# Mathematics of nomography

Dominique Tournès

# Mathematics of nomography


## Dominique Tournès

LIM, EA 2525, University of La Réunion, France

SPHERE, UMR 7219, CNRS, University Paris-Diderot, France


Traditionally, history of mathematics was more interested in the production of abstract concepts and major theories than in the one of methods and tools of calculation. As a result, it has often neglected to consider the mathematical practices of engineers. I want to sketch here a sample of these practices through the example of nomography in the second half of the 19th century and the first half of the 20th century, precisely at the time when the professional community of engineers structures itself and deepens its mathematical culture.

### 1. Mathematics of engineers

The mathematical needs of engineers seem very different from those of mathematicians. To illustrate this with a significant example, consider the problem of the numerical solution of equations, a pervasive problem in all areas of mathematics intervention. Léon-Louis Lalanne (1811-1892), a French civil engineer who, throughout his career, sought to develop practical methods for solving equations, wrote what follows (see Tournès 2012, p. 382):

> “The applications have been, until now, the stumbling block of all the methods devised for solving numerical equations, not that, nor the rigor of these processes, nor the beauty of the considerations on which they are based, could have been challenged, but finally it must be recognized that, while continuing to earn the admiration of geometers, the discoveries of Lagrange, Cauchy, Fourier, Sturm, Hermite, etc., did not always provide easily practicable means for the determination of the roots”.

Lalanne says that as politely as possible, but his conclusion is clear: the methods advocated by mathematicians are not satisfactory. They are complicated to understand, long to implement and sometimes totally impracticable for ground engineers, foremen and technicians, who, moreover, did not always receive a high-level mathematical training.

Given such situation, 19th-century engineers were often forced to imagine by themselves the operational methods and the calculation tools that mathematicians could not provide them. The objectives of the engineer are not the same as those of the mathematician, the physicist or the astronomer: the engineer rarely needs high accuracy in his calculations, he is rather sensitive to the speed and simplicity of their implementation, especially since he often has to perform numerous and repetitive operations.

## 2. About the training of engineers during 19th century

The 19th century is the moment of the first industrial revolution, which spreads throughout the Western world at different rates in different countries. Industrialization causes profound transformations of society. In this process, the engineering world acquires a new identity, marked by its implications in the economic development of industrial states and the structuration of new professional relationships that transcend national boundaries. Linked to the Industrial Revolution, enormous computational requirements appeared during the 19th century in all areas of engineering sciences and caused an increasing mathematization of these sciences. This led naturally to the question of engineering education: how were engineers prepared to use high-level mathematics in their daily work and, if necessary, to create by themselves new mathematical tools?

The French model of engineering education in the early 19th century is that of the *École polytechnique*, founded in 1794. Although it had initially the ambition to be comprehensive and practice-oriented, this school quickly promoted a high-level teaching dominated by mathematical analysis. This teaching only theoretical was then completed, from the professional point of view, by two years in application schools with civil and military purposes. Such a training model, which subordinates practice to theory, has produced a corporation of “scholarly engineers” capable of using the theoretical resources acquired during their studies to achieve an unprecedented mathematization of the engineering art.

This model is considered to have influenced the creation of many polytechnic institutes throughout Europe and to the United States. However, unlike France, the teaching of the theory and practice are held together in these new schools, and the training they offer is far from monolithic. These institutes propose courses directly adapted to the future professional practice of engineering students, such as courses of descriptive geometry, graphic statics, or graphical calculation.

Unlike continental Europe, there are no polytechnic institutes in England for the training of civil engineers; the latter is organized by employers inside an apprenticeship system. This is done so first and foremost in the business, before being integrated later, towards the end of the century, into the new universities of industrial cities: the "red-brick universities". The result is a much less intense standardization of the engineering profession than on the Continent, and a less formal training.

## 3. The paradigmatic case of nomography

The main purpose of nomography is to construct graphical tables to represent any relationship between three variables, and, more generally, relationships between any number of variables. Why I am especially interested in nomography is that it is the paradigmatic example of a corpus of mathematical tools, constituting an autonomous discipline, which was created from scratch by engineers themselves to meet their needs.

### 3.1 The birth of nomography through the "cut and fill" problem

Among the "Founding Fathers" of nomography, four were students of the *École polytechnique*: Lalanne, Charles Lallemand (1857-1938), Maurice d'Ocagne (1862-1938) and Rodolphe Soreau (1865-1936). The only exception in this list is the Belgian engineer Junius Massau (1852-1909), an ancient student and then professor at the school of civil engineering of the University of Ghent, but, in this school of civil engineering, the training was comparable to that of the École polytechnique, with high-level courses of mathematics and mechanics.

During the years 1830-1860, the sector of public works experiences a boom in France and more generally in Europe. The territories of the different countries are covered progressively by vast networks of roads, canals, and, after 1842, of railways. These achievements require many tedious calculations of surfaces of "cut and fill" on cross-sections of the ground. Cut and fill is the process of earthmoving needed to construct a road, a canal or a railway. You have to cut land where the ground level is too high and transport this land to fill the places where the ground level is too low. And to calculate roughly the volume of land to be transported, you have to decompose this volume in thin vertical slices, evaluate the area of each slice and sum all these elementary areas.

Civil engineers tried then different methods of calculation more or less expeditious. Some, like Gaspard-Gustave Coriolis (1792-1843), have calculated numerical tables giving the surfaces directly based on a number of features of the road and its environment. Other engineers, especially in Germany and Switzerland, designed and built several kinds of planimeters, that is mechanical instruments used to quickly calculate the area of any plane surface. These planimeters, which concretize the continuous summation of infinitesimal surfaces, had significant applications in many other scientific fields beyond cuts and fills. Still others, like Lalanne, have imagined replacing numerical tables by graphical tables, cheaper and easier to use. It is within this framework that nomography developed itself and was deepened throughout the second half of the 19th century.

### 3.2. First principles of nomography

The departure point of nomography lies in the fact that a relationship between three variables $\alpha$, $\beta$ and $\gamma$ can be considered, under certain conditions, as the result of the elimination of two auxiliary variables *x* and *y* between three equations, each containing only one of the initial variables. One can then represent the equation by three sets of lines in the plane *x-y*, one of them parametrized by $\alpha$, the second by $\beta$ and the third by $\gamma$. On this kind of graphical table, called a "concurrent-line abaque", a solution of the equation corresponds to an intersection point of three lines (see Figure 1).

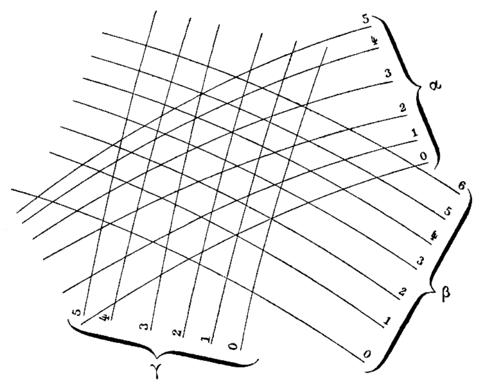


Fig. 1: Concurrent-line abaque

Isolated examples of graphical translation of double-entry tables are found already in the first half of the 19th century, mainly in the scope of artillery, but this is especially Lalanne who gave a decisive impetus to the theory of graphical tables. In 1843, he provided consistent evidence that any law linking three variables can be graphed in the same manner as a topographic surface using its marked level lines. His ideas came to a favorable moment. Indeed, the Act of June 11, 1842 had decided to establish a network of major railway lines arranged in a star from Paris. To run the decision quickly, one felt the need for new ways of evaluating the considerable earthworks to be carried out. In 1843, the French government sent to all engineers involved in this task a set of graphical tables for calculating the areas of cut and fill on the profile of railways and roads.

Curves other than straight lines are difficult to construct on paper. For this reason, Lalanne imagined the use of non-regular scales on the axes for transforming curves into straight lines. By analogy with the well-known optical phenomenon previously used by certain painters, he called “anamorphosis” this general transformation process. After Lalanne, the graphical tables resting on the principle of concurrent lines spread rapidly until becoming, in the third quarter of the 19th century, very common tools in the world of French engineers.

Massau succeeded Lalanne to enrich the method and its scope of applications. For that, he introduced a notion of generalized anamorphosis, seeking what are the functions that can be represented using three pencils of lines. Massau put in evidence that a given relationship between three variables can be represented by a concurrent-straight-line abaque if, and only if, it can be put into the form of a determinant of the type

$$\begin{vmatrix} f_1(\alpha) & f_2(\alpha) & f_3(\alpha) \\ g_1(\beta) & g_2(\beta) & g_3(\beta) \\ h_1(\gamma) & h_2(\gamma) & h_3(\gamma) \end{vmatrix} = 0.$$

These determinants, called “Massau determinants”, played an important role in the subsequent history of nomography; they are encountered in research until today. As an application of this new theory, Massau succeeded in simplifying Lalanne's abaques for cuts and fills. With Massau’s publications, the theory of abaques was entering into a mature phase, but in the same time a new character intervened to orient this theory towards a new direction.

### 3.3. From concurrent-straight-line abaques to alignment nomograms

In 1884, when he is only 22 years old, d'Ocagne observes that most of the equations encountered in practice can be represented by an abaque with three systems of straight lines and that three of these lines, each taken in one system, correspond when they meet into a point. His basic idea is then to construct by duality, by substituting the use of tangential coordinates to that of punctual coordinates, a figure in correlation with the previous one: each line of the initial chart is thus transformed into a point, and three concurrent lines are transformed into three aligned points. The three systems of marked straight lines become three marked curves. Through this transformation, a concurrent-straight-line abaque becomes an “alignment abaque”, which is easier to use (see Figure 2).

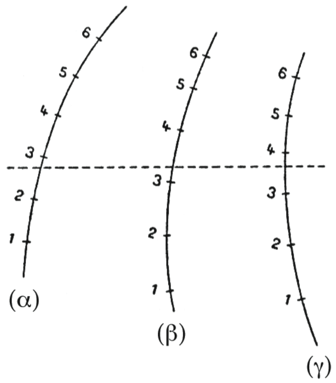


Fig. 2: Alignment abaque

A given relationship between three variables is representable by an alignment abaque if, and only if, it can be put into the form of a Massau determinant, because it is clear that the problem of the concurrency of three straight lines and the problem of the alignment of three points, dual to each other, are mathematically equivalent. As his predecessors, d’Ocagne applied immediately his new ideas to the problem of cuts and fills, actually one of the main problems of civil engineering.

**3.4. Diffusion of nomography**

After this first achievement in 1891, d’Ocagne deepened the theory and applications of alignment abaques until the publication of a large treatise in 1899, the famous *Traité de nomographie*, which became for a long time the reference book of the new discipline. A little later, he introduced the generic term “nomogram” to replace “abaque”, and the science of graphical tables became “nomography”. From there, alignment nomograms were quickly adopted by many engineers for the benefit of the most diverse applications. At the turn of the 20th century, nomography was already an autonomous discipline well established in the landscape of applied sciences.

Nomography was transmitted to Germany by Friedrich Schilling (1868-1950), who translated and adapted d’Ocagne's work in 1900. Carl Runge (1856-1927) also played an important role to popularize graphical methods in general, through his teaching in Göttingen and New York, and several important books in English and in German printed from his lessons. In the first half of the 20th century, Germany became the paradise of nomography, with even more applications and publications than in France.

To appreciate rightly the social and economic importance of nomography at this time, the best thing is to look at the inventory made in 1950 by Douglas Payne Adams. He collected 700 references on nomography from 97 American journals and he classified these papers into 21 themes. These themes cover not only the classical parts of civil engineering, but also a lot of other human activities. Conceived at the origin for the cut and fill problem and more generally for civil engineering problems, nomography extended its application field to unexpected new domains, like statistics, chemistry or medicine.

## 4. Mathematical problems issued from nomography

The mathematical practices of engineers are often identified only as “applications”, which is equivalent to consider them as independent from the development of mathematical knowledge in itself. In this perspective, the engineer is not supposed to develop a truly mathematical activity. We want to show, through the example of nomography, that this representation is somewhat erroneous: it is easy to realize that the engineer is sometimes a creator of new mathematics, and, in addition, that some of the problems which he arises can in turn irrigate the theoretical research of mathematicians.

### 4.1. The anamorphosis problem

Firstly, the problem of general anamorphosis, that is to say, of characterizing the relationships between three variables that can be put in the form of a Massau determinant, has inspired many theoretical research to mathematicians and engineers: Cauchy, Saint-Robert, Massau, Lecornu, and Duporcq have brought partial responses to this problem before that in 1912 the Swedish mathematician Thomas Hakon Gronwall (1877-1932) gives a complete solution resulting in the existence of a common integral to two very complicated partial differential equations. But, as one can easily imagine, this solution was totally inefficient, except in very simple cases.

After Gronwall, other mathematicians considered the problem of anamorphosis in a different way, with a more algebraic approach that led to study the important theoretical problem of linear independence of functions of several variables. These mathematicians, like Kellogg in the US, wanted to find a more practical solution not involving partial differential equations. A complete and satisfactory solution was finally found by the Polish mathematician Mieczyslaw Warmus (1918-2007). In his Dissertation of 1958. Warmus defined precisely what is a nomographic function, that is a function of two variables that can be represented by an alignment nomogram, and classified nomographic functions through homography into 17 equivalence classes of Massau determinants. Moreover, he gave an effective algorithm for determining if a function is nomographic and, if true, for representing it explicitly as a Massau determinant.

**4.2 The Hilbert's 13th problem**

Beyond the central problem of nomographic representation of relationships between three variables, which define implicit functions of two variables, there is the more general problem of the representation of functions of three or more variables. Engineers have explored various ways in this direction, the first consisting in decomposing a function of any number of variables into a finite sequence of functions of two variables, which results in the combined use of several nomograms with three variables, each connected to the next by means of a common variable.

Such a practical concern was echoed unexpectedly in the formulation of the Hilbert's 13th problem, one of the famous 23 problems that were presented at the International Congress of Mathematicians in 1900. The issue, entitled "Impossibility of the solution of the general equation of the 7th degree by means of functions of only two arguments", is based on the initial observation that up to the sixth degree, algebraic equations are nomographiable.

Indeed, up to the fourth degree, the solutions are expressed by a finite combination of additions, subtractions, multiplications, divisions, square root extractions and cube root extractions, that is to say by functions of one or two variables. For the degrees 5 and 6, the classical Tschirnhaus transformations lead to reduced equations whose solutions depend again on one or two parameters only. The seventh degree is then the first actual problem, as Hilbert (1902, p. 462) remarks:

> "Now it is probable that the root of the equation of the seventh degree is a function of its coefficients which does not belong to this class of functions capable of nomographic construction, i. e., that it cannot be constructed by a finite number of insertions of functions of two arguments. In order to prove this, the proof would be necessary that the equation of the seventh degree is not solvable with the help of any continuous functions of only two arguments."

In 1901, Ocagne had found a way to represent the equation of the seventh degree by a nomogram involving an alignment of three points, two being carried by simple scales and the third by a double scale. Hilbert rejected this solution because it involved a mobile element. Without going into details, we will retain that there has been an interesting dialogue between an engineer and a mathematician reasoning in two different perspectives.

In the terms formulated by Hilbert, it was only in 1957 that the 13th problem is solved negatively by Vladimir Arnold (1937-2010), who proved to everyone's surprise that every continuous function of three variables could be decomposed into continuous functions of two variables only.

## 5. Final remarks

In conclusion, recent research by historians shows more clearly that mathematical knowledge and mathematical representations are part of various social groups in interaction, in which they find various legitimacies. Within this new framework, history of mathematics should enrich itself by taking greater account of the engineering community, within which specific mathematical practices, original and fruitful, did exist.

Moreover, as these old practices are often based on numerical, graphical and instrumental methods translating in a simple and concrete manner the key concepts of mathematics, these practices should constitute a fruitful source of inspiration for creating relevant activities to be exploited nowadays in mathematics education.

## Literature

Adams, D. P. (1950): An Index of Nomograms. New York: Wiley.

Hilbert, D. (1902). Mathematical problems. In: Bulletin of the American Mathematical Society 8, p. 437-479.

Ocagne, M. d' (1899): Traité de nomographie. Théorie des abaques, applications pratiques. Paris: Gauthier-Villars.

Runge, C. (1912): Graphical Methods. New York: Columbia University Press.

Schilling, F. (1900): Über die Nomographie von M. d'Ocagne. Eine Einführung in dieses Gebiet. Leipzig: Teubner.

Tournès, D. (2011): Une discipline à la croisée de savoirs et d'intérêts multiples: la nomographie. In: Circulation Transmission Héritage, P. Ageron et É. Barbin (eds), Caen: Université de Caen-Basse-Normandie, p. 415-448.

Tournès, D. (2012): Mathematics of the 19th-century engineers: methods and instruments. In: Proceedings of HPM 2012, Daejon: KSME, p. 381-393.