Linear and nonlinear characterisation of loading systems under piecewise discontinuous disturbances voltage: analytical and numerical approaches

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Linear and Nonlinear Characterisation of Loading Systems under Piecewise Discontinuous Disturbances Voltage: Analytical and Numerical Approaches

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Abstract – The characteristics of the dynamic systems exerted by discontinuous piecewise constant voltage loading are investigated. Compared to the systems under continuous voltage exertions, diverse behaviours of the systems with discontinuous exertions are found. Responses of the systems to various piecewise constant exertions under different initial conditions are analyzed theoretically and numerically. Conditions of oscillation and stability are also presented for such piecewise systems.

Keywords – Energy saving, nonlinear-dynamic loading systems, discontinuous exertions voltage, piecewise constant arguments, oscillation, numerical simulation.

INTRODUCTION

In many electronic systems, loadings acting on the systems are discontinuous and can be considered as piecewise constants. Numerous mathematical investigations on the differential equations with piecewise constant arguments of retarded and advanced types such as $q([t])$ or $q([t \pm n])$ were reported [1,2]. The behaviour of the linear first-order differential equations involving piecewise constant arguments has attracted great attention in the past years. However, in the current literature, there is still a lack of systematic studies on the properties of charging of the dynamic systems subjected to piecewise loading voltage. In 1997, Dai and Singh [3] introduced a novel piecewise constant argument $[Nt]/N$ for analytically and numerically solving the second order differential equations which govern the nonlinear dynamic systems exerted by piecewise constant voltages. With this piecewise constant argument, the gap between the dynamic systems subjected to continuous loadings and the systems under piecewise exertions was filled. Employing the piecewise constant argument $[Nt]/N$, the present work investigates the dynamic systems subjected to piecewise constant voltage with focus on the extraordinary behaviour of charging systems.

Theoretical analysis of the properties of loading of dynamic systems under a piecewise constant exertion voltage is to be undertaken. The results of the corresponding loadings of the systems will also be studied numerically and graphically with various combinations of coefficients in the equations of loading and different initial conditions so that the behaviour of the systems under piecewise constant voltage may be visualized and comprehensively understood. Oscillatory loading of the dynamic systems will then be examined with the help of the diagrams of charge, current and piecewise constant voltage exertion against time. The solutions obtained for the systems will be compared with those of the corresponding continuous systems.

Due to the characteristics of the discontinuous exertions, loading of a dynamic system disturbed by the piecewise constant voltage shows an entirely different behaviour from that of the corresponding continuous system. For instance, the loading of the systems acted by piecewise constant loaded voltage can be very sensitive to initial conditions even for the linear dynamic systems. The peculiar behaviour will be demonstrated for several linear and nonlinear dynamic systems under piecewise constant exertions. As will be seen in the present work, the exponential matrices to be derived characterize the oscillatory behaviour of a system subjected to piecewise constant voltage.

Analysis of the properties of loading for these systems may therefore be conveniently performed on the basis of the attributes of the exponential matrices. The oscillatory condition for the loading of the system will be given with respect to the eigenvalues of the corresponding exponential matrices. Oscillatory and asymptotic properties of loading of the dynamic systems subjected to piecewise constant voltage will be theoretically analyzed in detail on the basis of the exponential matrices derived in the present work.

LOADING OF DYNAMICAL SYSTEMS UNDER PIECEWISE CONSTANT VOLTAGE

For the sake of clarification, we first consider a system governed by the following equation in which a sinusoidally varying piecewise constant exertion voltage acting on the system is independent of the charge $q(t)$

$$L\ddot{q} + R\dot{q} + C^{-1}q = A\cos\left(\Omega\frac{[Nt]}{N}\right) \qquad (1)$$

In the equation, L is the inductance of the system, R denotes the resistor coefficient, C the capacitance constant, A the amplitude of the piecewise constant exertion, $[\bullet]$ represents a function of greatest integer and N is an integer which controls the size of the time period on which the external exertion is a constant. Assume that $R^2 > 4LC^{-1}$ in equation (1). The value of the piecewise constant voltage shown in the governing equation can be calculated for any given time. It is therefore desirable to start the analysis of the properties of loading with initial conditions:

$$q(t=t_0)=q_0 \text{ and } \dot{q}(t=t_0)=i(t=t_0)=i_0 \quad (2)$$

Within an arbitrary time segment $[Nt]/N \le t < ([Nt]+1)/N$, the solution of the linear system described by equation (1) is readily available. This implies that the loading in any time interval represents a portion of a linear charging. At the ending point of an interval, the charge and current of the system are given by $q(([Nt]+1)/N)$ and $\dot{q}(([Nt]+1)/N)$ respectively. These specific results in turn become the starting conditions or the local initial conditions for the loading of the next time interval. With the local initial conditions, the loading in the follow-up intervals can then be consequently obtained. Considering that the loading of the system is continuous, i.e., $q(t)$ and $\dot{q}(t)$ are continuous on $t \in [0,\infty[$, the following conditions of continuity must be satisfied

$$\begin{cases} q_{[Nt]}\left(\dfrac{[Nt]}{N}\right) = q_{[Nt]-1}\left(\dfrac{[Nt]}{N}\right) \\ \dot{q}_{[Nt]}\left(\dfrac{[Nt]}{N}\right) = \dot{q}_{[Nt]-1}\left(\dfrac{[Nt]}{N}\right) \end{cases} \quad (3)$$

With the continuity conditions and the solutions for each of the time intervals, the solution of equation (1) for the entire time range considered can be derived as follows:

$$\begin{aligned} q(t) = e^{-\alpha\left(t-\frac{[Nt]}{N}\right)} & \Bigg[\cos\left(\tau\left(t-\frac{[Nt]}{N}\right)\right) \\ & + \frac{\alpha}{\tau^2}\sin^2\left(\tau\left(t-\frac{[Nt]}{N}\right)\right)\sin\left(\tau\left(t-\frac{[Nt]}{N}\right)\right)\Bigg]B \\ & + \frac{A}{\omega_0^2}\cos\left(\Omega\frac{[Nt]}{N}\right)e^{-\alpha\left(t-\frac{[Nt]}{N}\right)}\Bigg[e^{-\alpha\left(t-\frac{[Nt]}{N}\right)} \\ & - \cos\left(\tau\left(t-\frac{[Nt]}{N}\right)\right) - \frac{\alpha}{\tau}\sin\left(\tau\left(t-\frac{[Nt]}{N}\right)\right)\Bigg] \end{aligned} \quad (4)$$

where $\omega_0 = (LC)^{-\frac{1}{2}}$, $\alpha = \dfrac{R}{2L}$ and $\tau = \sqrt{\omega_0^2 - \alpha^2}$.

In equation (4), the matrix B takes the form

$$\begin{aligned} B = {} & e^{-\alpha\frac{[Nt]}{N}} D^{[Nt]} \begin{bmatrix} q_0 \\ i_0 \end{bmatrix} \\ & + \sum_{k=1}^{[Nt]} e^{-\alpha\frac{k}{N}} D^{k-1} \begin{bmatrix} e^{\frac{\alpha}{N}} - \cos\frac{\tau}{N} - \frac{\alpha}{\tau}\sin\frac{\tau}{N} \\ \left(\frac{\alpha^2}{\tau}+\tau\right)\sin\frac{\tau}{N} \end{bmatrix} \frac{A}{\omega_0^2}\cos\left(\Omega([Nt]-k)\right) \end{aligned} \quad (5)$$

in which the square matrix D has the form

$$D = \begin{bmatrix} \cos\frac{\tau}{N} + \frac{\alpha}{\tau}\sin\frac{\tau}{N} & \frac{1}{\tau}\sin\frac{\tau}{N} \\ -\left(\frac{\alpha^2}{\tau}+\tau\right)\sin\frac{\tau}{N} & \cos\frac{\tau}{N} - \frac{\alpha}{\tau}\sin\frac{\tau}{N} \end{bmatrix} \quad (6)$$

To visualize the loading of the systems governed by equation (1), we study a simple case of loading without damping numerically by substituting the values of system parameters and time into the analytical solution equation (4). Based on the solution of equation (1), corresponding to this specific case, the diagram of charge versus time is plotted in fig. 1 on which the corresponding time plots of current and piecewise constant voltage are illustrated separately in the figures (a), (b) and (c). It can be seen from the figures and the solution in equation (4), the curves of charge and current are continuous everywhere for $t \ge 0$ under the discontinuous piecewise constant exertion. It should be noted in the diagram that the charge and its slope are continuous, but the current shows slope discontinuities at the integer points of time. The discontinuities are a consequence of the discontinuous piecewise constant voltage acting on the system.

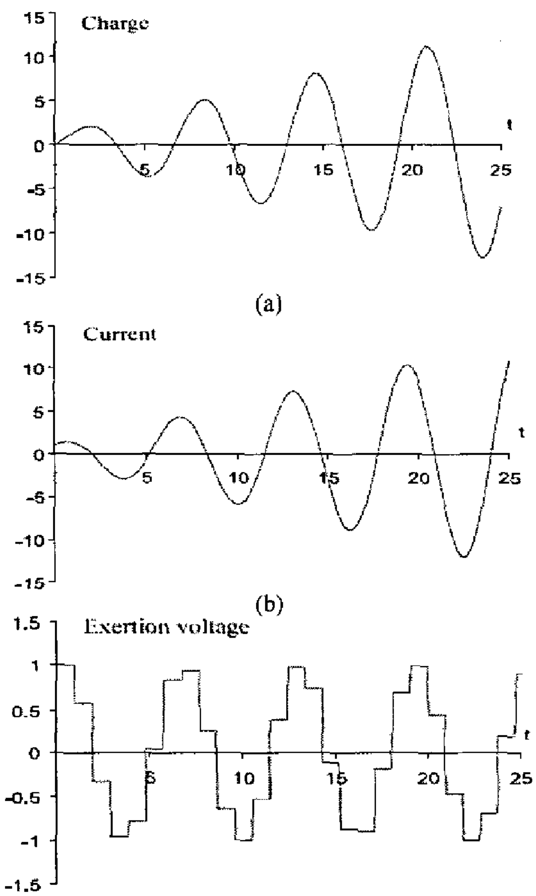


Fig. 1 : Time plots of charge (a), current (b) and piecewise constant voltage (c) acting on a loading system governed by

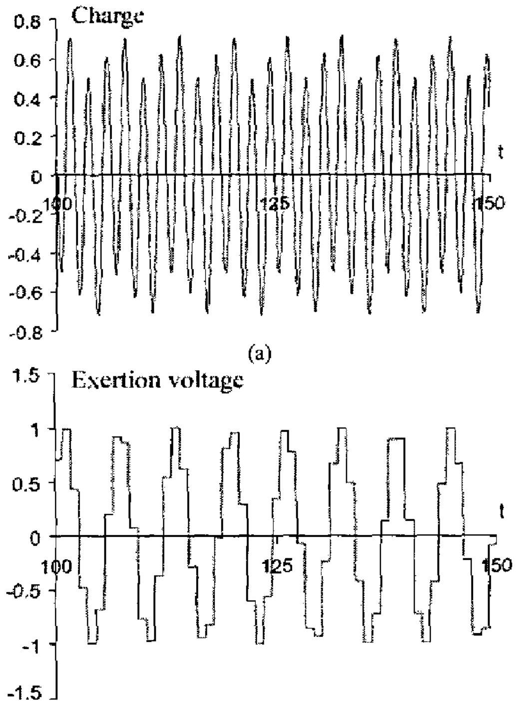
$$L\ddot{q} + R\dot{q} + C^{-1}q = A\cos\left(\Omega\frac{[Nt]}{N}\right), \text{ L=1, R=0, N=1, } C^{-1}=1,$$
$$\text{A=1}, \Omega = 1.$$

Fig. 1 also reveals the asymptotically divergent oscillatory behaviour of the system.

Under certain conditions, the dynamic system subjected to piecewise constant voltage may behave as a simple loading system with stable loading of constant amplitude. Consider a similar system as the case shown in fig. 1 with a higher C^{-1}. Under the same conditions and after a relatively long time from t=0, this system shows a stable oscillatory loading as exhibited in fig. 2. It can be seen from fig. 2, the waveforms of the charge are repeating precisely with a period of 2π, although the shapes of the stepwise exertions are different from period to period.

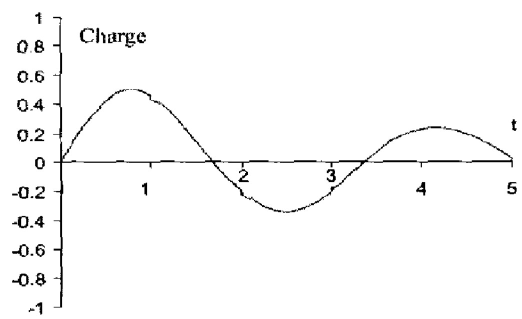
When the external piecewise constant exertion is charge related, the complete solution for a time interval of the system must first be obtained so that the end conditions for this interval, and the starting conditions of the consecutive interval, may be determined accordingly. Consider the following dynamic system governed by the differential equation,

$$L\ddot{q}+C^{-1}q=Aq\left(\frac{[Nt]}{N}\right) \qquad (7)$$

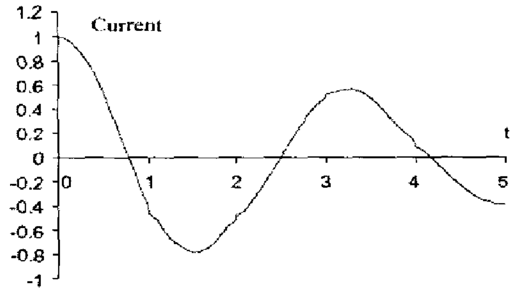


(b)

Fig. 2 : Waveform (a) and the corresponding piecewise constant voltage (b) acting on a loading system governed by $L\ddot{q}+R\dot{q}+C^{-1}q=A\cos\left(\Omega\frac{[Nt]}{N}\right)$, L=1, R=0, N=1, $C^{-1}=9$, A=1, $\Omega=1$.



(a)



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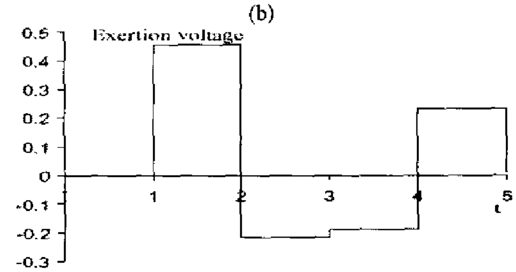


(c)

Fig. 3 : Time plots of charge (a), current (b) and piecewise constant voltage (c) acting on a loading system governed by $L\ddot{q}+R\dot{q}+C^{-1}q=Aq\left(\frac{[Nt]}{N}\right)$, L=1, R=0, N=1, $C^{-1}=4$, A=1, $\Omega=1$. Convergence of the charge is as shown in figure (a).

The charge of such a loading system is graphically demonstrated in fig. 3. As displayed, the piecewise constant voltage exerting on the system has the same value as the charge at the integer point of time, and the corresponding loading in this case is asymptotically convergent.

Loading of a dynamic system subjected to a piecewise constant voltage of the type shown in equation (7) exhibits, in general, a dramatically different behavior from those of the corresponding system exerted by continuous voltage. Compare a damped loading system subjected to a piecewise constant voltage described by the equation of charge:

$$L\ddot{q}(t)+R\dot{q}(t)+C^{-1}q(t)=Aq\left(\frac{[Nt]}{N}\right) \qquad (8)$$

with a similar linear loading system governed by the following equation of charge

$$L\ddot{q}(t)+R\dot{q}(t)+C^{-1}q(t)=Aq(t) \qquad (9)$$

where R, L, C^{-1} and A are constants of the system's physical properties. $R^2>4LC^{-1}$ in equations (8) and (9).

The solution of the linear system in equation (9) is readily available and the solution of equation (8) can be derived as follows:

$$q(t)=e^{-\alpha t}\left\{(1-AC)\left[\cos\left(\tau\left(t-\frac{[Nt]}{N}\right)\right)+\frac{\alpha}{\tau}\sin\left(\tau\left(t-\frac{[Nt]}{N}\right)\right)\right]\right.$$
$$\left.+ACe^{-\alpha\left(t-\frac{[Nt]}{N}\right)}\frac{1}{\tau}\sin\left(\tau\left(t-\frac{[Nt]}{N}\right)\right)\right\}G\begin{bmatrix}q_0\\ i_0\end{bmatrix} \qquad (10)$$

In equation (10), the constants α and τ are the same as those defined in equation (4), and the square matrix G takes the form

$$G=\begin{bmatrix} (1-AC)\left[\cos\tau-\dfrac{\alpha}{\tau}\sin\tau\right]+ACe^{-\alpha} & \dfrac{\sin\tau}{\tau} \\ -(1-AC)\left[\left(\dfrac{\alpha^2}{\tau}+\tau\right)\sin\tau\right] & \cos\tau-\dfrac{\alpha}{\tau}\sin\tau \end{bmatrix}^{[Nt]} \quad (11)$$

A comparison of the continuous system and the system with piecewise constant voltage is illustrated in fig. 4 in which the loading with the piecewise constant voltage $-8q\left(\frac{[Nt]}{N}\right)$ diverges asymptotically and, on the contrary, the charge under the continuous voltage $-8q(t)$ is damped out rapidly with time.

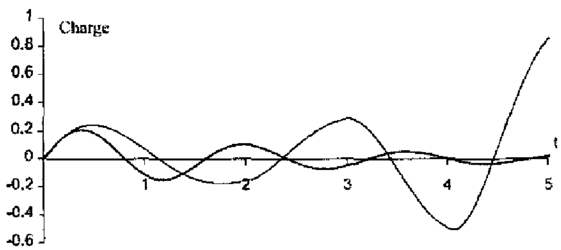


Fig. 4 : Comparison of the loading of the system subjected to piecewise constant voltage governed by $L\ddot{q}(t)+R\dot{q}(t)+C^{-1}q(t)=Aq\left(\frac{[Nt]}{N}\right)$ and the continuous system $L\ddot{q}(t)+R\dot{q}(t)+C^{-1}q(t)=Aq(t)$, L=1, R=0, N=1, $C^{-1}=1.5$, A=-8. In this case, the thinner solid line illustrates the solution of the piecewise constant system, and the thicker line is that of the continuous system.

One may note that the solutions in the form of equations (4) and (10) are unique. When a finite value of N is given (say N=1), the systems governed by equation (1) or equation (8) becomes the specific piecewise constant systems as those studied in [1,2]. Solutions for the systems with finite N values are readily available as expressed in equations (4) and (10). Moreover, the systems expressed in equations (1) and (8) may also be continuous systems when N tends to infinity as described in [3]. As such, the solutions in the forms of equations (4) and (10) bridge the gap between the piecewise constant systems and continuous systems.

For the loading systems subjected to known forms of piecewise constant and continuous voltage respectively, such as $A\cos\left(\Omega\frac{[Nt]}{N}\right)$ and $A\cos(\Omega t)$ which are merely time dependent, the contrast is also evident. As an example, the lading of the system governed by equation (1) is compared with the following system subjected to a continuous voltage,

$$L\ddot{q}(t)+R\dot{q}(t)+C^{-1}q(t)=A\cos(\Omega t), \quad (12)$$

as shown in fig. 5. It can be observed in the figure that the loading of the system subjected to the continuous voltage becomes a steady-state oscillation with identical shape as time increases, whereas the system under the piecewise constant voltage oscillates with the waveforms of distinct appearances and amplitudes.

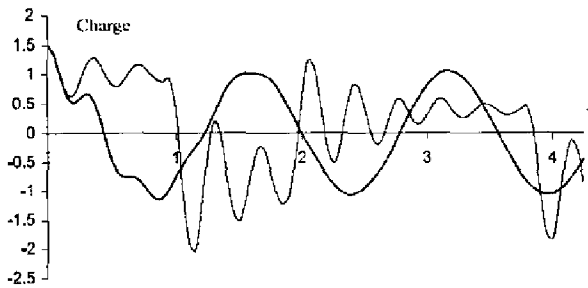
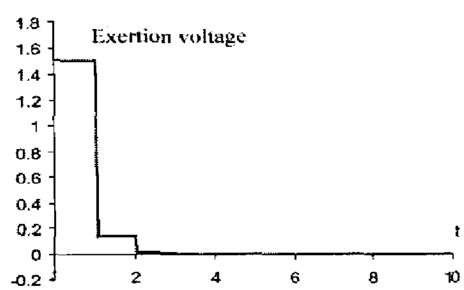


Fig. 5 : Comparison of the loading of the system subjected to piecewise constant voltage governed by $L\ddot{q}(t)+R\dot{q}(t)+C^{-1}q(t)=A\cos\left(\Omega\frac{[Nt]}{N}\right)$ and the continuous system $L\ddot{q}(t)+R\dot{q}(t)+C^{-1}q(t)=A\cos(\Omega t)$, L=1, R=3.1, N=1, $C^{-1}=311$, A=311, $\Omega=4$. In this case, the thinner solid line illustrates the solution of the piecewise constant system, and the thicker line is that of the continuous system.

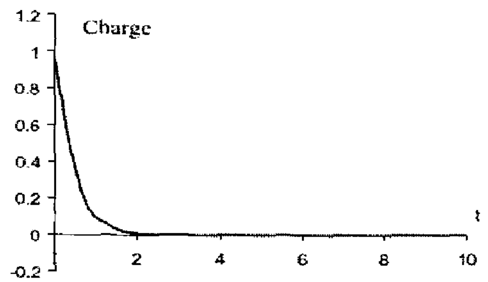
BEHAVIOUR OF DYNAMICAL SYSTEMS SUBJECTED TO PIECEWISE CONSTANT VOLTAGE

It has been demonstrated in the previous section that a dynamic system subjected to a piecewise constant voltage exhibits different oscillatory behaviour from that of the corresponding continuous system. In fact, as will be shown in the following sections, the loading of a dynamic system with piecewise constant exertions voltage may be oscillatory, non-oscillatory, stable or unstable with different coefficients and initial conditions.

An interesting phenomenon of a dynamic system subjected to a piecewise constant voltage is that its loading may asymptotically vanish even when the system is free of damping. Because of this, an undamped dynamic system under piecewise constant voltage cannot be considered as a conservative system. A simple example is shown in fig. 6 in which the loading of a capacitor disappears under the piecewise constant voltage shown. Solution for this problem can be obtained from the solution presented in equation (10). The fading of the loading of this system depends upon the voltage amplitude A, number of iteration $[Nt]$ and initial conditions. Once the local initial conditions become zeros, the loading of the capacitor will permanently stay at rest.



(a)



(b)

Fig. 6: Charge (b) and its corresponding piecewise constant voltage (a) acting on a system governed by $\ddot{q}(t) + Aq\left(\frac{[Nt]}{N}\right) = 0$, A=-1.5; Initial conditions: $q_0 = 1.0$ and $i_0 = -1.656$.

In general, when starting perturbation acting on the dynamical system is preserved, fundamental loadings of linear charging systems subjected to continuous voltage are not sensitive to initial conditions. However, under certain conditions, a dynamic system subjected to piecewise constant voltages may show sensitive dependence upon initial conditions. The above system can again be taken as an example. The sensitivity of the system to the initial conditions is illustrated in fig. 7 by starting two loadings from adjacent states. Under the piecewise constant exertions voltage, the two adjacent starts appear to remain close to each other for a time and then rapidly move apart. It may also be pointed out that a small perturbation of a parameter of such a system may also lead to a greater variation of the loading.

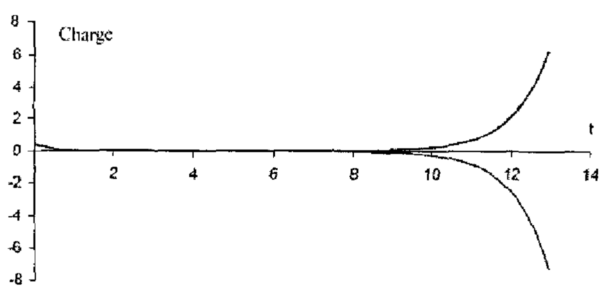


Fig. 7 : Sensitivity of the solution of the dynamic system governed by $\ddot{q}(t) + Aq\left(\frac{[Nt]}{N}\right) = 0$, A=-2.1; Initial conditions for the loading of the thinner solid line: $q_0 = 0.48395$ and $i_0 = -1.0$. Initial conditions for the loading of the thicker line: $q_0 = 0.48396$ and $i_0 = -1.0$.

So far, only the loadings with linear piecewise constant voltages are investigated. If nonlinear piecewise constant voltages are involved in the dynamic systems, the corresponding loading will become complex and nonlinear properties will be brought into the systems. Replace the piecewise constant voltage $A\cos\left(\Omega\frac{[Nt]}{N}\right)$ equation (1) by $Aq^3\left(\frac{[Nt]}{N}\right)$ and compare its solution with the corresponding continuous system governed by the following equation of loading:

$$L\ddot{q}(t) + R\dot{q}(t) + C^{-1}q(t) = Aq^3(t), \qquad (13)$$

The differential equation is actually the famous Duffing equation [4], for which the numerical results are given, among others by Fang and Dowell [5] and Christopher [6]. A solution with a cubic piecewise constant exertion voltage is illustrated in fig. 8 in comparison with a continuous system presented in equation (13). The numerical results for the nonlinear systems are calculated with Runge–Kutta method of forth order [8]. The step length used for the numerical calculations is 0.03. It can be seen from fig. 8 that the loading of the system with the piecewise constant exertion voltage are highly distinct from the one under the continuous voltage.

Quantitatively, the loading under the piecewise constant voltage varies much conspicuously in terms of amplitude and slope in comparison with the oscillation of the continuous nonlinear system corresponding to the Duffing equation. As may be observed from fig. 8, at t=0 and the integer points t=1, 2, 3 the magnitude of the exertion $Aq^3\left(\frac{[Nt]}{N}\right)$ does not vary much during the time period $0 \le t < 4$ due to the small charge q calculated by using the solution. However, the charge q is rapidly increased in absolute value on $4 \le t < 5$ as shown in the figure.

Corresponding to the increased charge, the magnitude of the cubic voltage $Aq^3\left(\frac{[Nt]}{N}\right)$ becomes much greater than that of the voltage at t=3, and the sign of the voltage $Aq^3\left(\frac{[Nt]}{N}\right)$ at t=4 is opposite to the sign of the voltage $Aq^3\left(\frac{[Nt]}{N}\right)$ at t=3. A sharp rise of the solution is thus produced at the integer point t=4, as is clearly shown in the diagram. Although there is also a cubic term in the continuous system of equation (13), oscillation of the system is relatively gentle and smooth since all the voltages involved are varying continuously.

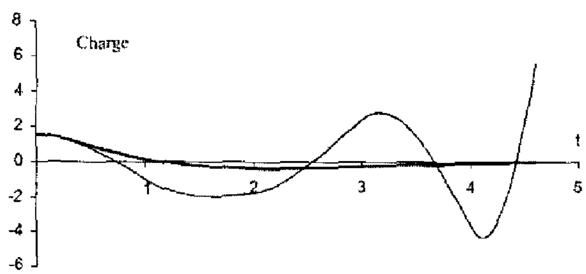


Fig. 8 : Comparison of the solutions of the nonlinear dynamic systems described by the continuous systems $L\ddot{q}(t) + R\dot{q}(t) + C^{-1}q(t) = Aq^3(t)$ and the piecewise constant system of $L\ddot{q}(t) + R\dot{q}(t) + C^{-1}q(t) = Aq^3\left(t\frac{[Nt]}{N}\right)$. $A/L = 2.0$, N=1, $R/L = 1.5$ and $1/CL = 1.0$. Initial conditions for the loadings of the systems: $q_0 = 1.5$ and $i_0 = 0.0$. The thinner solid line is for the piecewise system and the thicker solid line for the continuous system.

Sensitive dependence of a dynamic system upon initial conditions may also occur, under certain conditions, in a system subjected to piecewise constant voltages. The sensitivity to initial conditions is clear in fig. 9 in which the two loadings of a charging system exerted by piecewise constant voltages are plotted. The two loadings from nearby states (slightly different initial current and the same initial charge) remain close for a time, and then diverge from each other and become uncorrelated. Sensitivity of a dynamic system to initial conditions is a significant measure in nonlinear and chaotic dynamics.

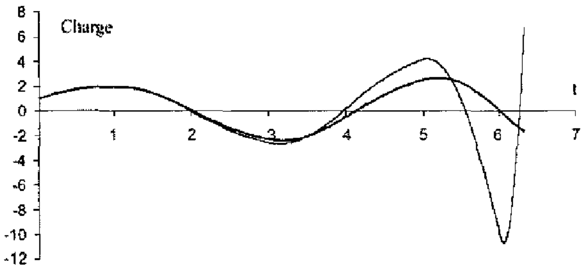


Fig. 9 : Sensitivity of the solution of the piecewise constant system $L\ddot{q}(t)+R\dot{q}(t)+C^{-1}q(t)=Aq^{3}\left(t\frac{[Nt]}{N}\right)$ to initial conditions. $A/L=-0.46$, N=1, $R/L=0.52$ and $1/CL=0.5$. Initial conditions for the loading of the thicker solid line: $q_0=1.0$ and $i_0=2.0$. Initial conditions for the loading of the thinner solid line: $q_0=1.0$ and $i_0=2.01$.

OSCILLATORY PROPERTIES OF THE DYNAMIC SYSTEMS

As can be seen from the solutions presented in the previous sections that the charge and the current of a dynamic system subjected to piecewise constant exertions voltages are continuous everywhere though the exertions applying on the system is discontinuous. An advantage that one may take from the continuous solutions is that the properties of the loading of a system subjected to piecewise constant voltages can be analytically studied.

Regarding the properties of the solutions given in the previous section, the following points may be remarked:

a. The square matrices in equations (6) and (11) are associated with the local initial conditions. In each time interval $[Nt]/N \le t < ([Nt]+1)/N$, loading of the dynamic system is largely affected by the local initial conditions $q_{[Nt]}$ and $i_{[Nt]}$ especially in the cases that the parameter N is large.
b. Independent of the linear loading between the two ends of a time interval $[Nt]/N \le t < ([Nt]+1)/N$, the values of the charge and current at the instants of $t=[Nt]$ are given by the square matrices.
c. In each time interval $[Nt]/N \le t < ([Nt]+1)/N$, there is a linear loading system corresponding to it.
d. The term $e^{-\alpha t}$ vanishes as t increases.
e. The absolute values of the sine and cosine functions in the solutions lie between 0 and unit.
f. q_0 and i_0 in the solutions are independent of the time t.

On accounting the above remarks, the properties of the dynamic systems such as that governed by equations (1) and (8) are characterized by the matrices with exponents. Therefore, in order to find the behaviour of loading for the dynamic systems, one needs only finding the properties of the exponential matrices. This significantly simplifies the investigation on the characteristics of the behaviour of the dynamic systems subjected to piecewise constant voltage.

Designating the square matrices as Q and the corresponding eigenvalues as λ_0 and λ_1, the square matrices can be written in the following form:

$$Q=M\begin{bmatrix}\lambda_0 & 0\\ 0 & \lambda_1\end{bmatrix}M^{-1} \tag{14}$$

where the matrix M represents a set of linearly independent eigenvectors for the corresponding eigenvalues and M^{-1} expresses the inverse of M.

For the real non-symmetrical matrix Q, it is well known [3,7] that

a. For $|\lambda|=1$, the solutions derived for the dynamic systems under piecewise constant exertions are stable.
b. For $|\lambda|<1$, with increase of time t, the corresponding solutions will be convergent or asymptotically convergent, and therefore, can be considered as another case of a stable solution.
c. For $|\lambda|>1$, solutions are monotonically or asymptotically divergent.

Through an iterative procedure, the exponential square matrix in equation (5) or equation (11) can be expressed in the following form:

$$Q^{[Nt]}=M\begin{bmatrix}\lambda_0^{[Nt]} & 0\\ 0 & \lambda_1^{[Nt]}\end{bmatrix}M^{-1} \tag{15}$$

M and are M^{-1} constant matrices, therefore, only the diagonal matrix in equation (15) varies with argument $[Nt]$ as time t increases. As such, the following conclusions can be stated as

a. In the case that the eigenvalues λ_0 and λ_1 are both negative, the values of $\lambda_0^{[Nt]}$ and $\lambda_1^{[Nt]}$ are both positive for even $[Nt]$; and the signs of $\lambda_0^{[Nt]}$ and $\lambda_1^{[Nt]}$ are both negative with odd $[Nt]$. Due to the varying signs of the exponential square matrices, the corresponding solution will be oscillatory with increasing time t.
b. In the case that the eigenvalues λ_0 and λ_1 are positive, the values of $\lambda_0^{[Nt]}$ and $\lambda_1^{[Nt]}$ are always positive and the absolute value of the corresponding solution will then divergent with increasing time t.
c. The eigenvalues may also be complex. In this case, the eigenvalues are expressible as a polar form $\lambda=\rho e^{i\theta}$, where ρ and θ are constants. The corresponding loadings of the dynamic systems in this case are then oscillatory due to the alternating signs of $\lambda_0^{[Nt]}$ and $\lambda_1^{[Nt]}$ with changing time t.

Based on the above conclusions, the properties of the loadings of the dynamic systems subjected to piecewise constant voltages can be conveniently analyzed with considerations of various values of the coefficients of physical properties of the systems, such as L, R, N, C^{-1}, A, Ω as used in those governing differential equations.

CONCLUSIONS

This study is conducted to investigate the characteristics of the loading of the linear and nonlinear dynamic systems subjected to piecewise constant exertions voltages. Solutions for the dynamic systems with piecewise constant exertions voltages are developed with the piecewise constant argument $\frac{[Nt]}{N}$. Loadings of the piecewise constant systems are analyzed theoretically and numerically.

Extraordinary behaviours of the dynamic systems are found in the study in comparing with the dynamic systems under continuous voltages. Based on the results obtained in the investigation, it is significant and important to summarize the main characteristics of the behaviour of the piecewise constant systems as the following:

a. Loading of the dynamic systems disturbed by piecewise discontinuous exertions voltages is continuous.
b. Dynamic systems show greatly different behaviour from the regular systems when piecewise constant exertions are involved. Solutions of the piecewise constant systems can be determined via the procedures described in Section 2 with the piecewise constant argument $\frac{[Nt]}{N}$.
c. The piecewise constant systems may exhibit harmonic behaviour even in the cases in which the magnitude of the piecewise constant exertions varies from period to period.
d. Loading of dynamic systems subjected to piecewise constant exertions shows sensitivity to initial conditions under certain conditions.
e. Loading of the piecewise constant systems may vanish with the piecewise constant exertions.

Oscillatory and asymptotic properties of the loading of the piecewise constant systems are analyzed in the present work. The exponential matrices with the exponents $[Nt]$ have a great advantage in theoretically analyzing the properties of the piecewise constant systems. Detailed processes of the analysis are presented for the solutions obtained.

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