Ballistics during 18th and 19th centuries: What kind of mathematics?

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Two recent papers ([1], [7]) have studied the scientific and social context of ballistics during and around the First World War, and have put in evidence the collaborations and tensions that have been existing between two major milieus, the one of artillerymen, that is engineers and officers in the military schools and on the battlefield, and the other one of mathematicians that were called to solve difficult theoretical problems. My aim is to give a similar survey for the previous period, that is to say during the second half of the 18th century and the 19th century.

The main problem of exterior ballistics – I won't speak of interior ballistics, which is nearer to physics and chemistry than mathematics – is to determine the trajectory of a projectile launched from a cannon with a given angle and a given velocity. The principal difficulty encountered here is that the differential equations of motion involve the air resistance $F(v)$, which is an unknown function of the velocity v. In fact, the problem is more complex because we must take into account other factors like the variations of the atmospheric pressure and temperature, the rotation of the Earth, the wind, the geometric form of the projectile and its rotation around its axis, etc. However these effects could be often neglected in the period considered here, because the velocities of projectiles remained small.

For a long time, artillerymen have made the assumption that the trajectory is parabolic, but this was not in agreement with the experiments. Newton was the first to research this topic taking into account the air resistance. In his *Principia* of 1687, he solved the problem with the hypothesis of a resistance proportional to the velocity, and he got quite rough approximations when the resistance is proportional to the square of the velocity. After Newton, Jean Bernoulli discovered the general solution in the case of a resistance proportional to any power of the velocity, but his solution, published in the *Acta Eruditorum* of 1719, was not convenient for numerical computation. After Bernoulli, many attempts have been done to treat mathematically the ballistic equation. We may organize these attempts throughout two main strategies, one analytical and one numerical.

The analytical strategy consists in integrating the differential equation in finite terms or, alternatively, by quadratures. Reduction to an integrable equation can be achieved in two ways: 1) choose an air resistance law so that the equation can be solved in finite form, leaving it to the artillerymen to decide after if this law can satisfy their needs; 2) if a law of air resistance is imposed through experience, change the other coefficients of the equation to make it integrable, with of course the risk that modifying the equation could modify also the solution in a significant way.

In 1744, D'Alembert restarts the problem of integrability of the equation. Acting here as a geometer, concerned only with progress of pure analysis, he finds four new cases of integrability. His work went relatively unnoticed at first: Legendre in 1782, and Jacobi in 1842 have found again certain of the same cases of integrability, but without quoting D'Alembert.

During the 19th century, we can observe a parallelism between the increasing velocities of bullets and cannonballs, and the appearance of new instruments to measure these velocities [2]. Ballisticians have therefore felt the necessity of proposing new air resistance laws for certain intervals of velocity [3]. Thus, certain previous theoretical developments, initially without applications, led to tables that were actually used by the artillerymen. The fact that some functions determined by artillerymen from experimental measurements fell within the scope of integrable forms has reinforced the idea that it might be useful to continue the search for such forms.

It is within this context that Francesco Siacci resumes the theoretical search for integrable forms of the law of resistance. In two papers published in 1901, he discovers ten families of air resistance laws corresponding to new integrable equations. The question of integrability by quadratures of the ballistic equation is finally resolved in 1920 by Jules Drach [5], a brilliant mathematician who has contributed much in Galois theory of differential equations. Drach exhausts therefore the problem from a theoretical point of view, but his very complicated results are greeted without enthusiasm by the ballisticians, who do not see at all how to transform them into practical applications.

Another way was explored by theoreticians who accepted Newton's law of the square of the velocity, and tried to act on other terms of the ballistic equation to make it integrable. In 1769, Borda proposes to assume that the medium density is variable and to choose, for this density, a function that does not stray too far from a constant and makes the equation integrable. Legendre deepens Borda's ideas in his essay on the ballistic question [8], with which he won in 1782 the prize of the Berlin Academy. After Legendre, many other people, for example Siacci at the end of the 19th century [9], have developed similar ideas to obtain very simple, general, and practical methods of integration.

The second strategy for integrating the ballistic differential equation, that is to say the numerical approach, contains three main procedures: 1) calculate the integral by successive small arcs; 2) develop the integral into an infinite series and keep the first terms; 3) construct graphically the integral curve.

Euler is truly at the starting point of the calculation of firing tables in the case of the square of the velocity [6]. In 1755, he resumes Bernoulli's solution and puts it in a form that will be convenient for numerical computation. The integration is then done by successive arcs: each small arc of the curve is replaced by a small straight line, whose inclination is the mean of the inclinations at the extremities of the arc. A little later, Grävenitz achieves the calculations of the program conceived by Euler and publishes firing tables in Rostock in 1764. In

1834, Otto improves Euler's method and calculates new range tables that will experience a great success, and will be in use until the early 20th century.

Another approach is that of series expansions. In the second half of the 18th century and early 19th, we are in the era of calculation of derivations and algebraical analysis. The expression of solutions by infinite series whose law of formation of terms is known, is considered to be an acceptable way to solve a problem exactly, despite the philosophical question of the infinite and the fact that the series obtained, sometimes divergent or slowly convergent, do not always allow an effective numerical computation. Lambert, in 1765, is one of the first to express as series the various quantities involved in the ballistic problem. On his side, Français applies the calculation of derivations for obtaining a number of explicit new formulas. However, he himself admits that these series are too complicated for applications.

Let us mention finally graphical approaches providing artillerymen with an easy and economic tool. Lambert in 1767, and Obenheim in 1818 have the similar idea of replacing some previous ballistic tables by a set of curves carefully drawn by points. In 1848, Didion [4], following some of Poncelet's ideas, constructs some ballistic curves that are not a simple graphic representation of numerical tables, but are obtained directly from the differential equation by a true graphical calculation. Artillery was so the first domain of engineering science in which graphical tables, called "abaques" in French, were commonly used.

In conclusion, throughout the 18th and 19th centuries, there has been an interesting interaction between analytic theory of differential equations, numerical and graphical integration, and empirical research through experiments and measurements. Mathematicians, ballisticians and artillerymen, although part of different worlds, collaborated and inspired each other regularly. All that led however to a relative failure, both experimentally to find a good law of air resistance, and mathematically to find a simple solution of the ballistic differential equation.

Mathematical research on the ballistic equation has nevertheless played the role of a laboratory where the modern numerical analysis was able to develop. Mathematicians have indeed been able to test on this recalcitrant equation all possible approaches to calculate the solution of a differential equation. There is no doubt that these tests, joined with the similar ones conceived for the differential equations of celestial mechanics, have helped to organize the domain into a separate discipline at the beginning the 20th century.

References

[1] D. Aubin, *'I'm just a mathematician': Why and how mathematicians collaborated with military ballisticians at Gâvre*, 2010, http://hal.upmc.fr/hal-00639895/fr/.

[2] F. Bashforth, *A Mathematical Treatise on the Motion of Projectiles, founded chiefly on the results or experiments made with the author's chronograph* London, 1873.

[3] C. Cranz and E. Vallier, *Balistique extérieure*, in: Encyclopédie des sciences mathématiques pures et appliquées, J. Molk and P. Appel (eds.), tome IV, 6^{e} volume, 1er fascicule, Paris: Gauthier-Villars and Leipzig: Teubner, 1913.

[4] I. Didion, *Traité de balistique*, Paris: Leneveu, 1848.

[5] J. Drach, *L'équation différentielle de la balistique extérieure et son intégration par quadratures*, Annales scientifiques de l'École normale supérieure **37** (1920), 1–94

[6] L. Euler, *Recherches sur la véritable courbe que décrivent les corps jettés dans l'air ou dans un autre fluide quelconque*, Mémoires de l'Académie des sciences de Berlin, **9** (1755), 321-352.

[7] A. Gluchoff, *Artillerymen and mathematicians: Forest Ray Moulton and changes in American exterior ballistics, 1885-1934*, Historia Mathematica **38** (2011), 506–547.

[8] A.-M. Legendre, *Dissertation sur la question de balistique*, Berlin: Decker, 1782.

[9] F. Siacci, *Balistica*, Torino: Casanova, 1888.