**} \to F^{**}$ is $\mathbb{K}$-linear and for every $\Phi \in F^{**}$, $T^t(can^{tt}(\Phi)) = can^{tt}(\Phi) \circ T = \Phi$. □

It follows that $\mathbf{D}_{\mathbb{K}}$ implies (and is equivalent to) the following statement: "For every vector space E, for every vector subspace F of of E, the canonical linear mapping $F^{**} \to E^{**}$ is one-to-one and there exists a $\mathbb{K}$-vector space G such that G is a complement of F^{**} in E^{**}.

Remark 3. Corollary 1 is equivalent to its "multiple form": given a family $(E_i)_{i\in I}$ of $\mathbb{K}$-vector spaces and a family $(F_i)_{i\in I}$ such that for each $i \in I$, F_i is a vector subspace of E_i, then for each $i \in I$, the canonical mapping $can_i : F_i^{**} \to E_i^{**}$ is one-to-one and there exists a family $(r_i)_{i\in I}$ such that for each $i \in I$, $r_i : E_i^{**} \to F_i^{**}$ is a $\mathbb{K}$-linear retraction of $can_i : F_i^{**} \to E_i^{**}$.

3. Other statements equivalent to $\mathbf{D}_{\mathbb{K}}$

3.1. $\mathbb{K}$-linearity and additive retractions.

Proposition 2. *Let $\mathbb{K}$ be a commutative field and let a be a non-null element of a $\mathbb{K}$-vector space E. Let $j_a : \mathbb{K} \hookrightarrow E$ be the mapping $\lambda \mapsto \lambda.a$. Given any additive mapping $r : E \to \mathbb{K}$ such that $r(a) = 1$, then r is $\mathbb{K}$-linear if and only if r is a retraction of the mapping $j_a : \mathbb{K} \hookrightarrow E$.*

Proof. The direct implication is easy to prove. We show the converse statement. Let k be the prime field of $\mathbb{K}$ (if the characteristic of $\mathbb{K}$ is 0, then $k = \mathbb{Q}$, else the characteristic of $\mathbb{K}$ is a prime number p and k is the field $\mathbb{F}_p$ with p elements). Since $r : E \to \mathbb{K}$ is additive, r is k-linear. Since r is a retraction of $j_a : \mathbb{K} \hookrightarrow E$, $E = \ker(r) \oplus \mathbb{K}.a$ where $\ker(r)$, the kernel of the additive mapping r, is a vector subspace of the k-vector space E. Let N be the $\mathbb{K}$-vector subspace generated by $\ker(r)$ in E. Then $N \cap \mathbb{K}.a = \{0\}$: given some $x \in \mathbb{K}.a \cap N$, then $x = \mu.a$ where $\mu \in \mathbb{K}$, thus $r(x) = \mu$ and $x = \lambda y$ where $y \in \ker(r)$ and $\lambda \in \mathbb{K}$; if $\lambda = 0$ then $x = 0$, else $y = \frac{\mu}{\lambda} a \in \mathbb{K}.a$ thus $r(y) = y$ so (since $y \in \ker(r)$), $y = 0$ thus $\mu = 0$ so $x = 0$. Thus N and $\ker(r)$ are two complementary subspaces of the k-vector space $\mathbb{K}.a$ in the k-vector space E, and $\ker(r) \subseteq N$: it follows that $\ker(r) = N$. Thus $\ker(r)$

is a $\mathbb{K}$-vector subspace of the $\mathbb{K}$-vector space E. It follows that the additive mapping r is $\mathbb{K}$-linear: given $z \in E$, $z = x \oplus t.a$ where $x \in \ker(r)$ and $t \in \mathbb{K}$; thus for every $\lambda \in \mathbb{K}$, $r(\lambda.z) = r(\lambda.x \oplus \lambda t.a) = r(\lambda.x) + r(\lambda t.a) = \lambda t = \lambda.r(z)$. $\square$

3.2. **Additivity statements equivalent to $\mathbf{D}_{\mathbb{K}}$.** Given a commutative field $(\mathbb{K}, +, \times, 0, 1)$, we consider the following statements:

$\mathbf{A}_{\mathbb{K}}$: *"For every $\mathbb{K}$-vector space E and every subgroup F of $(E, +)$, for every additive mapping $f : F \to (\mathbb{K}, +)$, there exists an additive mapping $\tilde{f} : E \to \mathbb{K}$ extending f.*

$\mathbf{A}'_{\mathbb{K}}$: *"For every $\mathbb{K}$-vector space E and every vector subspace F of E, for every $\mathbb{K}$-linear mapping $f : F \to \mathbb{K}$, there exists an additive mapping $\tilde{f} : E \to \mathbb{K}$ extending f.*

$\mathbf{A}''_{\mathbb{K}}$: *"For every $\mathbb{K}$-vector space E and every $a \in E\backslash\{0\}$, there exists an additive mapping $f : E \to \mathbb{K}$ such that for every $\lambda \in \mathbb{K}$, $f(\lambda.a) = \lambda$."*

Proposition 3. *For every commutative field $\mathbb{K}$, $\mathbf{A}_{\mathbb{K}} \Leftrightarrow \mathbf{A}'_{\mathbb{K}} \Leftrightarrow \mathbf{A}''_{\mathbb{K}} \Leftrightarrow \mathbf{D}_{\mathbb{K}}$.*

Proof. The implications $\mathbf{A}_{\mathbb{K}} \Rightarrow \mathbf{A}'_{\mathbb{K}}$ and $\mathbf{A}'_{\mathbb{K}} \Rightarrow \mathbf{A}''_{\mathbb{K}}$ are trivial. We prove $\mathbf{A}''_{\mathbb{K}} \Rightarrow \mathbf{D}_{\mathbb{K}}$. In view of Proposition 1, we prove that for every filter $\mathcal{F}$ on a set I, the canonical mapping $j_{\mathcal{F}} : \mathbb{K} \hookrightarrow \mathbb{K}_{\mathcal{F}} := \mathbb{K}^I/\mathcal{F}$ has a $\mathbb{K}$-linear retraction. Let $f : \mathbb{K} \to \mathbb{K}$ be the identity mapping. Using $\mathbf{A}''_{\mathbb{K}}$, let $\tilde{f} : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be an additive mapping extending f. Using Proposition 2, $\tilde{f}$ is $\mathbb{K}$-linear. It follows that $\tilde{f}$ is a $\mathbb{K}$-linear retraction of $j_{\mathcal{F}} : \mathbb{K} \hookrightarrow \mathbb{K}_{\mathcal{F}}$.
$\mathbf{D}_{\mathbb{K}} \Rightarrow \mathbf{A}_{\mathbb{K}}$. Let E be a $\mathbb{K}$-vector space, let F be a subgroup of the additive group $(E, +)$, and let $f : F \to (\mathbb{K}, +)$ be an additive mapping. Using a concurrent relation (the proof is similar to the proof of Lemma 1 in [9]), let $\mathcal{F}$ be a filter on the set $I = \mathbb{K}^E$ and let $\iota : E \to \mathbb{K}_{\mathcal{F}}$ be an additive mapping extending f. Using $\mathbf{D}_{\mathbb{K}}$, let $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be an additive retraction of $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$. Then $\tilde{f} := r \circ \iota : E \to \mathbb{K}$ is additive and extends f. $\square$

3.3. **A consequence of $\mathbf{D}_{\mathbb{K}}$.**

Proposition 4. *Given a commutative field $\mathbb{K}$, the following statements are equivalent:*

i) $w\mathbf{D}_{\mathbb{K}}$*:* "For every $\mathbb{K}$-vector space E and every $a \in E\backslash\{0\}$, there exists an additive mapping $f : E \to \mathbb{K}$ such that $f(a) = 1$."
ii) "For every non null $\mathbb{K}$-vector space E, there exists a non null additive mapping $f : E \to \mathbb{K}$."
iii) "For every filter $\mathcal{F}$ on a set I, there exists a non null additive mapping $f : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$."

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are easy. We prove (iii) $\Rightarrow$ (i) Given a non null element a of a $\mathbb{K}$-vector space E, let $\mathcal{F}$ be a filter on a set I and $g : E \to \mathbb{K}_{\mathcal{F}}$ be a $\mathbb{K}$-linear mapping such that $g(a) = 1$. Using (iii), let $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be a non null additive mapping. Let $\alpha \in \mathbb{K}_{\mathcal{F}}$ such that $r(\alpha) \neq 0$. Let $m_\alpha : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}_{\mathcal{F}}$ be the additive mapping $x \mapsto \alpha x$. Then $g_1 := r \circ m_\alpha \circ g : E \to \mathbb{K}$ is additive; let $f := \frac{1}{r(\alpha)}.g_1 : E \to \mathbb{K}$; then f is additive and $f(a) = 1$. $\square$

Remark 4. Given a commutative field $\mathbb{K}$, then $w\mathbf{D}_{\mathbb{K}} \Leftrightarrow \mathbf{D}_k$ where k is the prime field of $\mathbb{K}$.

Proof. Given a $\mathbb{K}$-vector space E, a mapping $f : E \to \mathbb{K}$ is additive if and only if f is k-linear. $\square$

Question 3. Given a commutative field $\mathbb{K}$ with null characteristic, does $\mathbf{D}_{\mathbb{Q}}$ imply $\mathbf{D}_{\mathbb{K}}$?

Question 4. Given a prime number p and a commutative field $\mathbb{K}$ with characteristic p, and denoting by $\mathbb{F}_p$ the finite field with p elements, does $\mathbf{D}_{\mathbb{F}_p}$ imply $\mathbf{D}_{\mathbb{K}}$? Does **BPI** (which implies $\mathbf{D}_{\mathbb{F}_p}$) imply $\mathbf{D}_{\mathbb{K}}$?

4. Ingleton's statement for ultrametric valued fields

4.1. **Semi-norms on vector spaces over a valued field.**

4.1.1. *Pseudo metric spaces.* Given a set X, a *pseudo-metric* on X is a mapping $d : X\times X \to \mathbb{R}_+$ such that for every $x, y, z \in X$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$. If d satisfies the extra property $(d(x, y) = 0 \Rightarrow x = y)$, then d is a *metric* on X. A pseudo-metric d on X is said to be *ultrametric* if for every $x, y, z \in X$, $d(x, z) \leq \max(d(x, y), d(y, z))$.

Given a pseudo-metric space (X, d), for every $a \in X$ and every $r \in \mathbb{R}_+$, we denote by $B_s(a, r)$ the "strict" ball $\{x \in X : d(x, a) < r\}$ and we denote by $B(a, r)$ the "large" ball $\{x \in X : d(x, a) \leq r\}$. Notice that large balls of a pseudo-metric space are nonempty. A pseudo-metric space (X, d) is *spherically complete* if every chain (*i.e.* set which is linearly ordered for the inclusion) of large balls of X has a nonempty intersection.

Example 2. Given a nonempty set X, the discrete metric d_{disc} on X, associating to each $(x, y) \in X \times X$ the real number 1 if $x \neq y$ and 0 else is ultrametric and the associated metric space (X, d_{disc}) is spherically complete since large balls for this metric are singletons and the whole space X.

4.1.2. *Group semi-norms.* Given a commutative group $(G, +, 0)$, a *group semi-norm* on G is a mapping $N : G \to \mathbb{R}_+$ which is sub-additive (for every $x, y \in G$, $N(x+y) \leq N(x)+N(y)$) and symmetric (for every $x \in G$, $N(-x) = N(x)$) and such that $N(0) = 0$. If for every $x \in G$, $(N(x) = 0 \Rightarrow x = 0)$, then N is a *norm*. Given a group semi-norm N on an abelian group $(G, +, 0)$, the mapping $d : G \times G \to \mathbb{R}_+$ associating to each $(x, y) \in G \times G$ the real number $N(x - y)$ is a pseudo-metric on G. Moreover, if N is a norm, then d is a metric on G. The semi-norm N is said to be ultrametric if the pseudo-metric d_N is ultrametric, which is equivalent to say that for every $x, y \in G$, $N(x + y) \leq \max(N(x), N(y))$. The topology on G associated to the pseudo-metric d_N is a group topology on G, which is Hausdorff if and only if N is a norm.

4.1.3. *Absolute values on a commutative field.* Given a commutative unitary ring $(R, +, \times, 0, 1)$, a *ring semi-norm* (see [11, p. 137]) on R is a group semi-norm on $(R, +, 0)$ which is sub-multiplicative: for every $x, y \in R$, $N(x \times y) \leq N(x)N(y)$; if a ring semi-norm $N : R \to \mathbb{R}_+$ is non null, then $N(1) \geq 1$. If R is a commutative field and if N is a ring semi-norm on R which is multiplicative (for every $x, y \in R$, $N(x \times y) = N(x)N(y)$) and non null, then $N(1) = 1$, N is a norm and N is called an *absolute value* on the commutative field R, and (R, N) is a *valued field*.

Example 3. Given a commutative field $\mathbb{K}$, the mapping $|.|_{disc} : \mathbb{K} \to \mathbb{R}_+$ associating to each $x \in \mathbb{K}$ the real number 1 if $x \neq 0$ and 0 else is an absolute value on $\mathbb{K}$, which is called the *trivial* absolute value on $\mathbb{K}$. The metric associated to this absolute value is the discrete metric d_{disc} on $\mathbb{K}$, thus, the discrete field $(\mathbb{K}, |.|_{disc})$ is spherically complete.

4.1.4. *Vector semi-norms on vector spaces over a valued field.* Given a valued field $(\mathbb{K}, |.|)$ and a $\mathbb{K}$-vector space E, a *vector semi-norm* on E (see [11, p. 210]) is a group semi-norm $p : E \to \mathbb{R}_+$ on the abelian group $(E, +)$ such that for every $\lambda \in \mathbb{K}$ and every $x \in E$, $p(\lambda.x) = |\lambda| p(x)$; if $(\mathbb{K}, |.|)$ is ultrametric, then the semi-norm $p : E \to \mathbb{R}_+$ is said to be *ultrametric* if and only if for every $x, y \in E$, $p(x + y) \leq \max(p(x), p(y))$. Given a $\mathbb{K}$-algebra $(A, +, \times, 0, 1, \lambda.)$, an *algebra semi-norm* on A is a ring semi-norm on the ring A which is also a vector semi-norm on the $\mathbb{K}$-vector space A.

4.2. **Ingleton's statement for spherically complete ultrametric valued fields.**

Lemma (Ingleton, [6]). *Let $(\mathbb{K}, |.|)$ be a ultrametric spherically complete valued field, let E be a $\mathbb{K}$-vector space and let $p : E \to \mathbb{R}$ be an ultrametric vector semi-norm. Let F be a vector subspace of E and let $f : F \to \mathbb{K}$ be a linear mapping such that $|f| \leq p$. Then, for every $a \in E \backslash F$, there exists a linear mapping $\tilde{f} : F \oplus \mathbb{K}a \to \mathbb{K}$ extending f such that $|f| \leq p$.*

Proof. Ingleton's proof of this Lemma holds in **ZF**. For sake of completeness, we give a proof of this Lemma. We search for some $\alpha \in \mathbb{K}$ such that:

$$\forall z \in F \; \forall \lambda \in \mathbb{K} \; |f(z) + \lambda.\alpha| \leq p(z + \lambda.a) \tag{1}$$

Given some $\alpha \in \mathbb{K}$, and denoting by $\mathbb{K}^*$ the set $\mathbb{K} \backslash \{0\}$, Condition (1) is equivalent to:

$$\forall z \in F \; \forall \lambda \in \mathbb{K}^* \; |\lambda.(f(\frac{z}{\lambda}) + \alpha)| \leq p(\lambda(\frac{z}{\lambda} + a))$$

$$\forall z \in F \; \forall \lambda \in \mathbb{K}^* \; |\lambda|.|f(\frac{z}{\lambda}) + \alpha)| \leq |\lambda|.p(\frac{z}{\lambda} + a)$$

$$\forall z \in F \; \forall \lambda \in \mathbb{K}^* \; |(f(\frac{z}{\lambda}) + \alpha)| \leq p(\frac{z}{\lambda} + a)$$

$$\forall z' \in F \; |f(z') + \alpha)| \leq p(z' + a)$$

$$\forall z \in F \; |f(z) - \alpha)| \leq p(z - a)$$

Let $\mathcal{B}_a$ the set of large balls $B(f(z), p(z - a))$ for $z \in F$. The element $\alpha \in \mathbb{K}$ satisfies (1) if and only if $\alpha \in \cap \mathcal{B}_a$. Given two elements $z_1, z_2 \in F$, then $|f(z_1) - f(z_2)| = |f(z_1 - z_2)| \leq p(z_1 - z_2) \leq \max(p(z_1 - a), p(z_2 - a))$ since p is ultrametric. If $p(z_1 - a) \leq p(z_2 - a)$, it follows that $f(z_1) \in B(f(z_2), p(z_2 - a))$ thus, since $(\mathbb{K}, |.|)$ is ultrametric, $B(f(z_1), p(z_2 - a)) \subseteq B(f(z_2), p(z_2 - a))$, thus $B(f(z_1), p(z_1 - a)) \subseteq B(f(z_2), p(z_2 - a))$, so the set of large balls $\mathcal{B}_a$ is a chain. Since $(\mathbb{K}, |.|)$ is spherically complete, it follows that $\cap \mathcal{B}_a$ is nonempty. □

Given a spherically complete ultrametric valued field $(\mathbb{K}, |.|)$, we now consider Ingleton's statement:
$\mathbf{I}_{(\mathbb{K},|.|)}$: *"For every $\mathbb{K}$-vector space E endowed with a ultrametric vector semi-norm $p : E \to \mathbb{R}_+$, for every vector subspace F of E and every linear mapping $f : F \to \mathbb{K}$ such that $|f| \leq p$, there exists a linear mapping $\tilde{f} : E \to \mathbb{K}$ extending f such that $|f| \leq p$.*

Corollary (Ingleton, [6]). *For every ultrametric valued field $(\mathbb{K}, |.|)$, $\mathbf{AC} \Rightarrow \mathbf{I}_{(\mathbb{K},|.|)}$.*

Proof. Use Zorn's lemma and the previous Lemma. □

4.3. **MC implies Ingleton's statement for null characteristic fields.**

Proposition 5. *Given a spherically complete ultrametric valued field* $(\mathbb{K}, |.|)$ *with null characteristic,* **MC** *implies* $\mathbf{I}_{(\mathbb{K},|.|)}$. *It follows that in* **ZFA***, the full Ingleton statement restricted to null characteristic ultrametric valued fields does not imply* **AC***.*

Proof. Assume that E is a $\mathbb{K}$-vector space endowed with a ultrametric semi-norm $p : E \to \mathbb{R}_+$, and assume that F is a vector subspace of E and that $f : F \to \mathbb{K}$ is a linear mapping such that $|f| \le p$. Using **MC**, there exists an ordinal α and some partition $(F_i)_{i\in\alpha}$ in finite sets of $E\backslash F$. This implies that there is an ordinal β and a strictly increasing family $(V_i)_{i\in\beta}$ of vector subspaces of E such that $V_0 = F$, for every $i \in \beta$ such that $i + 1 \in \beta$, V_{i+1}/V_i is finite-dimensional, and for every non null limit ordinal $i \in \beta$, $V_i = \cup_{j<i} V_j$. Let Z be the set of 3-uples (V, W, l) such that V, W are subspaces of E satisfying $V \subseteq W$, W/V is finite-dimensional and $l : V \to \mathbb{K}$ is a linear mapping satisfying $|l| \le p$. For each $(V, W, l) \in Z$, using the previous Lemma (which holds in **ZFA**), the set $A_{(V,W,l)}$ of linear mappings $\tilde{l} : W \to \mathbb{K}$ extending l and satisfying $|\tilde{l}| \le p$ is non-empty; using **MC**, there is a mapping associating to every $i = (V, W, l) \in Z$ a nonempty finite subset B_i of $A_{(V,W,l)}$. Then, for every $i \in Z$, define $\Phi(i) := \frac{1}{|B_i|}\sum_{u\in B_i} u$ (here we use the fact that the characteristic of $\mathbb{K}$ is null). Using the choice function Φ, we define by transfinite recursion a family $(f_i)_{i\in\beta}$ such that for each $i \in \beta$, $f_i : V_i \to \mathbb{K}$ is linear, f_i extends f, $|f_i| \le p$, and for every $i < j \in \beta$, f_j extends f_i. Let $\tilde{f} := \cup_{i\in\beta} f_i$. Then $\tilde{f} : E \to \mathbb{K}$ is linear, $\tilde{f}$ extends f and $|\tilde{f}| \le p$. □

Question 5. Given a spherically complete ultrametric valued field $(\mathbb{K}, |.|)$, does $\mathbf{B}_{\mathbb{K}}$ imply $\mathbf{I}_{(\mathbb{K},|.|)}$? Does $\mathbf{BE}_{\mathbb{K}}$ imply $\mathbf{I}_{(\mathbb{K},|.|)}$?

Question 6 (van Rooij, see [10])**.** Does the full Ingleton statement (*i.e.* $\mathbf{I}_{(\mathbb{K},|.|)}$ for every spherically complete ultrametric valued field $(\mathbb{K}, |.|)$) imply **AC**?

Remark 5. If a commutative field $\mathbb{K}$ is endowed with the trivial absolute value $|.|_{disc}$, then $\mathbf{I}_{(\mathbb{K},|.|_{disc})}$ is the statement $\mathbf{D}_{\mathbb{K}}$, thus the "full Ingleton statement" implies $\mathbf{D}_{\mathbb{K}}$ for every commutative field $\mathbb{K}$.

4.4. **Ingleton's statement for ultrametric fields with compact large balls.** Given a valued field $(\mathbb{K}, |.|)$ and a filter $\mathcal{F}$ on a set I, we denote by $|.|_{\mathcal{F}} : \mathbb{K}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ the quotient mapping associating to each $x \in \mathbb{K}_{\mathcal{F}}$ which is the equivalence class of some $(x_i)_{i\in I} \in \mathbb{K}^I$, the class of $(|x_i|)_{i\in I}$ in $\mathbb{R}_{\mathcal{F}}$. We denote by $(\mathbb{K}_{\mathcal{F}})_b$ the following unitary subalgebra of "bounded elements" of the unitary $\mathbb{K}$-algebra $\mathbb{K}_{\mathcal{F}}$: $\{x \in \mathbb{K}_{\mathcal{F}} : \exists t \in \mathbb{R}_+ : |x|_{\mathcal{F}} \le_{\mathcal{F}} t\}$. We also denote by $N_{|.|,\mathcal{F}} : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{R}_+$ the mapping associating to each $x \in (\mathbb{K}_{\mathcal{F}})_b$ the real number $\inf\{t \in \mathbb{R}_+ : |x|_{\mathcal{F}} \le_{\mathcal{F}} t\}$. The mapping $N_{|.|,\mathcal{F}}$ is a unitary algebra semi-norm on $(\mathbb{K}_{\mathcal{F}})_b$; moreover, if the valued field $(\mathbb{K}, |.|)$ is ultrametric, the vector semi-norm $N_{|.|,\mathcal{F}}$ is also ultrametric.

Lemma 1. *Let* $(\mathbb{K}, |.|)$ *be a ultrametric valued field, let* E *be a* $\mathbb{K}$*-vector space and let* $p : E \to \mathbb{K}$ *be a ultrametric vector semi-norm. Assume that* F *is a vector subspace of* E *and that* $f : F \to \mathbb{K}$ *is a linear form such that* $|f| \le p$*. Let* $I := \mathbb{K}^E$*. There exists a filter* $\mathcal{F}$ *on* I *and a* $\mathbb{K}$*-linear mapping* $\iota : E \to (\mathbb{K}_{\mathcal{F}})_b$ *definable from* E*,* p *and* f *such that* ι *extends* f *and such that* $N_{|.|,\mathcal{F}} \circ \iota \le p$*.*

Proof. Let $R \subseteq (fin(E) \times I)$ be the following binary relation: given $F \in fin(E)$ and given some mapping $u : E \to \mathbb{K}$, then $R(F, u)$ iff u extends f, $|u| \le p$ and $u_{\restriction F}$ is linear *i.e.* for every $x, y \in F$ and $\lambda \in \mathbb{K}$, $(x + y \in F \Rightarrow u(x + y) = u(x) + u(y))$ and $(\lambda x \in F \Rightarrow u(\lambda x) = \lambda u(x))$. Using Ingleton's Lemma in Section 4.2, the binary relation R is concurrent, thus it generates a filter $\mathcal{F}$ on I. Let $\iota : E \to \mathbb{K}_{\mathcal{F}}$ be the mapping associating to each $x \in E$, the equivalence class of $(i(x))_{i \in I}$ in $\mathbb{K}_{\mathcal{F}}$. Then, for every $x \in F$, $\iota(x) = f(x)$ and for every $x \in E$, $|\iota(x)|_{\mathcal{F}} \leq p(x)$ whence $N_{|.|,\mathcal{F}}(\iota(x)) \leq p(x)$. Moreover, ι is $\mathbb{K}$-linear: given $x, y \in E$ and $\lambda \in \mathbb{K}$, let $F := \{x, y, \lambda y, x + \lambda y\}$; by definition of ι, the set $J := \{i \in I : R(F, i)\}$ belongs to $\mathcal{F}$, and $J \subseteq \{i \in I : i(x + \lambda y) = i(x) + \lambda i(y)\}$; thus $\iota(x + \lambda y) = \iota(x) + \lambda \iota(y)$ so ι is $\mathbb{K}$-linear. □

Lemma 2. *Given a spherically complete ultrametric valued field* $(\mathbb{K}, |.|)$*, the following statements are equivalent:*

(i) $\mathbf{I}_{(\mathbb{K},|.|)}$

(ii) For every filter $\mathcal{F}$ *on a set* I*, there exists an additive mapping* $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$ *fixing every element of* $\mathbb{K}$ *such that for every* $x \in (\mathbb{K}_{\mathcal{F}})_b$, $|r(x)| \leq N_{|.|,\mathcal{F}}(x)$.

Proof. (i) $\Rightarrow$ (ii) Given a filter $\mathcal{F}$ on a set I, consider the $\mathbb{K}$-linear mapping $f : \mathbb{K}.1_{\mathcal{F}} \to \mathbb{K}$ associating 1 to $1_{\mathcal{F}}$. Then $|f| \leq p_{\mathcal{F}}$. Since the vector semi-norm $p_{\mathcal{F}} : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{R}_+$ is ultrametric, (ii) implies a $\mathbb{K}$-linear mapping $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$ extending f such that $|\tilde{f}| \leq p_{\mathcal{F}}$. Since r is $\mathbb{K}$-linear and fixes $1_{\mathcal{F}}$, r fixes every element of $\mathbb{K}$ in $\mathbb{K}_{\mathcal{F}}$.
(ii) $\Rightarrow$ (i) Given a ultrametric vector semi-norm p on a $\mathbb{K}$-vector space E, and a linear mapping f defined on a vector subspace F of E such that $|f| \leq p$, using Lemma 1, consider a linear mapping $\iota : E \to (\mathbb{K}_{\mathcal{F}})_b$ extending f such that $N_{|.|,\mathcal{F}} \circ \iota \leq p$. Using (ii), let $r : \mathbb{K}_{\mathcal{F}} \to \mathbb{K}$ be a $\mathbb{K}$-linear retraction such that $|r| \leq N_{|.|,\mathcal{F}}$. Then the linear mapping $\tilde{f} := r \circ \iota : E \to \mathbb{K}$ extends f and $|\tilde{f}| \leq p$. □

Remark 6 (Hahn-Banach). Consider the Hahn-Banach statement **HB**: "Given a vector space E over $\mathbb{R}$, given a subadditive mapping $p : E \to \mathbb{R}$ such that for every $\lambda \in \mathbb{R}_+$ and every $x \in E$, $p(\lambda.x) = \lambda p(x)$, and given a linear form f defined on a vector subspace F of E satisfying $|f| \leq p$, there exists a linear mapping $\tilde{f} : E \to \mathbb{R}$ extending f such that $|\tilde{f}| \leq p$". It is known (see [4]) that **BPI** $\Rightarrow$ **HB** and that **HB** $\not\Rightarrow$ **BPI**. It is also known (see [8]) that **HB** is equivalent to the following statement: "For every filter $\mathcal{F}$ on a set I, there exists a $\mathbb{R}$-linear mapping $r : (\mathbb{R}_{\mathcal{F}})_b \to \mathbb{R}$ such that $r(1) = 1$ and r is positive.

Question 7. Is there a ultrametric spherically complete valued field $(\mathbb{K}, |.|)$ such that **HB** is equivalent to $\mathbf{I}_{(\mathbb{K},|.|)}$? Given two distinct prime numbers p and q, are the statements $\mathbf{I}_{\mathbb{Q}_p}$ and $\mathbf{I}_{\mathbb{Q}_q}$ equivalent? Are they equivalent to **HB**? Here we denote by $\mathbb{Q}_p$ (see [11, p. 186]) the valued field which is the Cauchy-completion of $\mathbb{Q}$ endowed with the p-adic absolute value.

Remark 7. Every ultrametric valued field in which every large ball is compact is spherically complete.

Corollary (van Rooij, [10])**.** *For every ultrametric valued field* $(\mathbb{K}, |.|)$ *such that every large ball of* $\mathbb{K}$ *is compact, then* **BPI** *implies* $\mathbf{I}_{(\mathbb{K},|.|)}$.

Proof. For sake of completeness, we give the proof sketched by van Rooij. Using Lemma 2, it is sufficient to show that given a filter $\mathcal{F}$ on a set I, there is a $\mathbb{K}$-linear mapping $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$ such that $|r| \leq N_{|.|,,\mathcal{F}}$. Using **BPI**, let $\mathcal{U}$ be a ultrafilter on I such that $\mathcal{F} \subseteq \mathcal{U}$. For every $x \in (\mathbb{K}_{\mathcal{F}})_b$, the large ball $B(0, N_{|.|,\mathcal{F}}(x))$ of $\mathbb{K}$ is compact and Hausdorff, whence for every $(u_i)_{i \in I} \in \mathbb{K}^I$ and $(v_i)_{i \in I} \in \mathbb{K}^I$ such that x is the class of $(u_i)_{i \in I}$ and the class of $(v_i)_{i \in I}$ in

$\mathbb{K}_{\mathcal{F}}$, then $(u_i)_{i\in I}$ and $(v_i)_{i\in I}$ both converge through the ultrafilter $\mathcal{U}$ to the same element of the ball $B(0, N_{|.|,\mathcal{F}}(x))$: we denote by $r(x)$ this element of $\mathbb{K}$. We have defined a mapping $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$. Then $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$ is additive and fixes every element of $\mathbb{K}$ thus r is $\mathbb{K}$-linear. And $|r| \leq N_{|.|,,\mathcal{F}}$ by construction. □

Remark 8. In particular, given a finite field $\mathbb{K}$ endowed with the trivial absolute value, then $\mathbb{K}$ is compact thus **BPI** implies $\mathbf{D}_{\mathbb{K}}$: this is Howard and Tachtsis's result (see [5, Theorem 3.14]).

Question 8 (van Rooij, see [10])**.** Does **BPI** imply the full Ingleton statement?

5. Isometric linear extenders

5.1. Continuous dual of a semi-normed vector space over a valued field.

Proposition. *Given a valued field* $(\mathbb{K}, |.|)$ *and two semi-normed* $\mathbb{K}$*-vector spaces* (E, p) *and* (F, q)*, a linear mapping* $T : E \to F$ *is continuous with respect to the topologies associated to the semi-norms* p *and* q *if and only if there exists a real number* $M \in \mathbb{R}_+$ *satisfying* $q(T(x)) \leq p(x)$ *for every* $x \in E$*.*

Given a valued field $(\mathbb{K}, |.|)$, and two semi-normed $\mathbb{K}$-vector spaces (E, p) and (F, q), we denote by $CL(E, F)$ the vector space of continuous linear mappings from E to F. Given some continuous linear mapping $T : E \to F$ and some real number $M \in \mathbb{R}_+$, the following properties are equivalent:

(1) $\forall x \in E\ q(x) \leq Mp(x)$
(2) $\forall x \in E\ (p(x) \leq 1 \Rightarrow q(T(x)) \leq M)$

The real number $\sup\{q(T(x)) : x \in E \text{ and } p(x) \leq 1\}$ is called the *norm* of the operator T, and is denoted by $\|T\|_{CL(E,F)}$ (or $\|T\|$). The mapping $N : L(E, F) \to \mathbb{R}_+$ associating to each continuous operator $T \in L(E, F)$ its norm $\|T\|$ is a vector semi-norm, which is ultrametric is the semi-norm q of F is ultrametric. We denote by E' the normed vector space $CL(E, \mathbb{K})$, which is called the *continuous dual* of the semi-normed space E.

Given a valued field $(\mathbb{K}, |.|)$, a semi-normed $\mathbb{K}$-vector space (E, p) and a vector subspace F of E, a *continuous linear extender* from F' to E' is a continuous linear mapping $T : F' \to E'$ such that for every $f \in F'$, $T(f)$ extends f; moreover, if for every $f \in F'$, $T(f)$ has the same norm as f, then the continuous linear extender T is *isometric*.

5.2. Ultrametric isometric linear extenders.

Given a spherically complete valued field $(\mathbb{K}, |.|)$, we consider the “isometric linear extender” statement:
$\mathbf{LE}_{(\mathbb{K},|.|)}$: *“For every vector subspace* F *of a ultrametric semi-normed* $\mathbb{K}$*-vector space* (E, p)*, there exists an isometric linear extender* $T : F' \to E'$*.*

Theorem 2. *Given a spherically complete ultrametric valued field* $(\mathbb{K}, |.|)$*, the statements* $\mathbf{I}_{(\mathbb{K},|.|)}$ *and* $\mathbf{LE}_{(\mathbb{K},|.|)}$ *are equivalent.*

Proof. $\mathbf{I}_{(\mathbb{K},|.|)} \Rightarrow \mathbf{LE}_{(\mathbb{K},|.|)}$: Given some vector subspace F of a ultrametric semi-normed vector space E, let I be the set of mappings $\Phi : F' \to E'$ and let R be the binary relation on $fin(F') \times I$ such that for every $Z \in fin(F')$ and every $\Phi \in I$, $R(Z, \Phi)$ if and only if for every $f \in Z$, the continuous linear form $\Phi(f)$ extends f and $\|\Phi(f)\| = \|f\|$, and Φ is $\mathbb{K}$-linear on $\text{span}_{F'}(Z)$. Then the binary relation R is concurrent: given m finite subsets $Z_1, \ldots, Z_m \in fin(F')$, let $B = \{f_1, \ldots, f_p\}$ be a (finite) basis of the $\mathbb{K}$-vector subspace of F' generated by the finite set $\cup_{1\leq i\leq m} Z_i$; then using $\mathbf{I}_{(\mathbb{K},|.|)}$, let $\tilde{f}_1, \ldots, \tilde{f}_p$ be continuous linear forms on E

extending $f_1, \ldots, f_p$ such that for each i, $\|\tilde{f}_i\| = \|f_i\|$, let $L : \mathrm{span}(\{f_1, \ldots, f_p\}) \to E'$ be the linear mapping such that for each $i \in \{1, \ldots, p\}$, $\Phi(f_i) = \tilde{f}_i$, and let $\Phi : F' \to E'$ be some mapping extending L (for example, define $\Phi(f) = 0$ for every $f \in F' \setminus \mathrm{span}(\{f_1, \ldots, f_p\})$. Then for every $i \in \{1, \ldots, m\}$, $R(Z_i, \Phi)$. Consider the filter $\mathcal{F}$ on I generated by $\{R(Z); Z \in fin(F')\}$. Then the mapping $\Phi : F' \to L(E, \mathbb{K}_{\mathcal{F}})$ associating to each $f \in F'$ the $\mathbb{K}$-linear mapping $\Phi(f) : E \to \mathbb{K}_{\mathcal{F}}$ associating to each $x \in E$ the class of $(f(x))_{f \in I}$ in $\mathbb{K}_{\mathcal{F}}$ is linear, and for every $f \in F'$, $\Phi(f) : E \to \mathbb{K}_{\mathcal{F}}$ extends f and for every $x \in E$, $N_{|.|,\mathcal{F}}(\Phi(f)(x)) \leq \|f\| \|x\|$. Using $\mathbf{I}_{(\mathbb{K},|.|)}$ and Lemma 2, consider a $\mathbb{K}$-linear retraction $r : (\mathbb{K}_{\mathcal{F}})_b \to \mathbb{K}$ of $j_{\mathcal{F}} : \mathbb{K} \to \mathbb{K}_{\mathcal{F}}$ such that for every $x \in (\mathbb{K}_{\mathcal{F}})_b$, $|r(x)| \leq N_{|.|,\mathcal{F}}(x)$. Let $T : F' \to E'$ be the mapping $f \mapsto r \circ \Phi(f)$. Then the mapping $T : F' \to E'$ is an isometric linear extender.
The implication $\mathbf{LE}_{(\mathbb{K},|.|)} \Rightarrow \mathbf{I}_{(\mathbb{K},|.|)}$ is trivial. $\square$

Remark 9. It can be proved in a similar way, that **HB** is equivalent to the following statement: "Given a real normed space $(E, \|.\|)$ and given a subspace F of E, there exists an isometric linear extender $T : F' \to E'$."

References

[1] A. Blass. Existence of bases implies the axiom of choice. In *Axiomatic set theory (Boulder, Colo., 1983)*, volume 31 of *Contemp. Math.*, pages 31–33. Amer. Math. Soc., Providence, RI, 1984.

[2] M. N. Bleicher. Some theorems on vector spaces and the axiom of choice. *Fund. Math.*, 54:95–107, 1964.

[3] W. Hodges. *Model theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[4] P. Howard and J. E. Rubin. *Consequences of the Axiom of Choice*, volume 59. American Mathematical Society, Providence, RI, 1998.

[5] P. Howard and E. Tachtsis. On vector spaces over specific fields without choice. *MLQ Math. Log. Q.*, 59(3):128–146, 2013.

[6] A. W. Ingleton. The Hahn-Banach theorem for non-Archimedean valued fields. *Proc. Cambridge Philos. Soc.*, 48:41–45, 1952.

[7] T. J. Jech. *The Axiom of Choice*. North-Holland Publishing Co., Amsterdam, 1973.

[8] W. A. J. Luxemburg. Reduced powers of the real number system and equivalents of the Hahn-Banach extension theorem. In *Applications of Model Theory to Algebra, Analysis, and Probability (Internat. Sympos., Pasadena, Calif., 1967)*, pages 123–137. Holt, Rinehart and Winston, New York, 1969.

[9] M. Morillon. Linear forms and axioms of choice. *Comment. Math. Univ. Carolin.*, 50(3):421–431, 2009.

[10] A. C. M. van Rooij. The axiom of choice in p-adic functional analysis. In *p-adic functional analysis (Laredo, 1990)*, volume 137 of *Lecture Notes in Pure and Appl. Math.*, pages 151–156. Dekker, New York, 1992.

[11] S. Warner. *Topological fields*, volume 157 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1989. Notas de Matemática [Mathematical Notes], 126.

Laboratoire d'Informatique et Mathématiques, Parc Technologique Universitaire, Bâtiment 2, 2 rue Joseph Wetzell, 97490 Sainte Clotilde

E-mail address: Marianne.Morillon@univ-reunion.fr

URL: http://lim.univ-reunion.fr/staff/mar/